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Sasakian Finsler structures on pull-back bundles

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Abstract. Under a pull-back approach given by T. Aikou and L. Kozma and firstly presented by D. Bao, S. S. Chern and Z. Shen, we introduce, in this paper, the concepts of almost contact and normal almost contact Finsler structures on the pull-back bundles. Properties of structures partly Sasakians are studied. Using the hh-curvature tensor of Chern connection given by D. Bao, S. S. Chern and Z. Shen, we obtain some characterizations of horizontal Sasakian Finsler structures and K-contact structures via the horizontal Ricci tensor and the flag curvature.

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§1. Introduction

Let (M, F) be a Finsler manifold and TM_0 be its slit tangent bundle. There exist, in the literature, several frameworks for the study of Finsler geometry. For instance, an approach through the double tangent bundle TTM_0 (see Grifone's approach in [7]), an approach via the vertical subbundle of TTM_0 (Bejancu-Farran, Abate-Patrizio, see [1]) and the pull-back bundle approach (Bao-Chern-Shen, see [3]). The latter motivates this paper. In fact, let π be a canonical submersion from TM_0 onto M. The pull-back bundle π^*TM, which is nothing but a collection of fibers of TM on TM_0, offers an adequate framework of this study. The key idea is to construct, like in Riemannian case, the Sasakian structures in Finsler geometry. Thus by using the horizontal part of the curvature associated to the Chern connection, we obtain some characterizations of Sasakian structures on the pull-back bundle, generalizing their Riemannian's analogues.

The paper is organized as follows. In Section 2, we recall some basic definitions and necessary geometric concepts that are used throughout this paper.

We also define in an adapted tensorial formalism, an almost contact Finsler structure (ϕ, η, ξ), on pull-back bundle π^*TM. We introduce in Section 3 almost contact and contact metric Finsler structures, and obtain some characterizations. We define horizontal and vertically Sasakian Finsler structures and K-contact Finsler structures. Under some conditions, we prove that contact metric Finsler structure on π^*TM is horizontally K-contact and vertically K-contact. Using the Chern connection, we determine the covariant derivative of ϕ and ξ, in any direction, thereby generalizing the Riemannian case for the Levi-civita connection. In the same section, we discuss some aspects of hh-curvature with respect to the Chern connection of contact metric Finsler pull-back bundle. We obtain the horizontal Ricci $(1,1;0)$-tensor relatively to ξ and his horizontal representative. Finally, with the aid of the flag curvature with transverse edge ξ, we obtain a Finslerian analogous characterization of Hatakeyama, Ogawa and Tanno results on K-contact metric structures (see [6] for more details).

§2. Preliminaries

Let $\pi : TM \longrightarrow M$ be a tangent bundle of connected smooth Finsler manifold M of odd-dimension $m = 2n+1$. We denote by $v = (x, y)$ the points in TM if $y \in \pi^{-1}(x) = T_xM$. We denote by $O(M)$ the zero section of TM, and by TM_0 the slit tangent bundle $TM \setminus O(M)$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate $(x^1, ..., x^m)$. By setting $v = y^i \frac{\partial}{\partial x^i}$ for every $v \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, ..., x^m, y^1, ..., y^m)$ on $\pi^{-1}(U)$.

Definition 1. A function $F : TM \longrightarrow [0, +\infty)$ is called a Finsler structure or Finsler metric on M if:

(i) $F \in C^\infty(TM_0)$,

(ii) $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$,

(iii) The $m \times m$ Hessian matrix (g_{ij}), where $g_{ij} := \dfrac{1}{2}(F^2)_{y^i y^j}$ is positive-definite at all (x, y) of TM_0.

The pair (M, F) is called *Finsler manifold*. The pull-back bundle π^*TM is a vector bundle over the slit tangent bundle TM_0, defined by

$$\pi^*TM := \{(x, y, v) \in TM_0 \times TM : v \in T_{\pi(x,y)}M\}.$$

By the objects (g_{ij}) given in Definition 1, the pull-back vector bundle π^*TM admits a natural fibre metric

$$g := g_{ij} dx^i \otimes dx^j.$$

This is the Finslerian fundamental tensor in a sense that we will specify later. Likewise, there is the Finslerian Cartan tensor

$$A = A_{ijk}dx^i \otimes dx^j \otimes dx^k \text{ with } A_{ijk} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k}. \tag{2.1}$$

Note that, with a slight abuse of notation, $\frac{\partial}{\partial x^i}$ and dx^i are regarded as sections of π^*TM and π^*T^*M, respectively.

Now we will give some geometric tools for understanding the intrinsic formulation of geometric objects that we used in this paper.

It is well known that the kernel of π_* spans the vertical subbundle $\mathcal{V}$ of TTM_0. As shown in [10], an Ehresmann connection is the choice of the horizontal complementary $\mathcal{H} \subset TTM_0$ such that

$$TTM_0 = \mathcal{H} \oplus \mathcal{V}.$$

In this paper, we shall consider the choice of Ehresmann connection which arises from the Finsler structure F: As shown in [11], every Finslerian structure F induces a spray

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

in which the spray coefficients G^i are defined by

$$G^i(x, y) := \frac{1}{4} g^{il} \left[F^2_{x^k y^l} y^k - F^2_{x^l} \right]$$

where the matrix (g^{ij}) means the inverse of (g_{ij}).

Define a π^*TM-valued smooth 1-form on TM_0 by

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j) \tag{2.2}$$

where functions $N^i_j(x, y)$ are given by $N^i_j(x, y) := \dfrac{\partial G^i}{\partial y^j}(x, y)$. This π^*TM-valued smooth 1-form θ is globally well-defined on TM_0 (see [5]).

By the 1-form θ, defined in (2.2) which is called Finsler-Ehresmann form, we can define a Finsler-Ehresmann connection as follow.

Definition 2. ([10]) A Finsler-Ehresmann connection of the submersion $\pi : TM_0 \longrightarrow M$ is the subbundle $\mathcal{H}$ of TTM_0 given by $\mathcal{H} = \ker \theta$, where $\theta : TTM_0 \longrightarrow \pi^*TM$ is the bundle morphism defined in (2.2), and which is complementary to the vertical subbundle $\mathcal{V}$.

It is well known that π^*TM can be naturally identified with the horizontal subbundle $\mathcal{H}$ and the vertical one $\mathcal{V}$ (see [2]). Thus, any section $\overline{X}$ of π^*TM is considered as a section of $\mathcal{H}$ or a section of $\mathcal{V}$. We denote by $\overline{X}^H$ and $\overline{X}^V$ respectively, the section of $\mathcal{H}$ and the section of $\mathcal{V}$ corresponding to $\overline{X} \in \Gamma(\pi^*TM)$ (see [2]):

$$\overline{X} = \overline{X}^i \frac{\partial}{\partial x^i} \in \pi^*TM \Longleftrightarrow \overline{X}^H = \overline{X}^i \frac{\delta}{\delta x^i} \in \Gamma(\mathcal{H}),$$

and

$$\overline{X} = \overline{X}^i \frac{\partial}{\partial x^i} \in \pi^*TM \Longleftrightarrow \overline{X}^V = \overline{X}^i F \frac{\partial}{\partial y^i} \in \Gamma(\mathcal{V})$$

where

$$\left\{F\frac{\partial}{\partial y^i} := (\frac{\partial}{\partial x^i})^V\right\}_{i=1,\dots,m} \quad \text{and} \quad \left\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j\frac{\partial}{\partial y^i} = (\frac{\partial}{\partial x^i})^H\right\}_{i=1,\dots,m}$$

are the vertical and horizontal lifts of natural local frame field $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\}$ with respect to the Finsler-Ehresmann connection $\mathcal{H}$, respectively.

Proposition 2.1 ([2]). *The bundle morphism π_* and θ satisfy*

$$\pi_*(\overline{X}^H) = \overline{X}, \quad \pi_*(\overline{X}^V) = 0 \ \text{ and } \ \theta(\overline{X}^H) = 0, \quad \theta(\overline{X}^V) = \overline{X}$$

for every $\overline{X} \in \Gamma(\pi^*TM)$.

Proposition 2.1 means that $\mathcal{H}TM_0$, as well as $\mathcal{V}TM_0$, can be naturally identified with the bundle π^*TM, that is,

$$\mathcal{H}TM_0 \cong \pi^*TM \ \text{ and } \ \mathcal{V}TM_0 \cong \pi^*TM. \tag{2.3}$$

Next, we recall the definition of the Chern connection on the pull-back bundle which is going to be used throughout the paper. This connection is symmetric but not always compatible with the metric of the underlying manifold.

Theorem 2.2 ([9]). *Let (M, F) be a Finsler manifold, g a fundamental tensor of F and θ the vector form defined in* (2.2). *There exist a unique linear connection ∇ on π^*TM such that, for all $X, Y \in \Gamma(TTM_0)$ and $\overline{Y}, \overline{Z} \in \Gamma(\pi^*TM)$, we have,*

(a) *Symmetry*

$$\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X, Y],$$

(b) Almost g-compatibility

$$(\nabla_X g)(\overline{Y}, \overline{Z}) = 2A(\theta(X), \overline{Y}, \overline{Z})$$

where A is the Cartan tensor defined in (2.1).

The connection ∇ in Theorem 2.2 is called the *Chern connection.*

Remark 1. If F is Riemannian (i.e., $g_{(x,y)}$ is independent of y), then $A = 0$ and hence ∇ is the Levi-Civita connection of the Riemannian metric $g(x \mapsto g_x(= g_{(x,y)}))$.

A Finslerian tensor field T of type $(p_1, p_2; q)$ on (M, F) is a map:

$$T : \Gamma((\pi^*TM)^{p_1}) \times \Gamma((TTM_0)^{p_2}) \longrightarrow \Gamma((\pi^*TM)^q),$$

which is $C^\infty(TM_0)$-linear in each argument.

If T is of the type $(p_1, 1; q)$, then the tensor T has the following decomposition

$$T = T^H + T^V,$$

and for any $\xi_1, \cdots, \xi_{p_1}$ and $X \in \Gamma(TTM_0)$, we have

$$T^H(\xi_1, \cdots, \xi_{p_1}, X) = T(\xi_1, \cdots, \xi_{p_1}, X^H)$$
$$\text{and } T^V(\xi_1, \cdots, \xi_{p_1}, X) = T(\xi_1, \cdots, \xi_{p_1}, X^V).$$

If T is of type $(p, 0; 0)$ and completely skew-symmetric, then it can be seen as a differential p-form on π^*TM and its exterior differential according to the horizontal and vertical directions are given by the following definition.

Definition 3 (Horizontal and vertical exterior differential [12])**.** Let T be a p-form on π^*TM. Then the horizontal and vertical exterior differentials $d^H T$ and $d^V T$ of T are the $(p+1, 0; 0)$-form given, respectively, by

$$d^H T(\overline{X}_1, \ldots, \overline{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{\overline{X}_i^H} T(\overline{X}_1, \ldots, \breve{\overline{X}}_i, \ldots, \overline{X}_{p+1})$$
$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j} T(\pi_*[\overline{X}_i^H, \overline{X}_j^H], \overline{X}_1, \ldots, \breve{\overline{X}}_i, \ldots, \breve{\overline{X}}_j, \ldots, \overline{X}_{p+1})$$

and

$$d^V T(\overline{X}_1, \ldots, \overline{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{\overline{X}_i^V} T(\overline{X}_1, \ldots, \breve{\overline{X}}_i, \ldots, \overline{X}_{p+1})$$
$$+ \sum_{1 \le i < j \le p+1} (-1)^{i+j} T(\theta([\overline{X}_i^V, \overline{X}_j^V]), \overline{X}_1, \ldots, \breve{\overline{X}}_i, \ldots, \breve{\overline{X}}_j, \ldots, \overline{X}_{p+1})$$

for any $\overline{X}_i \in \Gamma(\pi^*TM)$, $i = 1, \ldots, p+1$.

In particular for $p = 1$, we have: $\forall \overline{X}, \overline{Y} \in \Gamma(\pi^*TM)$,

$$d^H T(\overline{X}, \overline{Y}) = \nabla_{\overline{X}^H} T(\overline{Y}) - \nabla_{\overline{Y}^H} T(\overline{X}) - T(\pi_*[\overline{X}^H, \overline{Y}^H])$$

and

$$d^V T(\overline{X}, \overline{Y}) = \nabla_{\overline{X}^V} T(\overline{Y}) - \nabla_{\overline{Y}^V} T(\overline{X}) - T(\theta[\overline{X}^V, \overline{Y}^V]).$$

In a similar way, for all $\overline{X} \in \Gamma(\pi^*TM)$, the Lie derivative $\mathcal{L}_{\overline{X}}$ of the $(p, 0; r)$-Finslerian tensor is decomposed into the horizontal $\mathcal{L}^H_{\overline{X}}$ and vertical $\mathcal{L}^V_{\overline{X}}$. Note that, for $f \in C^\infty(TM_0)$ and $\overline{Y} \in \Gamma(\pi^*TM)$, we have

$$\mathcal{L}^H_{\overline{X}} f = \overline{X}^H(f), \quad \mathcal{L}^V_{\overline{X}} f = \overline{X}^V(f), \quad \mathcal{L}^H_{\overline{X}} \overline{Y} = \pi_*[\overline{X}^H, \overline{Y}^H], \quad \mathcal{L}^V_{\overline{X}} \overline{Y} = \theta[\overline{X}^V, \overline{Y}^V].$$

Now more generally, we adapt the formula of the Lie derivative of tensors given in [8] on a $(p, 0; r)$-Finslerian. tensor. Then we have:

Definition 4 (Horizontal and vertical Lie derivative)**.** Let $\overline{X} \in \Gamma(\pi^*TM)$ and T be a $(p, 0; r)$-tensor. The horizontal and vertical Lie derivative of T relative to $\overline{X}$ are given respectively by

$$\begin{aligned}\left(\mathcal{L}^H_{\overline{X}} T\right)(\overline{Y_1}, \ldots, \overline{Y_p}, \alpha_1, \ldots, \alpha_r) &= \overline{X}^H \left(T(\overline{Y_1}, \ldots, \overline{Y_p}, \alpha_1, \ldots, \alpha_r)\right) \\ &- \sum_{i=1}^{p} T(\overline{Y_1}, \ldots, \mathcal{L}^H_{\overline{X}} \overline{Y_i}, \ldots, \overline{Y_p}, \alpha_1, \ldots, \alpha_r) \\ &- \sum_{j=1}^{p} T(\overline{Y_1}, \ldots, \overline{Y_i}, \ldots, \overline{Y_p}, \alpha_1, \ldots, \mathcal{L}^H_{\overline{X}} \alpha_i, \ldots \alpha_r)\end{aligned}$$

and

$$\begin{aligned}\left(\mathcal{L}^V_{\overline{X}} T\right)(\overline{Y}_1, \ldots, \overline{Y}_p, \alpha_1, \ldots, \alpha_r) &= \overline{X}^V \left(T(\overline{Y}_1, \ldots, \overline{Y}_p, \alpha_1, \ldots, \alpha_r)\right) \\ &- \sum_{i=1}^{p} T(\overline{Y}_1, \ldots, \mathcal{L}^V_{\overline{X}} \overline{Y_i}, \ldots, \overline{Y}_p, \alpha_1, \ldots, \alpha_r) \\ &- \sum_{j=1}^{p} T(\overline{Y_1}, \ldots, \overline{Y_i}, \ldots, \overline{Y_p}, \alpha_1, \ldots, \mathcal{L}^V_{\overline{X}} \alpha_i, \ldots \alpha_r)\end{aligned}$$

where

$$\begin{aligned}(\mathcal{L}^H_{\overline{X}} \alpha_i)(\overline{Y}) &= \overline{X}^H \alpha_i(\overline{Y}) - \alpha_i(\pi_*[\overline{X}^H, \overline{Y}^H]), \\ \text{and } (\mathcal{L}^V_{\overline{X}} \alpha_i)(\overline{Y}) &= \overline{X}^V \alpha_i(\overline{Y}) - \alpha_i(\theta[\overline{X}^V, \overline{Y}^V]).\end{aligned}$$

In the same way, we adapt the definition of the covariant derivative of tensors in [8] to the $(p, q; r)$-tensor with respect to the Chern connection :

Definition 5 (Covariant Chern derivative)**.** Let T a Finslerian tensor of type $(p,q;r)$ and let $X\in\Gamma(TTM_0)$. Then, we define the covariant Chern derivative of T in the direction of X by the formula

$$\begin{aligned}
(\nabla_X T)(\overline{Y}_1,\ldots,\overline{Y}_p,X_1,\ldots,X_q,\alpha_1,\ldots,\alpha_r)&\\
=X(T(\overline{Y}_1,\ldots,\overline{Y}_p,X_1,\ldots,X_q,\alpha_1,\ldots,\alpha_r))&\\
-\sum_{i=1}^{p}T(\overline{Y}_1,\ldots,\nabla_X\overline{Y}_i,\ldots,\overline{Y}_p,X_1,\ldots,X_q,\alpha_1,\ldots,\alpha_r)&\\
-\sum_{j=1}^{q}T(\overline{Y}_1,\ldots,\overline{Y}_p,X_1,\ldots,(\nabla_X\pi_*X_j)^H,\ldots,X_q,\alpha_1,\ldots,\alpha_r)&\\
-\sum_{j=1}^{q}T(\overline{Y}_1,\ldots,\overline{Y}_p,X_1,\ldots,(\nabla_X\theta X_j)^V,\ldots,X_q,\alpha_1,\ldots,\alpha_r)&\\
-\sum_{k=1}^{r}T(\overline{Y}_1,\ldots,\overline{Y}_p,X_1,\ldots,X_q,\alpha_1,\ldots,\nabla_X\alpha_k,\ldots,\alpha_r)&
\end{aligned}$$

where $X_i\in\Gamma(TTM_0)$, $i=1,\ldots,p$; $\overline{Y}_j\in\Gamma(\pi^*TM)$, $j=1,\ldots,q$; $\alpha_k\in\Gamma(\pi^*T^*M)$, $k=1,\ldots,r$, and each of the quantities $\nabla_X\alpha_k$ is evaluated by $(\nabla_X\alpha_k)(\overline{Y})=X\alpha_k(\overline{Y})-\alpha_k(\nabla_X\overline{Y})$.

§3. Almost contact Finsler structures

In this section, we adapt the definition of almost contact structures given in [4] in the case of Finsler.

Let ϕ, ξ and η be the $(1,0;1)$-, $(1,0;0)$- and $(0,0;1)$-Finslerian tensor, respectively, such that

$$\phi^2=-\mathbf{I}+\eta\otimes\xi \ \text{ and } \ \eta(\xi)=1.$$

Then the triplet (ϕ,η,ξ) is called an almost contact Finsler structure on π^*TM and (π^*TM,ϕ,η,ξ) is called almost contact Finsler pull-back bundle.

First of all, we prove the following.

Proposition 3.1. *Let (π^*TM,ϕ,η,ξ) be an almost contact Finsler pull-back bundle. Then,*

$$\phi(\xi)=0 \quad \textit{and} \quad \eta\circ\phi=0.$$

Moreover, ϕ is of rank $2n$.

Proof. The proof is similar to the one given in [4]. □

Now, we define the normality condition on an almost contact Finsler structures on $(\pi^*TM, \phi, \eta, \xi)$. Let T be a $(1,0;1)$-Finslerian tensor. The Nijenhuis torsion of T is the $(2,0;1)$-Finslerian tensor decomposed into the horizontal and vertical part N_T^H and N_T^V given, respectively, by

$$\begin{aligned} N_T^H(\overline{X},\overline{Y}) =& T^2\pi_*[\overline{X}^H,\overline{Y}^H] + \pi_*[(T\overline{X})^H,(T\overline{Y})^H] - T\pi_*[(T\overline{X})^H,\overline{Y}^H] \\ & - T\pi_*[\overline{X}^H,(T\overline{Y})^H] \end{aligned}$$

and

$$\begin{aligned} N_T^V(\overline{X},\overline{Y}) =& T^2\theta[\overline{X}^V,\overline{Y}^V] + \theta[(T\overline{X})^V,(T\overline{Y})^V] - T\theta[(T\overline{X})^V,\overline{Y}^V] \\ & - T\theta[\overline{X}^V,(T\overline{Y})^V] \end{aligned}$$

for any $\overline{X},\overline{Y} \in \Gamma(\pi^*TM)$.

Definition 6. The almost contact Finsler structure (ϕ,η,ξ) on pull-back bundle π^*TM is horizontally normal if

$$\mathcal{N}_H^{(1)}(\overline{X},\overline{Y}) = N_\phi^H(\overline{X},\overline{Y}) + 2d^H\eta(\overline{X},\overline{Y})\xi = 0$$

and it is vertically normal if

$$\mathcal{N}_V^{(1)}(\overline{X},\overline{Y}) = N_\phi^V(\overline{X},\overline{Y}) + 2d^V\eta(\overline{X},\overline{Y})\xi = 0$$

for any $\overline{X},\overline{Y} \in \Gamma(\pi^*TM)$.

Next, we give some equivalent conditions for horizontal and vertical normality of the structure (ϕ,η,ξ). For this reason, we introduce six tensors $\mathcal{N}_H^{(2)}$, $\mathcal{N}_V^{(2)}$, $\mathcal{N}_H^{(3)}$, $\mathcal{N}_V^{(3)}$, $\mathcal{N}_H^{(4)}$ and $\mathcal{N}_V^{(4)}$ given by

$$\begin{aligned} \mathcal{N}_H^{(2)}(\overline{X},\overline{Y}) &:= (\mathcal{L}^H_{\phi\overline{X}}\eta)\overline{Y} - (\mathcal{L}^H_{\phi\overline{Y}}\eta)\overline{X}, \\ \mathcal{N}_V^{(2)}(\overline{X},\overline{Y}) &:= (\mathcal{L}^V_{\phi\overline{X}}\eta)\overline{Y} - (\mathcal{L}^V_{\phi\overline{Y}}\eta)\overline{X}, \\ \mathcal{N}_H^{(3)} &:= (\mathcal{L}^H_{\xi}\phi)\overline{X}, \\ \mathcal{N}_V^{(3)} &:= (\mathcal{L}^V_{\xi}\phi)\overline{X}, \\ \mathcal{N}_H^{(4)} &:= (\mathcal{L}^H_{\xi}\eta)\overline{X}, \\ \mathcal{N}_V^{(4)} &:= (\mathcal{L}^V_{\xi}\eta)\overline{X} \end{aligned}$$

for all $\overline{X},\overline{Y} \in \Gamma(\pi^*TM)$.

Theorem 3.2. *For an almost contact Finsler structure (ϕ,η,ξ), the vanishing of $\mathcal{N}_H^{(1)}$ implies the vanishing of $\mathcal{N}_H^{(2)}$, $\mathcal{N}_H^{(3)}$ and $\mathcal{N}_H^{(4)}$. Likewise, the vanishing of $\mathcal{N}_V^{(1)}$ implies the vanishing of $\mathcal{N}_V^{(2)}$, $\mathcal{N}_V^{(3)}$ and $\mathcal{N}_V^{(4)}$.*

Proof. The proof is similar to the one given in Riemannian case by Blair in [4]. □

Now, let (ϕ, η, ξ) be an almost contact Finsler structure on π^*TM. When the fundamental tensor g of the Finslerian structure F satisfies

$$g(\phi\overline{X}, \phi\overline{Y}) = g(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y})$$

for any sections $\overline{X}$ and $\overline{Y}$ on π^*TM, we said that g is compatible with the structure (ϕ, η, ξ). In this case, $(\pi^*TM, \phi, \eta, \xi, g)$ is called an almost contact metric Finsler pull-back bundle.

Moreover, we define the generalized second fundamental 2-form Φ by:

$$\Phi(\overline{X}, \overline{Y}) := g(\overline{X}, \phi\overline{Y}), \qquad \overline{X}, \overline{Y} \in \Gamma(\pi^*TM).$$

Because of the isomorphisms in (2.3), we defined an almost contact metric Finsler structure as follows.

Definition 7. An almost contact metric Finsler structure (ϕ, η, ξ, g) is a horizontal *contact Finsler structure* if

$$\Phi = 2d^H\eta, \text{ and vertically if } \Phi = 2d^V\eta.$$

Let (ϕ, η, ξ, g) be a contact metric Finsler structure on π^*TM. A section $\overline{X} \in \Gamma(\pi^*TM)$ is horizontally Killing and vertically Killing, if it satisfies, respectively

$$\mathcal{L}^H_{\overline{X}} g = 0 \text{ and } \mathcal{L}^V_{\overline{X}} g = 0.$$

Definition 8. Let (ϕ, η, ξ, g) be a contact metric Finsler structure. If ξ is horizontally Killing (respectively, vertically Killing), then (ϕ, η, ξ, g) is called horizontal K-contact Finsler structure (respectively, vertical K-contact Finsler structure).

Theorem 3.3. *Let (ϕ, η, ξ, g) be a contact metric Finsler structure on π^*TM. Then*

$$\mathcal{N}^{(4)}_H = \mathcal{N}^{(4)}_V = 0, \qquad \mathcal{N}^{(2)}_H = \mathcal{N}^{(2)}_V = 0.$$

Moreover $\mathcal{N}^{(3)}_H$ vanishes if and only if ξ is horizontally Killing and $\mathcal{N}^{(3)}_V$ vanishes if and only if ξ is vertically Killing.

Proof. The proof is similar to the one for the Riemannian case in [4]. □

Lemma 3.4. *For an almost contact metric Finsler structure (ϕ, η, ξ, g) on π^*TM with identification $\pi^*TM \cong \mathcal{H}TM_0$, the covariant derivative of ϕ with respect to the Chern connection is given by*

$$\begin{aligned} 2g((\nabla_X\phi)\overline{Y}, \overline{Z}) =& 3d^H\Phi(\pi_*X, \phi\overline{Y}, \phi\overline{Z}) - 3d^H\Phi(\pi_*X, \overline{Y}, \overline{Z}) \\ &+ g(\mathcal{N}_H^{(1)}(\overline{Y}, \overline{Z}), \phi\pi_*X) + \mathcal{N}_H^{(2)}(\overline{Y}, \overline{Z})\eta(\pi_*X) \\ &+ 2d^H\eta(\phi\overline{Y}, \pi_*X)\eta(\overline{Z}) - 2d^H\eta(\phi\overline{Z}, \pi_*X)\eta(\overline{Y}) \\ &- 2A(\theta(X), \phi\overline{Y}, \overline{Z}) - 2A(\theta(X), \overline{Y}, \phi\overline{Z}) \end{aligned} \tag{3.1}$$

where $X \in \Gamma(TTM_0)$ and $\overline{Y}, \overline{Z} \in \Gamma(\pi^*TM)$.

Proof. For $X, Y, Z \in \Gamma(TTM_0)$, it is well known, by the Chern connection that (see [9])

$$\begin{aligned} 2g(\nabla_X\pi_*Y, \pi_*Z) =& X.g(\pi_*Y, \pi_*Z) + Y.g(\pi_*Z, \pi_*X) - Z.g(\pi_*X, \pi_*Y) \\ &+ g(\pi_*[X, Y], \pi_*Z) - g(\pi_*[Y, Z], \pi_*X) + g(\pi_*[Z, X], \pi_*Y) \\ &- 2\mathcal{A}(X, Y, Z) \end{aligned}$$

where

$$\mathcal{A}(X, Y, Z) = A(\theta(X), \pi_*Y, \pi_*Z)) + A(\theta(Y), \pi_*Z, \pi_*X)) - A(\theta(Z), \pi_*X, \pi_*Y)).$$

Now we have,

$$\begin{aligned} 2g((\nabla_X\phi)\overline{Y}, \overline{Z}) =& 2g(\nabla_X\phi(\overline{Y}), \overline{Z}) + 2g(\nabla_X\overline{Y}, \phi(\overline{Z})) \\ =& X\Phi(\overline{Z}, \overline{Y}) + \phi\overline{Y}^H\left(\Phi(\overline{Z}, \pi_*X) + \eta(\overline{Z})\eta(\pi_*X)\right) \\ &- \overline{Z}^H\Phi(\pi_*X, \overline{Y}) - \Phi(\pi_*[X, \phi\overline{Y}^H], \phi\overline{Z}) + \eta(\pi_*[X, \phi\overline{Y}^H])\eta(\overline{Z}) \\ &- g(\phi\pi_*[\phi\overline{Y}^H, \overline{Z}^H], \phi\pi_*X) - \eta(\pi_*[\phi\overline{Y}^H, \overline{Z}^H])\eta(\pi_*X) \\ &+ \Phi(\pi_*[\overline{Z}^H, X], \overline{Y}) - 2A(\theta(X), \phi\overline{Y}, \overline{Z}) + X\Phi(\phi\overline{Y}, \phi\overline{Z}) \\ &- \overline{Y}^H\Phi(\overline{Z}, \pi_*X) - \phi\overline{Z}^H\left(\Phi(\phi\overline{Y}, \pi_*X) + \eta(\pi_*X, \eta(\overline{Y})\right) \\ &+ \Phi(\pi_*[X, \overline{Y}^H], \overline{Z}^H) - g(\phi\pi_*[\overline{Y}^H, \phi\overline{Z}^H], \phi\pi_*X) \\ &- \eta(\pi_*[\overline{Y}^H, \phi\overline{Z}^H])\eta(\pi_*X) - \Phi(\pi_*[\phi\overline{Z}^H, X], \phi\overline{Y}) \\ &+ \eta(\pi_*[\phi\overline{Z}^H, X])\eta(\overline{Y}) - 2A(\theta(X), \overline{Y}, \phi\overline{Z}) \\ &+ \Phi(\pi_*[\overline{Y}^H, \overline{Z}^H], \pi_*X) - g((\pi_*[\overline{Y}^H, \overline{Z}^H], \phi\pi_*X)) \\ &- \Phi(\pi_*[\phi\overline{Y}^H, \phi\overline{Z}^H], \pi_*X) + g((\pi_*[\phi\overline{Y}^H, \phi\overline{Z}^H], \phi\pi_*X)) \\ &+ g(2d^H\eta(\overline{Y}, \overline{Z})\xi, \phi\pi_*X). \end{aligned} \tag{3.2}$$

Note that, the sum of the last five terms of (3.2) is zero, and we keep it to write this expression in terms of horizontal exterior differential of Φ, and the tensors $\mathcal{N}_H^{(1)}$ and $\mathcal{N}_H^{(2)}$. It follows that

$$\begin{aligned}2g((\nabla_X\phi)\overline{Y},\overline{Z}) =& X\Phi(\phi\overline{Y},\phi\overline{Z}) + \phi\overline{Y}^H\Phi(\phi\overline{Z},\pi_*X) + \phi\overline{Z}^H\Phi(\pi_*X,\phi\overline{Y})\\
&- \Phi(\pi_*[X,\phi\overline{Y}^H],\phi\overline{Z}) - \Phi(\pi_*[\phi\overline{Y}^H,\phi\overline{Z}^H],\pi_*X)\\
&- \Phi(\pi_*[\phi\overline{Z}^H,X],\phi\overline{Y}) - X\Phi(\overline{Y},\overline{Z}) - \overline{Y}^H\Phi(\overline{Z},\pi_*X)\\
&- \overline{Z}^H\Phi(\pi_*X,\overline{Y}) + \Phi(\pi_*[X,\overline{Y}^H],\overline{Z})\\
&+ \Phi(\pi_*[\overline{Y}^H,\overline{Z}^H],\pi_*X) + \Phi(\pi_*[\overline{Z}^H,X],\overline{Y})\\
&+ \phi\overline{Y}^H\eta(\overline{Z})\eta(\pi_*X) - \eta(\pi_*[\phi\overline{Y}^H,\overline{Z}^H])\eta(\pi_*X)\\
&+ \phi\overline{Z}^H\eta(\overline{Y})\eta(\pi_*X) - \eta(\pi_*[\overline{Y}^H,\phi\overline{Z}^H])\eta(\pi_*X)\\
&- g(\phi\pi_*[\phi\overline{Y}^H,\overline{Z}^H],\phi\pi_*X) - g(\phi\pi_*[\overline{Y}^H,\phi\overline{Z}^H],\phi\pi_*X)\\
&- g(\pi_*[\overline{Y}^H,\overline{Z}^H],\phi\pi_*X) + g(\pi_*[\phi\overline{Y}^H,\phi\overline{Z}^H],\phi\pi_*X)\\
&+ g(2d^H\eta(\overline{Y},\overline{Z})\xi,\phi\pi_*X) + \eta(\pi_*[X,\phi\overline{Y}^H])\eta(\overline{Z})\\
&+ \eta(\pi_*[\phi\overline{Z}^H,X])\eta(\overline{Y}) - 2(A(\theta(X),\phi\overline{Y},\overline{Z}) + A(\theta(X),\overline{Y},\phi\overline{Z}))\\
&= 3d^H\Phi(\pi_*X,\phi\overline{Y},\phi\overline{Z}) - 3d^H\Phi(\pi_*X,\overline{Y},\overline{Z})\\
&+ \mathcal{N}_H^{(2)}(\overline{Y},\overline{Z})\eta(\pi_*X) + g(\mathcal{N}_H^{(1)}(\overline{Y},\overline{Z}),\phi\pi_*X)\\
&+ 2d^H\eta(\phi\overline{Y},\pi_*X)\eta(\overline{Z}) - 2d^H\eta(\phi\overline{Z},\pi_*X)\eta(\overline{Y})\\
&- 2(A(\theta(X),\phi\overline{Y},\overline{Z}) + A(\theta(X),\overline{Y},\phi\overline{Z})),\end{aligned}$$

which completes the proof. □

Remark 2. The relation (3.1) generalizes the one given by Blair in [4, page 82] for the case of almost contact metric structure (ϕ,ξ,η,g). Indeed, when the Finsler structure F is Riemannian, the Cartan tensor vanishes and the Chern connection reduces to the Levi-Civita connection of g.

Corollary 3.5. *For an almost contact metric Finsler structure (ϕ,η,ξ,g) on π^*TM with identification $\pi^*TM \cong \mathcal{H}TM_0$, the horizontal and vertical covariant derivatives of ϕ, are given respectively by:*

$$\begin{aligned}2g((\nabla^H_X\phi)\overline{Y},\overline{Z}) = & 3d^H\Phi(\pi_*X,\phi\overline{Y},\phi\overline{Z}) - 3d^H\Phi(\pi_*X,\overline{Y},\overline{Z})\\
&+ g(\mathcal{N}_H^{(1)}(\overline{Y},\overline{Z}),\phi\pi_*X) + \mathcal{N}_H^{(2)}(\overline{Y},\overline{Z})\eta(\pi_*X)\\
&+ 2d^H\eta(\phi\overline{Y},\pi_*X)\eta(\overline{Z}) - 2d^H\eta(\phi\overline{Z},\pi_*X)\eta(\overline{Y})\end{aligned}$$

and

$$2g((\nabla^V_X\phi)\overline{Y},\overline{Z}) = -2\left(A(\theta(X),\phi\overline{Y},\overline{Z}) + A(\theta(X),\overline{Y},\phi\overline{Z})\right)$$

where $\nabla^H_X = \nabla_{X^H}$ *and* $\nabla^V_X = \nabla_{X^V}$.

Proof. The proof follows from a straightforward calculation using Lemma 3.4. □

Definition 9. The horizontal normal contact Finsler structures (ϕ, η, ξ, g) are called *horizontal Sasakian Finsler structures*, and the vertical ones are called *vertical Sasakian Finsler structures.*

Theorem 3.6. *An almost contact metric Finsler structure* (ϕ, η, ξ, g) *on* π^*TM*, with identification* $\pi^*TM \cong \mathcal{H}TM_0$*, is horizontally Sasakian if and only if*

$$(\nabla_X\phi)\overline{Y} = g(\pi_*X, \overline{Y})\xi - \eta(\overline{Y})\pi_*X + \phi A^\sharp(\theta(X), \overline{Y}, \bullet) - A^\sharp(\theta(X), \phi\overline{Y}, \bullet),$$

where

$$g(A^\sharp(\theta(X), \overline{Y}, \bullet), \overline{Z}) = A(\theta(X), \overline{Y}, \overline{Z})$$

for all $X \in \Gamma(TTM_0)$ *and* $\overline{Y}, \overline{Z} \in \Gamma(\pi^*TM)$.

Proof. If (ϕ, η, ξ, g) is horizontally Sasakian, then by Lemma 3.4,

$$\begin{aligned}
2g((\nabla_X\phi)\overline{Y}, \overline{Z}) =& 2d^H\eta(\phi\overline{Y}, \pi_*X)\eta(\overline{Z}) - 2d^H\eta(\phi\overline{Z}, \pi_*X)\eta(\overline{Y}) \\
&- 2A(\theta(X), \phi\overline{Y}, \overline{Z}) - 2A(\theta(X), \overline{Y}, \phi\overline{Z}) \\
=& 2g(\overline{Y}, \pi_*X)\eta(\overline{Z}) - 2g(\pi_*X, \overline{Z})\eta(\overline{Y}) \\
&- 2g(A^\sharp(\theta(X), \phi\overline{Y}, \bullet), \overline{Z}) - 2g(A^\sharp(\theta(X), \overline{Y}, \bullet), \phi\overline{Z}) \\
=& 2g(g(\overline{Y}, \pi_*X)\xi - \eta(\overline{Y})\pi_*X + \phi A^\sharp(\theta(X), \overline{Y}, \bullet), \overline{Z}) \\
&- 2g(A^\sharp(\theta(X), \phi\overline{Y}, \bullet), \overline{Z}).
\end{aligned}$$

From which the result follows. Conversely, assuming that

$$(\nabla_X\phi)\overline{Y} = g(\pi_*X, \overline{Y})\xi - \eta(\overline{Y})\pi_*X + \phi A^\sharp(\theta(X), \overline{Y}, \bullet) - A^\sharp(\theta(X), \phi\overline{Y}, \bullet)$$

and taking $\overline{Y} = \xi$, we have

$$(\nabla_X\phi)\xi = \eta(\pi_*X)\xi - \pi_*X + \phi A^\sharp(\theta(X), \xi, \bullet).$$

Hence, one obtains

$$\nabla_X\xi = -\phi\pi_*X + \phi^2 A^\sharp(\theta(X), \xi, \bullet).$$

Therefore, we have

$$d^H\eta(\overline{X}, \overline{Y}) = \frac{1}{2}(g(\overline{Y}, \nabla_{\overline{X}^H}\xi) - g(\overline{X}, \nabla_{\overline{Y}^H}\xi)) = g(\overline{X}, \phi\overline{Y}) = \Phi(\overline{X}, \overline{Y}).$$

Thus (ϕ, ξ, η, g) is a contact metric Finsler structure. Now, by definition,

$$N_\phi^H(\overline{X}, \overline{Y}) = \phi(\nabla_{\overline{Y}^H}\phi)\overline{X} - \phi(\nabla_{\overline{X}^H}\phi)\overline{Y} + (\nabla_{(\phi\overline{X})^H}\phi)\overline{Y} - (\nabla_{(\phi\overline{Y})^H}\phi)\overline{X},$$

and using the hypothesis, we obtain

$$N_\phi^H(\overline{X}, \overline{Y}) = -2d^H\eta(\overline{X}, \overline{Y})\xi,$$

which completes the proof. □

Corollary 3.7. *For an almost contact metric Finsler structure (ϕ, η, ξ, g) on π^*TM, with identification $\pi^*TM \cong \mathcal{H}TM_0$, the following holds:*

$$\nabla_{\xi^H}\phi = 0 \quad \textit{and} \quad (\nabla_{\xi^V}\phi)\overline{Y} = \phi A^\sharp(\theta(\xi^V), \overline{Y}, \bullet) - A^\sharp(\theta(\xi^V), \phi\overline{Y}, \bullet)$$

for all $\overline{Y} \in \Gamma(\pi^*TM)$.

Lemma 3.8. *For a contact metric Finsler structure (ϕ, η, ξ, g) on π^*TM, with identification $\pi^*TM \cong \mathcal{H}TM_0$, the following holds:*

$$\begin{aligned}2g((\nabla_X\phi)\overline{Y}, \overline{Z}) &= g(\mathcal{N}_H^{(1)}(\overline{Y}, \overline{Z}), \phi\pi_*X) + 2d^H\eta(\phi\overline{Y}, \pi_*X)\eta(\overline{Z}) \\ &- 2d^H\eta(\phi\overline{Z}, \pi_*X)\eta(\overline{Y}) - 2\left(A(\theta(X), \phi\overline{Y}, \overline{Z}) + A(\theta(X), \overline{Y}, \phi\overline{Z})\right).\end{aligned}$$

Proof. The proof is obtained by Lemma 3.4 and Theorem 3.3. □

On a contact metric Finsler structure (Theorem 3.3), $\mathcal{N}_H^{(3)} = 0$ if and only if ξ is horizontally Killing, and $\mathcal{N}_V^{(3)} = 0$ if and only if ξ is vertically Killing. Then the horizontal Sasakian Finsler structure is horizontal K-contact structure and the vertical Sasakian Finsler structure is vertical K-contact structure.

Due to the use of tensors $\mathcal{N}_H^{(3)}$ and $\mathcal{N}_V^{(3)}$, it would be important to investigate some properties of these tensors.

Let

$$\mathbf{h} = \frac{1}{2}\mathcal{N}_H^{(3)} = \frac{1}{2}\mathcal{L}_{\xi^H}\phi \quad \text{and} \quad \mathbf{v} = \frac{1}{2}\mathcal{N}_V^{(3)} = \frac{1}{2}\mathcal{L}_{\xi^V}\phi.$$

We firstly notice that

$$\mathbf{h}\xi = 0 \quad \text{and} \quad \mathbf{v}\xi = 0.$$

Proposition 3.9. *For a contact metric Finsler (ϕ, η, ξ, g), $\mathbf{h}$ is a symmetric operator, whereas $\mathbf{v}$ is a symmetric operator if*

$$A(\theta(\overline{X}^V), \xi, \phi\overline{Y}) + A(\theta(\phi\overline{X}^V), \xi, \overline{Y}) = A(\theta(\overline{Y}^V), \xi, \phi\overline{X}) + A(\theta(\phi\overline{Y}^V), \xi, \overline{X})$$

for all $\overline{X}, \overline{Y} \in \Gamma(\pi^*TM)$.

Proof. Let $\overline{X}, \overline{Y} \in \Gamma(\pi^*TM)$. Then we have

$$\begin{aligned} 2g(\mathbf{h}\overline{X}, \overline{Y}) &= g((\mathcal{L}_{\xi^H}\phi)\overline{X}, \overline{Y}) = g(\pi_*[\xi^H, (\phi\overline{X})^H] - \phi(\pi_*[\xi^H, \overline{X}^H]), \overline{Y}) \\ &= g(-\nabla_{(\phi\overline{X})^H}\xi + \phi(\nabla_{\overline{X}^H}\xi), \overline{Y}). \end{aligned}$$

It follows that if $\overline{X}$ or $\overline{Y}$ is equal to ξ, then $g(\mathbf{h}\overline{X}, \overline{Y}) = 0$. Recall that $\mathcal{N}_H^{(2)} = 0$ for contact metric Finsler structures. Then for $\overline{X}$ and $\overline{Y}$ orthogonal to ξ,

$$\begin{aligned} 0 = \mathcal{N}_H^{(2)} &= (\mathcal{L}_{(\phi\overline{X})^H}\eta)\overline{Y} - (\mathcal{L}_{(\phi\overline{Y})^H}\eta)\overline{X} \\ &= \eta(\pi_*[(\phi\overline{Y})^H, \overline{X}^H]) - \eta(\pi_*[(\phi\overline{X})^H, \overline{Y}^H]). \end{aligned}$$

Thus

$$\eta(\nabla_{(\phi\overline{X})^H}\overline{Y}) + \eta(\nabla_{\overline{X}^H}\phi\overline{Y}) = \eta(\nabla_{(\phi\overline{Y})^H}\overline{X}) + \eta(\nabla_{\overline{Y}^H}\phi\overline{X}).$$

On the other hand, $g(\xi, \overline{Y}) = 0$ implies that $g(\xi, \nabla_{(\phi\overline{X})^H}\overline{Y}) = -g(\nabla_{(\phi\overline{X})^H}\xi, \overline{Y})$, so we have

$$\begin{aligned} 2g(\mathbf{h}\overline{X}, \overline{Y}) &= g(-\nabla_{(\phi\overline{X})^H}\xi + \phi(\nabla_{\overline{X}^H}\xi), \overline{Y}) \\ &= \eta(\nabla_{(\phi\overline{X})^H}\overline{Y}) + \eta(\nabla_{\overline{X}^H}\phi\overline{Y}) \\ &= \eta(\nabla_{(\phi\overline{Y})^H}\overline{X}) + \eta(\nabla_{\overline{Y}^H}\phi\overline{X}) \\ &= 2g(\mathbf{h}\overline{Y}, \overline{X}), \end{aligned}$$

which proves that $\mathbf{h}$ is symmetric. For the operator $\mathbf{v}$, using the fact that (see [2] for details)

$$\nabla_X\theta(Y) - \nabla_Y\theta(X) = \theta([X, Y]),$$

we obtain

$$\eta(\nabla_{(\phi\overline{X})^V}\overline{Y}) + \eta(\nabla_{\overline{X}^V}\phi\overline{Y}) = \eta(\nabla_{(\phi\overline{Y})^V}\overline{X}) + \eta(\nabla_{\overline{Y}^V}\phi\overline{X}).$$

By observing that

$$g(\xi, \nabla_{\overline{X}^V}\phi\overline{Y}) + g(\nabla_{\overline{X}^V}\xi, \phi\overline{Y}) = 2A(\theta(\overline{X}^V), \xi, \phi\overline{Y})),$$

one gets

$$\begin{aligned} 2g(\mathbf{v}\overline{X}, \overline{Y}) =& g(\xi, \nabla_{(\pi\overline{X})^V}\overline{Y}) + g(\xi, \nabla_{\overline{X}^V}\phi\overline{Y}) - 2A((\phi\overline{X})^V, \xi, \overline{Y}) \\ &- 2A(\theta(\overline{X}^V), \xi, \phi\overline{Y}) \\ =& \eta(\nabla_{(\pi\overline{X})^V}\overline{Y}) + \eta(\nabla_{\overline{X}^V}\phi\overline{Y}) - 2(A((\phi\overline{X})^V, \xi, \overline{Y}) \\ &+ A(\theta(\overline{X}^V), \xi, \phi\overline{Y})) \end{aligned}$$

$$=\eta(\nabla_{(\pi\overline{Y})^V}\overline{X})+\eta(\nabla_{\overline{Y}^V}\phi\overline{X})-2(A((\phi\overline{X})^V,\xi,\overline{Y})$$
$$+A(\theta(\overline{X}^V),\xi,\phi\overline{Y})).$$

Then, $\mathbf{v}$ is a symmetric operator if

$$A(\theta(\overline{X}^V),\xi,\phi\overline{Y})+A(\theta(\phi\overline{X}^V),\xi,\overline{Y})=A(\theta(\overline{Y}^V),\xi,\phi\overline{X})+A(\theta(\phi\overline{Y}^V),\xi,\overline{X}).$$

This completes the proof. □

Lemma 3.10. *For a contact metric Finsler structure (ϕ,η,ξ,g), the covariant derivative of ξ in the direction of $X\in\Gamma(TTM_0)$ is given by*

$$\nabla_X\xi=-\phi\mathbf{h}(\pi_*X)-\phi\pi_*X+A^\sharp(\theta(X),\xi,\bullet)-2A(\theta(X),\xi,\xi)\xi$$

where

$$g(A^\sharp(\theta(X),\xi,\bullet),\overline{Z})=A(\theta(X),\xi,\overline{Z})$$

for all $\overline{Z}\in\Gamma(\pi^*TM)$.

Proof. By Lemma 3.8, we have

$$\begin{aligned}&2g((\nabla_X\phi)\xi,\overline{Z})=g(\mathcal{N}_H^{(1)}(\xi,\overline{Z}),\phi\pi_*X)-2d^H\eta(\phi\overline{Z},\pi_*X)-2A(\theta(X),\xi,\phi\overline{Z})\\&=-g((\mathcal{L}_{\xi^H}\phi)\overline{Z},\pi_*X)-2g(\overline{Z},\pi_*X)+2g(\overline{Z},\eta(\pi_*X)\xi)-2A(\theta(X),\xi,\phi\overline{Z})\\&=-2g(\mathbf{h}\overline{Z},\pi_*X)-2g(\overline{Z},\pi_*X)+2g(\overline{Z},\eta(\pi_*X)\xi)-2A(\theta(X),\xi,\phi\overline{Z}).\end{aligned}$$

Thus, by symmetry of $\mathbf{h}$ we have

$$\phi\nabla_X\xi=\mathbf{h}\pi_*X+\pi_*X-\eta(\pi_*X)\xi+\phi A^\sharp(\theta(X),\xi,\bullet).$$

Applying ϕ to this equation, we obtain

$$\nabla_X\xi=-\phi\mathbf{h}(\pi_*X)-\phi\pi_*X+A^\sharp(\theta(X),\xi,\bullet)-2A(\theta(X),\xi,\xi)\xi,$$

which completes the proof. □

Remark 3. When the Finsler structure is Riemannian, the Cartan tensor A vanishes and the Chern connection coincides with the Levi-Civita ones. Then we find the covariant derivative of the characteristic tensor ξ given in [4, p. 84].

Corollary 3.11. *Let (ϕ,η,ξ,g) be a contact metric structure. Under the identification $\pi^*TM\cong\mathcal{H}TM_0$, the operator $\mathbf{h}$ anti-commutes with ϕ and*

$$\mathrm{trace}_g\mathbf{h}=0.$$

Proof. For all $\overline{X}, \overline{Y} \in \Gamma(\pi^*TM)$, we have

$$\begin{aligned} 2g(\overline{X}, \phi\overline{Y}) &= g(\nabla_{\overline{X}^H}\xi, \overline{Y}) - g(\nabla_{\overline{Y}^H}\xi, \overline{X}) \\ &= g(-\phi\mathbf{h}(\overline{X}) - \phi\overline{X}, \overline{Y}) - g(-\phi\mathbf{h}(\overline{Y}) - \phi\overline{Y}, \overline{X}) \\ &= 2g(\overline{X}, \phi\overline{Y}) - g(\phi\mathbf{h}(\overline{X}), \overline{Y}) + g(\phi\mathbf{h}(\overline{Y}), \overline{X}). \end{aligned}$$

Consequently,

$$-g(\phi\mathbf{h}(\overline{X}), \overline{Y}) + g(\phi\mathbf{h}(\overline{Y}), \overline{X}) = 0$$

and we obtain

$$\mathbf{h}\phi + \phi\mathbf{h} = 0.$$

So $\mathbf{h}$ anti-commutes with ϕ. For the last assertion, if $\mathbf{h}\overline{X} = \lambda\overline{X}$, then

$$\mathbf{h}\phi\overline{X} = -\phi\mathbf{h}\overline{X} = -\lambda\phi\overline{X}.$$

Thus if λ is an eigenvalue of $\mathbf{h}$, so is $-\lambda$ and hence $\mathrm{trace}_g\mathbf{h} = 0$. □

Now, we discuss some aspects of the hh-curvature R with respect to the Chern connection of contact metric Finsler pull-back bundle $(\pi^*TM, \phi, \eta, \xi, g)$.

Definition 10 ([10])**.** The full curvature $\mathcal{C}$ of Chern connection ∇ is defined by:

$$\mathcal{C}(X,Y)\overline{Z} = \nabla_X\nabla_Y\overline{Z} - \nabla_Y\nabla_X\overline{Z} - \nabla_{[X,Y]}\overline{Z},$$

where $X, Y \in \Gamma(TTM_0)$ and $\overline{Z} \in \Gamma(\pi^*TM)$.

Using the decomposition $\nabla_X = \nabla_{X^H} + \nabla_{X^V}$, we can write the full curvature as

$$\begin{aligned} \mathcal{C}(X,Y)\overline{Z} =& \mathcal{C}^{HH}(X,Y)\overline{Z} + \mathcal{C}^{HV}(X,Y)\overline{Z} + \mathcal{C}^{VH}(X,Y)\xi \\ &+ \mathcal{C}^{VV}(X,Y)\overline{Z} \end{aligned} \tag{3.6}$$

where $\mathcal{C}^{HH}(X,Y)\overline{Z} = \mathcal{C}(X^H,Y^H)\overline{Z}$, $\mathcal{C}^{HV}(X,Y)\overline{Z} = \mathcal{C}(X^H,Y^V)\overline{Z}$ and $\mathcal{C}^{VV}(X,Y)\overline{Z} = \mathcal{C}(X^V,Y^V)\overline{Z}$.

The component $\mathcal{C}^{HH}$ given in (3.6) is called hh-curvature tensor of Chern connection ∇ and will be denoted by R, i.e.,

$$R(X,Y)\overline{Z} = \mathcal{C}^{HH}(X,Y)\overline{Z}.$$

Proposition 3.12. *On contact metric Finsler pull-back bundle $(\pi^*TM, \phi, \eta, \xi, g)$, we have:*

$$\begin{aligned} \left(\nabla_{\xi^H}\mathbf{h}\right)\pi_*X =& \phi\pi_*X - \mathbf{h}^2\phi\pi_*X + \phi R(\xi^H, X)\xi \\ &+ \phi A^\sharp(\theta[\xi^H, X], \xi, \bullet) \end{aligned} \tag{3.7}$$

and

$$\frac{1}{2}\left(R(\xi^H, X)\xi - \phi R(\xi^H, (\phi\pi_* X)^H)\xi\right) = \mathbf{h}^2\pi_* X + \frac{1}{2}\phi^2 A^\sharp(\theta([\xi^H, X]), \xi, \bullet) + \phi^2\pi_* X \frac{1}{2}\phi A^\sharp(\theta([\xi^H, (\phi\pi_* X)^H]), \xi, \bullet) \tag{3.8}$$

for any $X \in \Gamma(TTM_0)$ *and any* $\xi \in \Gamma(\pi^* TM)$.

Proof. For any $X \in \Gamma(TTM_0)$ and since $\nabla_{\xi^H}\xi = 0$, one has

$$R(\xi^H, X)\xi = \nabla_{\xi^H}\nabla_X \xi - \nabla_{[\xi^H, X]}\xi.$$

By Lemma 3.10, we obtain

$$\begin{aligned} R(\xi^H, X)\xi = &-\phi\nabla_{\xi^H}\mathbf{h}(\pi_* X) - \phi\nabla_{\xi^H}\pi_* X + \phi\mathbf{h}(\pi_*[\xi^H, X]) \\ &+ \phi\pi_*[\xi^H, X] + A^\sharp(\theta([\xi,^H X]), \xi, \bullet) - 2A(\theta([\xi,^H X]), \xi, \xi)\xi. \end{aligned} \tag{3.9}$$

Applying ϕ to the equation (3.9) and using the fact that $\nabla_{\xi^H}\phi = 0$ we have,

$$\begin{aligned} \phi R(\xi^H, X)\xi =& \nabla_{\xi^H}(\pi_* X + \mathbf{h}\pi_* X) - \eta(\nabla_{\xi^H}(\pi_* X + \mathbf{h}\pi_* X))\xi - \mathbf{h}\pi_*[\xi^H, X] \\ &+ \eta(\pi_*[\xi^H, X])\xi - \pi_*[\xi^H, X] + \phi A^\sharp(\theta([\xi^H, X], \xi, \bullet) \\ =& \nabla_{\xi^H}(\pi_* X + \mathbf{h}\pi_* X) - \mathbf{h}\pi_*[\xi^H, X] - \pi_*[\xi^H, X] \\ &+ \phi A^\sharp(\theta([\xi^H, X], \xi, \bullet) \end{aligned}$$

by Corollary 3.11 and the fact that $\eta(\nabla_{\xi^H}\mathbf{h}\pi_* X) = 0$, we obtain

$$\begin{aligned} \phi R(\xi^H, X)\xi =& (\nabla_{\xi^H}\mathbf{h})\pi_* X - \phi\pi_* X + \mathbf{h}^2\phi\pi_* X \\ &- \phi A^\sharp(\theta([\xi^H, X], \xi, \bullet), \end{aligned} \tag{3.10}$$

which leads to the relation (3.7). Now, from (3.10) we have

$$\begin{aligned} \phi R(\xi^H, (\phi(\pi_* X))^H)\xi =& (\nabla_{\xi^H}\mathbf{h})\phi(\pi_* X) - \phi^2(\pi_* X) + \mathbf{h}^2\phi^2\pi_* X \\ &- \phi A^\sharp(\theta([\xi^H, (\phi(\pi_* X))^H], \xi, \bullet) \\ =& -\phi(\nabla_{\xi^H}\mathbf{h})(\pi_* X) - \mathbf{h}^2\pi_* X - \phi^2\pi_* X \\ &- \phi A^\sharp(\theta([\xi^H, (\phi(\pi_* X))^H], \xi, \bullet) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} R(\xi^H, X)\xi =& -\phi((\nabla_{\xi^H}\mathbf{h})\pi_* X) + \phi^2(\pi_* X) + \mathbf{h}^2\pi_* X \\ &- \phi^2(A^\sharp(\theta([\xi^H, X], \xi, \bullet)). \end{aligned} \tag{3.12}$$

Subtracting (3.12) by (3.11), we get (3.8). □

Theorem 3.13. *On the manifold with horizontal Sasakian Finsler structure, the hh-curvature and the characteristic section ξ satisfy*

$$R(X,Y)\xi = \eta(\pi_*X)\pi_*Y - \eta(\pi_*Y)\pi_*X + \mathcal{A}_\phi(Y,X) - \mathcal{A}_\phi(X,Y) + \nabla_X\mathcal{B}_\xi(Y) - \nabla_Y\mathcal{B}_\xi(X) - \mathcal{B}_\xi([X,Y]), \tag{3.13}$$

where $\mathcal{A}$ is the $(0,2;1)$-tensor given by

$$\mathcal{A}_\phi(X,Y) = \phi A^\sharp(\theta(X),\pi_*Y,\bullet) - A^\sharp(\theta(X),\phi\pi_*Y,\bullet)$$

and $\mathcal{B}_\xi$ the $(0,1;1)$-tensor given by

$$\mathcal{B}_\xi(X) = A^\sharp(\theta(X),\xi,\bullet) - 2A^\sharp(\theta(X),\xi,\xi)\xi$$

for any X, $Y \in \Gamma(TTM_0)$.

Proof. For any X, $Y \in \Gamma(TTM_0)$, we have

$$\begin{aligned} R(X,Y)\xi =& -\nabla_X(\phi\pi_*Y) + \nabla_X A^\sharp(\theta(Y),\xi,\bullet) - 2\nabla_X A(\theta(Y),\xi,\xi)\xi \\ &+ \nabla_Y(\phi\pi_*X) - \nabla_Y A^\sharp(\theta(X),\xi,\bullet) + 2\nabla_Y A(\theta(X),\xi,\xi)\xi \\ &+ \phi\pi_*([X,Y]) - A^\sharp(\theta([X,Y]),\xi,\bullet) + 2A(\theta([X,Y]),\xi,\xi)\xi \\ =& -(\nabla_X\phi)(\pi_*Y) + \nabla_X A^\sharp(\theta(Y),\xi,\bullet) - 2\nabla_X A(\theta(Y),\xi,\xi)\xi \\ &+ (\nabla_Y\phi)(\pi_*X) - \nabla_Y A^\sharp(\theta(X),\xi,\bullet) + 2\nabla_Y A(\theta(X),\xi,\xi)\xi \\ &- A^\sharp(\theta([X,Y]),\xi,\bullet) + 2A(\theta([X,Y]),\xi,\xi)\xi. \end{aligned}$$

Letting

$$\mathcal{A}_\phi(X,Y) = \phi A^\sharp(\theta(X),\pi_*Y,\bullet) - A^\sharp(\theta(X),\phi\pi_*Y,\bullet)$$

and

$$\mathcal{B}_\xi(X) = A^\sharp(\theta(X),\xi,\bullet) - 2A^\sharp(\theta(X),\xi,\xi)\xi,$$

we complete the proof. □

Remark 4. The formula (3.13) generalizes in Finsler case the curvature of Sasakian manifold $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$ given by Blair in [4, p. 113].

Recall that the pull-back bundle π^*TM is locally of dimension $(2n+1)$. Then, as in Riemannian case, we define a ϕ-basis of π^*TM as

$$\{\partial_i,\ \ \partial_{i^*} := \phi\partial_i,\ \ \xi\}_{i=1,\cdots,n}.$$

It is well known that (see [10] for more details), by the trace of hh-curvature of Chern connection, we have a Finslerian analogous of Ricci tensor, called horizontal Ricci (1,1;0)-tensor, denoted by Ric^H and given by

$$Ric^H(\overline{Z},X) = \mathrm{trace}_g(\overline{Y} \mapsto R(X,\overline{Y}^H)\overline{Z}) \tag{3.14}$$

for any $X \in \Gamma(TTM_0)$ and $\overline{Y},\overline{Z} \in \Gamma(\pi^*TM)$.

Proposition 3.14. *On contact metric Finsler pull-back bundle* $(\pi^*TM, \phi, \eta, \xi, g)$*, the horizontal Ricci* $(1,1;0)$*-tensor curvature with respect to* ξ *and* ξ^H *is given by*

$$\begin{aligned} Ric^H(\xi, \xi^H) =& -2n + 2\mathrm{trace}_g \mathbf{h}^2 \\ &+ \sum_{i=1}^{n} \frac{1}{2} \left\{ A(\theta([\xi^H, \delta_i]), \xi, \partial_i) - A(\theta([\xi^H, \delta_{i^*}]), \xi, \phi\partial_i) \right\}, \end{aligned}$$

where $\delta_i = \partial_i^H$ *and* $\delta_{i^*} = \partial_{i^*}^H$*, for all* $i = 1, \cdots, n$*.*

Proof. By (3.14), and the fact that $R(\xi^H, \xi^H)\xi = 0$, we have

$$\begin{aligned} Ric^H(\xi, \xi^H) &= \sum_{i=1}^{n} \left[g(R(\xi^H, \delta_i)\xi, \partial_i) + g(R(\xi^H, \delta_{i^*})\xi, \partial_{i^*}) \right] \\ &= \sum_{i=1}^{n} \left[g(R(\xi^H, \delta_i)\xi - \phi R(\xi^H, \phi\delta_i)\xi, \partial_i) \right]. \end{aligned}$$

Then, by Proposition 3.12, we obtain

$$\begin{aligned} Ric^H(\xi, \xi^H) =& \sum_{i=1}^{n} 2g \left(\mathbf{h}^2\partial_i + \phi^2\partial_i + \frac{1}{2}\phi^2 A^\sharp(\theta([\xi^H, \delta_i]), \xi, \bullet), \partial_i \right) \\ &+ \sum_{i=1}^{n} g \left(\phi A^\sharp(\theta([\xi^H, \phi\delta_i], \xi, \bullet)), \partial_i \right), \end{aligned}$$

which completes the assertion. □

Theorem 3.15. *A contact metric Finsler pull-back bundle* $(\pi^*TM, \phi, \eta, \xi, g)$ *is horizontally* K*-contact if and only if*

$$Ric^H(\xi, \xi^H) = -2n + \sum_{i=1}^{n} \left(A(\theta([\xi^H, \delta_i]), \xi, \partial_i) - A(\theta([\xi^H, \delta_{i^*}]), \xi, \phi\partial_i) \right).$$

Proof. The proof is derived from Proposition 3.14 and the fact that $\mathbf{h} = 0$ for the horizontal K-contact metric Finsler structures. □

Recall that there is a distinguished section l of π^*TM which is globally defined on the manifold TM_0 (see [3] for more details). It is defined by

$$l = l_{(x,y)} := \frac{y^i}{F(y)} \frac{\partial}{\partial x^i} = \frac{y^i}{F} \frac{\partial}{\partial x^i} =: l^i \frac{\partial}{\partial x^i}.$$

There is also a geometrical invariant that generalizes the sectional curvature of Riemannian geometry (see [3]). This is, in fact, the flag curvature defined by

$$K(l,V) := \frac{v^i(l^j R_{jiks} l^s)v^k}{g(l,l)g(V,V) - [g(l,V)]^2},$$

where V is the section of the pull-back bundle π^*TM.

Note that the flag curvature is the Finslerian analogous part of Hatakeyama, Ogawa and Tanno result (see [6]).

Theorem 3.16. *Let $(\pi^*TM, \phi, \eta, \xi, g)$ be a horizontal K-contact metric Finsler pull-back bundle. Then, the flag curvature with the transverse edge ξ is equal to* 1.

Proof. Assume that $(\pi^*TM, \phi, \eta, \xi, g)$ is horizontally K-contact then $\mathbf{h} = 0$. On the other hand, the flag curvature with the transverse edge ξ is given by

$$\begin{aligned} K(l,\xi) &= g(R(l^H, \xi^H)\xi, l) \\ &= g(\phi\nabla_{\xi^H}\mathbf{h}l - \phi^2 l - \mathbf{h}^2 l - \phi^2 A^\sharp(\theta[\xi^H, l^H], \xi, \bullet), l) \\ &= -g(\nabla_{\xi^H}\mathbf{h}l, \phi l) + 1 - g(\mathbf{h}^2 l, l), \end{aligned}$$

and the assertion follows by hypothesis. □

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