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Hecke L-functions of certain subextensions in an extraspecial extension

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Abstract. In 1925, Hecke found two different quadratic fields having the same L-functions attached to certain ray class groups. In this paper, we show that if $K/\mathbb{Q}$ is a Galois extension whose Galois group is isoclinic to an extraspecial group, then there are many elementary abelian extensions inside K whose L-functions coincide.

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§1. Introduction

Let F be a number field and K an abelian extension over F. By class field theory, a quotient group H of a ray class group is isomorphic to the Galois group $\mathrm{Gal}(K/F)$. Then a complex linear character $\chi\colon H \longrightarrow \mathbb{C}$ is called a ray class character. For a ray class character χ of the ray class group mod $\mathfrak{f}$, the Hecke L-function is defined by

$$L(s,\chi) = \prod \frac{1}{1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s}},$$

where $\mathfrak{p}$ runs over all prime ideals of F not dividing $\mathfrak{f}$. Hecke observed a phenomenon where there exist distinct quadratic fields F_1, F_2 and ray class characters χ_1, χ_2 of F_1, F_2, respectively, satisfying the equality between Hecke L-functions

$$L(s,\chi_1) = L(s,\chi_2)$$

up to a finite number of Euler factors. We call such a phenomenon the coincidence of L-functions. Hecke observed such a coincidence for a real quadratic field F_1 and an imaginary quadratic field F_2.

Ishii [8] studied the coincidence of cyclic Galois extensions over $\mathbb{Q}$. He showed that if the projective image of the representation is abelian, then the coincidence of L-functions of cyclic extensions occurs.

Kida and Namura [11] proved that if the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is isoclinic to the dihedral group D_4 of order 8, then L-functions of three distinct quadratic fields coincide. This is exactly the case Hecke previously studied.

In our previous paper [10], we studied the coincidence in a more general setting where the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is isoclinic to the Heisenberg group He_p of order p^3 with prime p and showed that the $p+1$ L-functions of $p+1$ distinct cyclic Galois extensions $F_i/\mathbb{Q}$ of degree p coincide. If $p = 2$, then there is an isomorphism $\mathrm{He}_2 \cong D_4$, and thus the result is a natural extension of [11].

The aim of this paper is to generalize these results further. The main theorem of this paper is the following.

Theorem 1.1. *Let p be a prime and n a positive integer. Let $K/\mathbb{Q}$ be a Galois extension with Galois group G isoclinic to an extraspecial group $\mathrm{ES}(n,p)$. For an irreducible character ψ of G of degree p^n, there exist $N = \prod_{i=1}^{n}(p^i+1)$ subfields $F_1, \ldots, F_N$ of K which are elementary abelian extensions over $\mathbb{Q}$ of degree p^n and a ray class character χ_i of a certain ray class field of each F_i such that the equalities of the Hecke L-functions and the Artin L-function*

$$L(s,\chi_1) = L(s,\chi_2) = \cdots = L(s,\chi_N) = L(s,\psi)$$

hold up to a finite number of Euler factors.

For the definitions of isoclinism and extraspecial p-groups $\mathrm{ES}(n,p)$, see Section 2.

Since the groups D_4 and He_p are extraspecial groups, therefore Theorem 1.1 is a generalization of the previous results in [10] and [11].

Throughout this paper, we use the following notation. The cyclic group of order n is denoted by C_n, and by D_n the dihedral group of order $2n$. For a finite group G, we denote by $Z(G)$ the center of G and by G' the commutator subgroup of G. A G-extension is a Galois extension over the rational field $\mathbb{Q}$ with Galois group isomorphic to G. All representations in this paper are complex linear representations. We denote by $\mathrm{Irr}(G)$ the set of all the irreducible characters of G and $\mathrm{Irr}(G)_i = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = i\}$. Let H be a subgroup of G. For $\psi \in \mathrm{Irr}(G)$, we denote by ψ_H the restriction of ψ to H and for $\chi \in \mathrm{Irr}(H)$, we denote by χ^G the induction of χ to G.

The paper is organized as follows. In Section 2, we give some group-theoretic preliminaries. In Section 3, we prove the main theorem and construct Galois extensions over $\mathbb{Q}$ with Galois groups isomorphic to extraspecial groups.

§2. Preliminaries

2.1. Isoclinism

In [4], P. Hall introduced the notion of isoclinism on finite groups, which is a weaker equivalence than isomorphism. The definition is given as follows.

Definition 2.1. Let G_1 and G_2 be finite groups. The groups G_1 and G_2 are *isoclinic* if there exist isomorphisms $\varphi\colon G_1/Z(G_1) \cong G_2/Z(G_2)$ and $\psi\colon G_1' \cong G_2'$ such that the following diagram is commutative:

$$\begin{array}{ccc} G_1/Z(G_1) \times G_1/Z(G_1) & \xrightarrow{k_{G_1}} & G_1{}' \\ \downarrow{\scriptstyle \varphi\times\varphi} & & \downarrow{\scriptstyle \psi} \\ G_2/Z(G_2) \times G_2/Z(G_2) & \xrightarrow{k_{G_2}} & G_2{}', \end{array}$$

where k_{G_1} and k_{G_2} are the commutator maps. If G_1 and G_2 are isoclinic, then we write $G_1 \sim G_2$, and we call the pair (φ, ψ) an *isoclinism.*

Clearly, the orders of G' and $G/Z(G)$ are invariants of the isoclinism class.

We need the following properties of irreducible representations under isoclinism in Section 3.

Proposition 2.2 ([1, III 5.7 Corollary])**.** *Let G_1 and G_2 be isoclinic groups with isoclinism (φ, ψ). We assume that G_1 is a stem group, which is by definition a group satisfying $Z(G_1) \subset G_1'$, and that there exists an injection $Z(G_1) \to Z(G_2)$. Let G be the fiber product $\{(g_1, g_2) \in G_1 \times G_2 \mid \varphi(g_1 Z(G_1)) = g_2 Z(G_2)\}$ of G_1 and G_2. Then each irreducible representation $\widetilde{\rho}$ of G_2 has the form $\widetilde{\rho}(g_2) = \mu^*(g_1, g_2)\tau(g_2)\rho(g_1)$, where $\mu^* \in \mathrm{Irr}(G)_1$, $\tau \in \mathrm{Irr}(G_2)_1$ and $\rho \in \mathrm{Irr}(G_1)$.*

The following corollary is obtained immediately by Proposition 2.2.

Corollary 2.3. *Let the notation and assumptions be as in Proposition 2.2. Let H_i be a subgroup of G_i containing $Z(G_i)$ such that $H_1/Z(G_1)$ and $H_2/Z(G_2)$ are isomorphic via φ. Then the following are equivalent:*

(i) ρ is induced from an irreducible representation of H_1;

(ii) $\widetilde{\rho}$ is induced from an irreducible representation of H_2.

2.2. Extraspecial groups

Let p be a prime. A p-group G is called an extraspecial p-group if $Z(G)$ and G' are isomorphic to C_p and $G/Z(G)$ is an elementary abelian p-group. From definition, all the extraspecial p-groups are nilpotent of class 2 and hence are monomial by [7, Corollary (6.14)]. To state some properties of extraspecial groups, we recall the definition of a central product of finite groups.

Definition 2.4 ([5, I, 9.10 Satz])**.** Let G_1 and G_2 be finite groups. If there exist subgroups H_1 of $Z(G_1)$ and H_2 of $Z(G_2)$ and an isomorphism $\sigma\colon H_1 \to H_2$, then the central product $G_1 \circ G_2$ is defined by

$$G_1 \circ G_2 = (G_1 \times G_2)/Z,$$

where $Z = \{(g_1, g_2) \in H_1 \times H_2 \mid \sigma(g_1) = g_2^{-1}\}$.

In this paper, we only consider the case $H_1 = Z(G_1)$ and $H_2 = Z(G_2)$.

The center of $G_1 \circ G_2$ is isomorphic to $Z(G_1)$ and the quotient $(G_1 \circ G_2)/Z(G_1 \circ G_2)$ is isomorphic to $G_1/Z(G_1) \times G_2/Z(G_2)$. Hence if G_1 and G_2 are extraspecial p-groups, then $G_1 \circ G_2$ is also an extraspecial p-group. For a positive integer n, we define the central product $G_1 \circ \cdots \circ G_n$ inductively.

Proposition 2.5 ([5, III, 13.7 Satz])**.** *An extraspecial p-group G has the following properties:*

(i) The commutator map k_G is a non-degenerate alternative bilinear form over the finite field $\mathbb{F}_p$ of p-elements.

(ii) The order of $G/Z(G)$ is a square.

(iii) G is a central product of extraspecial p-groups of order p^3.

(iv) If the order of $G/Z(G)$ is p^{2n}, then the orders of all maximal abelian normal subgroups of G are p^{n+1}.

By Proposition 2.5 (ii), the order of each extraspecial group is p^{2n+1} with a prime p and a positive integer n.

From [10, Section 2.2], non-abelian p-groups of order p^3 are mutually isoclinic extraspecial groups. If p is odd, then the groups are He_p and $C_{p^2} \rtimes C_p$ ([3, Section 5.5]), where He_p is the Heisenberg group, which is the unitriangular matrix group over $\mathbb{F}_p$ of degree 3

$$\mathrm{He}_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F}_p \right\}.$$

If $p = 2$, then extraspecial 2-groups of order 8 are D_4 and the quaternion group Q_8, and we can show $D_4 \cong \mathrm{He}_2$. Hence all the extraspecial p-groups of orders p^{2n+1} are isoclinic for a fixed pair (p, n). Since we can show that $\mathrm{He}_p \circ \mathrm{He}_p \cong (C_{p^2} \rtimes C_p) \circ (C_{p^2} \rtimes C_p)$ for odd prime p and $D_4 \circ D_4 \cong Q_8 \circ Q_8$, there are exactly two isomorphism classes of extraspecial p-groups of orders p^{2n+1} by Proposition 2.5 (iii). We denote the central product of n Heisenberg groups He_p by $\mathrm{ES}^+(n,p)$ and the other by $\mathrm{ES}^-(n,p)$. If we do not have to distinguish two groups, then we denote an extraspecial p-group of order p^{2n+1} by $\mathrm{ES}(n,p)$.

We need some facts on the irreducible characters of $\mathrm{ES}(n,p)$. From [6, Examples. 7.6 b)], the character degrees of $\mathrm{ES}(n,p)$ are 1 and p^n and the irreducible characters of $\mathrm{ES}(n,p)$ of degree p^n vanish outside $Z(\mathrm{ES}(n,p))$. For an irreducible character χ of degree p^n, since the kernel of χ is contained in $Z(\mathrm{ES}(n,p))$ and not containing $(\mathrm{ES}(n,p))'$, the kernel is trivial and hence χ is faithful. Since the maximal abelian quotient of $\mathrm{ES}(n,p)$ is isomorphic to C_p^{2n}, we have $|\mathrm{Irr}(\mathrm{ES}(n,p))_1| = p^{2n}$ and hence $|\mathrm{Irr}(\mathrm{ES}(n,p))_{p^n}| = p-1$.

§3. Coincidence of Hecke L-functions

In this section, we shall prove Theorem 1.1. By the well-known results of Artin L-functions on induced character, Theorem 1.1 follows from the theorem below.

Theorem 3.1. *Let p be a prime and n a positive integer. Let G be a finite group isoclinic to an extraspecial p-group $\mathrm{ES}(n,p)$ of order p^{2n+1}. Then for an irreducible character ψ of G of degree p^n, there exist exactly $N = \prod_{i=1}^{n}(p^i+1)$ abelian normal subgroups H_j of G containing $Z(G)$ and linear characters χ_j of H_j such that $\chi_1^G = \cdots = \chi_N^G = \psi$.*

In fact, for every irreducible character χ of $\mathrm{ES}(n,p)$ of degree p^n and every maximal abelian subgroup H of $\mathrm{ES}(n,p)$, χ is induced from an irreducible character of H. The following lemma gives the number of the maximal abelian subgroups of G.

Lemma 3.2. *The group $\mathrm{ES}(n,p)$ has $\prod_{i=1}^{n}(p^i+1)$ abelian subgroups of index p^n containing its center.*

Proof of Lemma 3.2. Since $G/Z(G)$ and G' are elementary abelian p-groups, we consider them as $\mathbb{F}_p$-vector spaces. From Proposition 2.5 (i), the commutator map k_G is a non-degenerate alternative bilinear form over $\mathbb{F}_p$. Hence we can consider $G/Z(G)$ as a hyperbolic space of dimension $2n$. Here we note that a subgroup H of G containing $Z(G)$ is abelian if and only if $H/Z(G)$ is an isotropic space with respect to the bilinear form k_G. From Proposition 2.5 (iv), the dimension of a maximal isotropic subspace of $G/Z(G)$ is n. Then by [12] the number of isotropic subspaces of $G/Z(G)$ is $\prod_{i=1}^{n}(p^i+1)$, which equals the number of abelian subgroups of index p^n containing $Z(G)$. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. First, we give several remarks. From the definition of the extraspecial group, $\mathrm{ES}(n,p)$ is a stem group. If G is isoclinic to $\mathrm{ES}(n,p)$, then G is a nilpotent group of nilpotency class 2 and hence we have $G' \subset Z(G)$. From the definition of isoclinism, there is an isomorphism $(\mathrm{ES}(n,p))' \cong G'$. This isomorphism induces the injection $Z(\mathrm{ES}(n,p)) \to Z(G)$ and hence Corollary 2.3 is valid for $\mathrm{ES}(n,p)$ and G. We therefore conclude that an irreducible character of G is induced from a subgroup of G containing $Z(G)$ if and only if the corresponding character of $\mathrm{ES}(n,p)$ is induced form the corresponding subgroup of $\mathrm{ES}(n,p)$ via the isoclinism. Hence we have only to consider the case $G = \mathrm{ES}(n,p)$.

We shall show that for a subgroup H of G, there exists an irreducible character φ of H such that $\varphi^G = \chi$ if and only if H is a maximal abelian subgroup of G.

If there exists a subgroup H of G and $\varphi \in \mathrm{Irr}(H)_1$ such that $\varphi^G = \chi$, then, because χ is faithful, it follows from [7, Problem (2.8)] that H is an abelian subgroup. From [7, Lemma (2.27)], it follows that H contains $Z(G)$. Since $G/Z(G)$ is an abelian group, every subgroup of G containing $Z(G)$ is a normal subgroup. Thus, we have shown that if χ is induced by a linear character of H, then H is an abelian normal subgroup containing $Z(G)$.

Conversely, let H be an abelian normal subgroup of G of index p^n containing $Z(G)$. Recall that $\chi_{Z(G)}$ has a unique irreducible constituent. Let λ be the unique irreducible constituent of $\chi_{Z(G)}$ and φ an irreducible character of H such that $\varphi_{Z(G)} = \lambda$. If φ^G is reducible, then, since the character degrees of G are 1 and p^n, every irreducible constituent of φ^G is a linear character and hence the kernel of φ^G contains G'. On the other hand, since χ is a faithful character and $(\varphi^G)_{Z(G)} = p^n\lambda = \chi_{Z(G)}$, the kernel of φ^G does not contain $Z(G)$ and this is a contradiction. Hence φ^G is an irreducible character. Since every irreducible character of G vanishes outside $Z(G)$, so does φ. Hence we have $\varphi^G = \chi$. We therefore conclude that the number of subgroups H of G such that $\varphi \in \mathrm{Irr}(H)_1$ satisfying $\varphi^G = \chi$ is equal to the number of the abelian subgroups of G of index p^n, which is N by Lemma 3.2. □

Before giving an example, some remarks are in order. Since there exists a surjection $(\mathrm{He}_p)^n \to \mathrm{ES}^+(n,p)$, we have if $L/\mathbb{Q}$ is a $(\mathrm{He}_p)^n$-extension, then there exists a subfield K of L which is an $\mathrm{ES}^+(n,p)$-extension over $\mathbb{Q}$. From [9, Corollary 6.6.6.], there exist infinitely many He_p-extensions over $\mathbb{Q}$. Hence there also exist infinitely many $\mathrm{ES}^+(n,p)$-extensions. For the constructions of special cases of He_p-extensions, see [11, Lemma 7.1] (for the case $p = 2$) and [10, Proposition 3.3] (for the case where p is odd).

Here we give a numerical example for the case $p = 2$ and $n > 1$ using Magma [2]. In this case, since the automorphism group $\mathrm{Aut}(Z(D_4))$ is trivial, every $(D_4)^n$-extension has a unique $\mathrm{ES}^+(n,2)$-subextension K over $\mathbb{Q}$. The field K is constructed as follows.

Proposition 3.3. *Let $L_1, \dots, L_n$ be linearly disjoint D_4-extensions over $\mathbb{Q}$. If we set $F_i = L_i^{Z(\mathrm{Gal}(L_i/\mathbb{Q}))}$ and we choose $\alpha_i \in F_i$ such that $F_i(\sqrt{\alpha_i}) = L_i$, then the extension $F_1 \cdots F_n(\sqrt{\alpha_1 \cdots \alpha_n})$ is an $\mathrm{ES}^+(n,2)$-extension over $\mathbb{Q}$.*

Proof. Let L be the composite field of $L_1, \dots, L_n$ and F be that of $F_1, \dots, F_n$. We let K be the unique subextension of L such that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to $\mathrm{ES}^+(n,2)$. Since the natural surjection $\mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q})$ induces an isomorphism $(D_4)^n/Z((D_4)^n) \cong \mathrm{ES}^+(n,2)/Z(\mathrm{ES}^+(n,2))$, we have $K^{Z(\mathrm{Gal}(K/\mathbb{Q}))} = L^{Z(\mathrm{Gal}(L/\mathbb{Q}))}$. In particular, F is a subfield of K. We define $g_i \in \mathrm{Gal}(L/F)$ by

$$g_i \colon \sqrt{\alpha_i} \mapsto -\sqrt{\alpha_i},\ \sqrt{\alpha_j} \mapsto \sqrt{\alpha_j},\ (i \neq j).$$

From the definition of central product, $\mathrm{Gal}(L/K)$ is generated by $g_i g_j$ $(1 \le i, j \le n)$. Since $\sqrt{\alpha_1 \cdots \alpha_n}^{g_i g_j} = \sqrt{\alpha_1 \cdots \alpha_n}$ for any i and j, the field $M = F(\sqrt{\alpha_1 \cdots \alpha_n})$ contains K. Because $\alpha_1, \dots, \alpha_n$ are linearly independent in $F^\times/(F^\times)^2$, we have $\sqrt{\alpha_1 \cdots \alpha_n} \notin F$. Hence we have $[M : F] = 2 = [K : F]$. We therefore conclude $K = M$ and complete the proof. $\square$

Example 3.4. Let $n \in \mathbb{N}$. Let q_i, l_i $(1 \le i \le n)$ be distinct odd primes such that $q_i \equiv 1 \pmod 4$ and $\left(\frac{q_i}{l_i}\right) = 1$. Set $l_i^* = (-1)^{(l_i-1)/2} l_i$. Then the Diophantine equation $x^2 - q_i y^2 - l_i^* z^2 = 0$ has integer solutions. Let x_i, y_i, z_i be primitive solutions. Since x_i is odd, then y_i and z_i have different parities. Let w_i be equal to either y_i or z_i which is even. We define integers $s_i, r_i \in \mathbb{Z}$ by

$$s_i = \begin{cases} 1, & z_i \text{ is even}, \\ 2, & y_i \text{ is even}, \end{cases}$$

$$r_i = \begin{cases} s_i, & x_i - w_i \equiv 1 \pmod 4, \\ -q_i^* s_i, & x_i - w_i \equiv 3 \pmod 4. \end{cases}$$

Then from [11, Lemma 7.1], the extension

$$\mathbb{Q}\left(\sqrt{r_i\left(x_i + z_i\sqrt{l_i^*}\right)}, \sqrt{q_i}\right)/\mathbb{Q}$$

is a D_4-extension ramified only at q_i and l_i. We set $\alpha_i = \sqrt{r_i(x_i + z_i\sqrt{l_i^*})}$. From Proposition 3.3, the extension

$$K = \mathbb{Q}\left(\sqrt{q_1}, \dots, \sqrt{q_n}, \sqrt{\alpha_1 \cdots \alpha_n}\right)/\mathbb{Q}$$

is an $\mathrm{ES}^+(n,2)$-extension unramified outside $q_i, l_i\,(1 \le i \le n)$.

Let us consider the case $n = 2$ and $(q_1, q_2, l_1, l_2) = (5, 13, 11, 3)$. Then we compute

$$K = \mathbb{Q}\left(\sqrt{5}, \sqrt{13}, \sqrt{33(1+2\sqrt{-3})(1+2\sqrt{-11})}\right).$$

From Lemma 3.2, there exist 15 biquadratic fields $F_1, \dots, F_{15}$ such that the Galois groups $\mathrm{Gal}(K/F_i)$ are abelian groups. The following table contains the fields F_i, the conductors $\mathfrak{f}_i$ of K/F_i, and the abelian invariants of the ray class groups $\mathrm{Cl}_{F_i}(\mathfrak{f}_i)$ of F_i modulo $\mathfrak{f}_i$.

i	Field F_i	Conductor $\mathfrak{f}_i$	Invariants of $\mathrm{Cl}_{F_i}(\mathfrak{f}_i)$
1	$\mathbb{Q}(\sqrt{-3}, \sqrt{5})$	(143)	$[2, 12, 24, 840]$
2	$\mathbb{Q}(\sqrt{-3}, \sqrt{-11})$	(65)	$[4, 12, 24, 168]$
3	$\mathbb{Q}(\sqrt{-3}, \sqrt{-55})$	(13)	$[12^3, 48]$
4	$\mathbb{Q}(\sqrt{5}, \sqrt{-39})$	(11)	$[10^3, 40]$
5	$\mathbb{Q}(\sqrt{-11}, \sqrt{13})$	(15)	$[2^2, 4, 120]$
6	$\mathbb{Q}(\sqrt{-11}, \sqrt{-39})$	(5)	$[4^3, 16]$
7	$\mathbb{Q}(\sqrt{13}, \sqrt{-55})$	(3)	$[2, 4, 16]$
8	$\mathbb{Q}(\sqrt{-15}, \sqrt{-143})$	(1)	$[2^2, 10]$
9	$\mathbb{Q}(\sqrt{-15}, \sqrt{429})$	(1)	$[2, 4]$
10	$\mathbb{Q}(\sqrt{33}, \sqrt{-195})$	(1)	$[2, 4]$
11	$\mathbb{Q}(\sqrt{-39}, \sqrt{-55})$	(1)	$[8^2]$
12	$\mathbb{Q}(\sqrt{-143}, \sqrt{165})$	(1)	$[2, 20]$
13	$\mathbb{Q}(\sqrt{5}, \sqrt{13})$	$(33)\infty_1\infty_2\infty_3\infty_4$	$[2^3, 120]$
14	$\mathbb{Q}(\sqrt{33}, \sqrt{65})$	$(1)\infty_1\infty_2\infty_3\infty_4$	$[2^3]$
15	$\mathbb{Q}(\sqrt{65}, \sqrt{165})$	$(1)\infty_1\infty_2\infty_3\infty_4$	$[2, 4]$

Let $\mathrm{Ar}_i : \mathrm{Cl}_{F_i}(\mathfrak{f}_i) \longrightarrow \mathrm{Gal}(K/F_i)$ be the surjections induced from the Artin maps and Ker_i the kernel of each Ar_i. From Section 2.2, there exists a unique irreducible character ψ of $\mathrm{Gal}(K/\mathbb{Q})$ of degree 4. Since ψ is a faithful character and vanishes outside $Z(\mathrm{Gal}(K/\mathbb{Q})) = \mathrm{Gal}(K/F_1F_5) \cong C_2$, we have $\psi(\sigma) = -1$ for the non-trivial element $\sigma \in Z(\mathrm{Gal}(K/\mathbb{Q}))$. Let $c_i = \mathrm{Ar}_i^{-1}(\sigma)$ and $\chi_i\colon \mathrm{Cl}_{F_i}(\mathfrak{f}_i)/\mathrm{Ker}_i \longrightarrow \mathbb{C}$ ray class characters such that $\chi_i(c_i) = -1$. We compute

$$c_1 = \left[\left(\frac{-37 + 13\sqrt{-3} + 123\sqrt{5} - 39\sqrt{-15}}{4}\right)\right],$$

$$c_2 = \left[\left(\frac{-45 - 23\sqrt{-3} + 7\sqrt{-11} - 9\sqrt{33}}{4}\right)\right],\ c_3 = \left[(-174 - \sqrt{-55})\right],$$

$$\begin{aligned}
c_4 &= \left[\left(\frac{179-13\sqrt{-195}}{2}\right)\right],\ c_5 = \left[\left(\frac{6+5\sqrt{-11}+3\sqrt{13}}{2}\right)\right],\\
c_6 &= \left[\left(-46-50\sqrt{-11}+8\sqrt{-39}\right)\right],\\
c_7 &= \left[\left(2704, \frac{5045+96\sqrt{13}+41\sqrt{-55}-16\sqrt{-715}}{2}\right)\right],\\
c_8 &= \left[\left(2, \frac{-82-4\sqrt{-15}+\sqrt{-143}+\sqrt{2145}}{2}\right)\right],\\
c_9 &= \left[\left(25, \frac{609+9\sqrt{-15}-18\sqrt{429}-4\sqrt{-715}}{2}\right)\right],\\
c_{10} &= \left[\left(1156, \frac{-32257+5633\sqrt{33}+193\sqrt{-195}+429\sqrt{-715}}{4}\right)\right],\\
c_{11} &= \left[\left(80, \frac{62+38\sqrt{-39}-39\sqrt{-55}-\sqrt{2145}}{2}\right)\right],\\
c_{12} &= \left[\left(49, \frac{-147-30\sqrt{-143}+46\sqrt{165}+19\sqrt{-195}}{2}\right)\right],\\
c_{13} &= \left[\left(8, \frac{2939-13728\sqrt{5}-7128\sqrt{13}+561\sqrt{65}}{2}\right)\right],\\
c_{14} &= \left[\left(\frac{-1797-199\sqrt{33}-141\sqrt{65}+39\sqrt{2145}}{4}\right)\right],\\
c_{15} &= \left[\left(25, \frac{201+13\sqrt{65}+2\sqrt{165}-2\sqrt{429}}{4}\right)\right].
\end{aligned}$$

Since any such χ_i induces the irreducible character ψ, we have a coincidence of the Hecke L-functions of the 15 biquadratic fields and the Artin L-function

$$L(s,\chi_1) = L(s,\chi_2) = \cdots = L(s,\chi_{15}) = L(s,\psi)$$

by Theorem 1.1. The Artin L-function $L(s,\psi)$ is given by

$$\begin{aligned}
L(s,\psi) =& \frac{1}{1^s} + \frac{2}{4^s} + \frac{1}{9^s} + \frac{3}{16^s} + \frac{-1}{25^s} + \frac{2}{36^s} + \frac{-2}{49^s} + \frac{4}{64^s} + \frac{1}{81^s} + \cdots \\
& \cdots + \frac{4}{751^s} + \frac{-6}{784^s} + \frac{-8}{796^s} + \frac{4}{829^s} + \frac{2}{841^s} + \frac{4}{859^s} + \cdots .
\end{aligned}$$

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