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Energy asymptotic expansion for a system of nonlinear Schrödinger equations with three wave interaction

Yuki Osada

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Abstract. In this paper we consider the asymptotic expansion of the energy $I_\alpha^\beta(\gamma, \mu, s)$ associated with a nonlinear Schrödinger system with three wave interaction as $\beta \to \infty$ with $\alpha = \beta^\kappa$ for a given $\kappa \in \mathbb{R}$. In particular, we classify the asymptotic expansion formula into five cases for the parameter κ.

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§1. Introduction

Recently, there are many studies on the existence of standing waves and their stability for the nonlinear Schrödinger system with three wave interaction (see Colin-Colin-Ohta [6, 7], Pomponio [14], Ardila [2], Kurata-Osada [10] and the references therein) and related systems (see e.g. Gou-Jeanjean [9], Bhattarai [3], Zhao-Zhao-Shi [15] and the references therein).

In particular, the L^2-constrained variational problems associated with the systems and the orbital stability of ground states have been studied by many works (e.g. Bhattarai [3], Gou-Jeanjean [9], Ardila [2], Kurata-Osada [10]). In this paper, we focus on the following L^2-constrained variational problem:

$$\begin{aligned} I_\alpha^\beta(\gamma, \mu, s) &:= \inf\{E_\alpha^\beta(\vec{u}) \mid \vec{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3), \\ &\qquad \|u_1\|_2^2 = \gamma,\ \|u_2\|_2^2 = \mu,\ \|u_3\|_2^2 = s\}, \\ E_\alpha^\beta(\vec{u}) &:= \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla u_j|^2\, dx + \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(x)|u_j|^2\, dx \end{aligned}$$

$$- \frac{\beta}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |u_j|^{p+1}\, dx - \alpha \mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3\, dx,$$

where $\vec{u} := (u_1, u_2, u_3)$, $\overline{u}_3$ is the complex conjugate of u_3, $\alpha, \beta > 0$, $N = 1, 2, 3$, $1 < p < 1 + 4/N$, $\gamma, \mu, s > 0$ and each potential V_j $(j = 1, 2, 3)$ satisfies some suitable conditions. In this paper, we assume only one of the following conditions for the potentials V_j $(j = 1, 2, 3)$.

(V1) $V \in L^\infty(\mathbb{R}^N; \mathbb{R})$.

(V2) $V \in C(\mathbb{R}^N; \mathbb{R})$ and $V(x) \le \lim_{|y| \to \infty} V(y) = 0, \quad$ for all $x \in \mathbb{R}^N$.

In the previous paper ([10]), for the case $\beta = 1$, we studied the energy asymptotic expansion of $I^1_\alpha(\gamma, \mu, s)$ as $\alpha \to \infty$. In this paper, we consider the asymptotic expansion of the energy $I^\beta_\alpha(\gamma, \mu, s)$ as $\beta \to \infty$ with $\alpha = \beta^\kappa$ for a given $\kappa \in \mathbb{R}$.

To state the main result in this paper in details, we define the following variational problems:

$$\begin{aligned}
\Sigma_0(\gamma, \mu, s) &:= \inf\{E^0(\vec{u}) \mid \vec{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3), \\
&\qquad \|u_1\|_2^2 = \gamma,\ \|u_2\|_2^2 = \mu,\ \|u_3\|_2^2 = s\}, \\
\Sigma_1(\gamma, \mu, s) &:= \sup\{E^1(\vec{u}) \mid \vec{u} \text{ is a minimizer for } \Sigma_0(\gamma, \mu, s)\}, \\
I_\infty(\gamma, \mu, s) &:= \inf\{E_\infty(\vec{u}) \mid \vec{u} \in H^1(\mathbb{R}^N; \mathbb{C}^3), \\
&\qquad \|u_1\|_2^2 = \gamma,\ \|u_2\|_2^2 = \mu,\ \|u_3\|_2^2 = s\}, \\
S_\infty(\gamma) &:= \inf\{J_\infty(u) \mid u \in H^1(\mathbb{R}^N),\ \|u\|_2^2 = \gamma\}, \\
S^1(\gamma, \mu, s) &:= \sup\{J^1(\vec{u}) \mid u_1, u_2, u_3 \text{ are minimizers for} \\
&\qquad S_\infty(\gamma), S_\infty(\mu), S_\infty(s) \text{ respectively}\},
\end{aligned}$$

where

$$E^0(\vec{u}) := \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla u_j|^2\, dx - \mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3\, dx,$$

$$E^1(\vec{u}) := \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |u_j|^{p+1}\, dx,$$

$$E_\infty(\vec{u}) := \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla u_j|^2\, dx - \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |u_j|^{p+1}\, dx - \mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3\, dx,$$

$$J_\infty(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx,$$

$$J^1(\vec{u}) := \mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3\, dx.$$

Remark 1.1. Let $N \leq 3$, $1 < p < 1+4/N$, $\alpha, \beta > 0$. Under the following three assumptions on V_j $(j = 1, 2, 3)$:

- $V \in L^\infty(\mathbb{R}^N; \mathbb{R})$,
- $V(x) \leq \lim_{|y|\to\infty} V(y) = 0$ (a.e. $x \in \mathbb{R}^N$),
- $V(-x_1, x') = V(x_1, x')$ (a.e. $x_1 \in \mathbb{R},\ x' \in \mathbb{R}^{N-1}$),
  $V(s, x') \leq V(t, x')$ (a.e. $s, t \in \mathbb{R}$ with $0 \leq s < t,$ a.e. $x' \in \mathbb{R}^{N-1}$),

the existence of a minimizer for $I^\beta_\alpha(\gamma, \mu, s)$ is known (see [10]).

See also [10] about the existence of minimizer for $\Sigma_0(\gamma, \mu, s)$ under the additional condition $N \leq 2$. Moreover, since it is easy to check that the set of minimizers for $\Sigma_0(\gamma, \mu, s)$ is uniformly bounded in $H^1(\mathbb{R}^N; \mathbb{C}^3)$, it follows that $\Sigma_1(\gamma, \mu, s) < \infty$.

Remark 1.2. When $N \in \mathbb{N}$, $1 < p < 1 + 4/N$, for all $\gamma > 0$, it is well-known that there exists a unique positive, radially symmetric and strictly decreasing minimizer $\Psi_\gamma \in H^1(\mathbb{R}^N)$ for $S_\infty(\gamma)$ such that for all minimizer u for $S_\infty(\gamma)$, there exist $y \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$ such that

$$u(x) = e^{i\theta}\Psi_\gamma(x + y)$$

(see [5, 8, 11]).

Unless otherwise noted, Ψ_γ means the one in Remark 1.2. Also, we set $\vec{\Psi} := (\Psi_\gamma, \Psi_\mu, \Psi_s)$. Note that $\vec{\Psi}$ is a maximizer for $S^1(\gamma, \mu, s)$. See Lemma 2.3 for the proof.

For a given $\kappa \in \mathbb{R}$, as $\alpha = \beta^\kappa$ we define for simplicity

$$E^\beta(\vec{u}) := E^\beta_{\beta^\kappa}(\vec{u}),$$
$$I^\beta(\gamma, \mu, s) := I^\beta_{\beta^\kappa}(\gamma, \mu, s).$$

We show that there exist two critical numbers

$$\kappa_1 := (4 - N)/(4 - N(p - 1)), \quad \kappa_2 := -N/(4 - N(p - 1))$$

such that the asymptotic expansion of $I^\beta(\gamma, \mu, s)$ as $\beta \to \infty$ are different in the following five cases:
(i) $\kappa > \kappa_1$, (ii) $\kappa = \kappa_1$, (iii) $\kappa_2 < \kappa < \kappa_1$, (iv) $\kappa = \kappa_2$, (v) $\kappa < \kappa_2$.
We say $\{\vec{u}_n\}_{n=1}^\infty$ is a minimizing sequence for $I^{\beta_n}(\gamma, \mu, s)$ with $\beta_n \to \infty$ if

$$\|u_{1,n}\|_2^2 = \gamma, \quad \|u_{2,n}\|_2^2 = \mu, \quad \|u_{3,n}\|_2^2 = s,$$
$$E^{\beta_n}(\vec{u}_n) = I^{\beta_n}(\gamma, \mu, s) + o(1), \quad \text{as } n \to \infty.$$

We also study the asymptotic behavior of minimizing sequences $\{\vec{u}_n\}$ by using the rescaled functions of two types:

$$\vec{w}_n(x) := \beta_n^{-\kappa N/(4-N)} \vec{u}_n(\beta_n^{-2\kappa/(4-N)} x) \tag{1.1}$$

for the case (i) and

$$\vec{v}_n(x) := \beta_n^{-N/(4-N(p-1))} \vec{u}_n(\beta_n^{-2/(4-N(p-1))} x) \tag{1.2}$$

for the cases (ii)–(v), respectively.

Now we state the main result in this paper.

Theorem 1.3. *Let $N = 1, 2, 3$, $1 < p < 1 + 4/N$ and let $\{\vec{u}_n\}_{n=1}^{\infty}$ be a minimizing sequence for $I^{\beta_n}(\gamma, \mu, s)$ with $\beta_n \to \infty$. Then we have the asymptotic expansion of $I^{\beta}(\gamma, \mu, s) = I^{\beta}_{\beta^\kappa}(\gamma, \mu, s)$ as $\beta \to \infty$ in the five cases as follows:*

(i) For the case $\kappa > \kappa_1$, assume $N \leq 2$ and the condition (V1) for each potential V_j $(j = 1, 2, 3)$. Then

$$I^{\beta}(\gamma, \mu, s) = \beta^{4\kappa/(4-N)} \Sigma_0(\gamma, \mu, s) - \beta^{\kappa N(p-1)/(4-N)+1} \Sigma_1(\gamma, \mu, s) + o(\beta^{\kappa N(p-1)/(4-N)+1}), \quad \text{as } \beta \to \infty.$$

Moreover, for the rescaled function $\vec{w}_n$ defined by (1.1), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ and a maximizer $\vec{w}$ for $\Sigma_1(\gamma, \mu, s)$ such that

$$\|\vec{w}_n(\cdot + y_n) - \vec{w}\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

(ii) For the case $\kappa = \kappa_1$, assume the condition (V2) for each potential V_j $(j = 1, 2, 3)$ and $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. Then it holds that

$$I^{\beta}(\gamma, \mu, s) = \beta^{4/(4-N(p-1))} I_{\infty}(\gamma, \mu, s) + \frac{1}{2} \min_{x \in \mathbb{R}^N} \{V_1(x)\gamma + V_2(x)\mu + V_3(x)s\} + o(1), \quad \text{as } \beta \to \infty.$$

Moreover, for the rescaled function $\vec{v}_n$ defined by (1.2), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, a minimizer $\vec{v}$ for $I_{\infty}(\gamma, \mu, s)$ and $z_0 \in \mathbb{R}^N$ such that

$$\|\vec{v}_n(\cdot + y_n) - \vec{v}\|_{H^1} \to 0, \quad y_n/\beta_n^{2/(4-N(p-1))} \to z_0 \text{ in } \mathbb{R}^N, \quad \text{as } n \to \infty,$$
$$\min_{x \in \mathbb{R}^N} \{V_1(x)\gamma + V_2(x)\mu + V_3(x)s\} = V_1(z_0)\gamma + V_2(z_0)\mu + V_3(z_0)s.$$

(iii) For the case $\kappa_2 < \kappa < \kappa_1$, assume the condition (V1) for each potential V_j $(j = 1, 2, 3)$. Then

$$I^{\beta}(\gamma, \mu, s) = \beta^{4/(4-N(p-1))} (S_{\infty}(\gamma) + S_{\infty}(\mu) + S_{\infty}(s))$$

$$- \beta^{N/(4-N(p-1))+\kappa} S^1(\gamma, \mu, s) + o(\beta^{N/(4-N(p-1))+\kappa}), \quad \text{as } \beta \to \infty.$$

Moreover, for the rescaled function $\vec{v}_n$ defined by (1.2), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that

$$\begin{aligned} &\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi_j\|_{H^1} \to 0, \quad \text{as } n \to \infty, \\ &\theta_1 + \theta_2 = \theta_3, \end{aligned}$$

where $\Psi_1 = \Psi_\gamma$, $\Psi_2 = \Psi_\mu$, $\Psi_3 = \Psi_s$.

(iv) *For the case $\kappa = \kappa_2$, assume that the condition (V2) for each potential V_j $(j = 1, 2, 3)$, $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. We also assume that V_j has a unique minimum point $z_{j,0}$ and $z_{1,0} = z_{2,0} = z_{3,0} =: z_0$. Then*

$$\begin{aligned} I^\beta(\gamma, \mu, s) = \ &\beta^{4/(4-N(p-1))}(S_\infty(\gamma) + S_\infty(\mu) + S_\infty(s)) \\ &- S^1(\gamma, \mu, s) + \frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)\gamma + V_2(x)\mu + V_3(x)s\} + o(1), \\ &\text{as } \beta \to \infty. \end{aligned}$$

Moreover, for the rescaled function $\vec{v}_n$ defined by (1.2), up to a subsequence, there exist $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$, and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that

$$\begin{aligned} &\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi_j\|_{H^1} \to 0, \quad \text{as } n \to \infty, \\ &\theta_1 + \theta_2 = \theta_3, \\ &y_n/\beta_n^{2/(4-N(p-1))} \to z_0 \text{ in } \mathbb{R}^N, \end{aligned}$$

where $\Psi_1 = \Psi_\gamma$, $\Psi_2 = \Psi_\mu$, $\Psi_3 = \Psi_s$.

(v) *For the case $\kappa < \kappa_2$, assume that the condition (V2) for each potential V_j $(j = 1, 2, 3)$ and $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. Then*

$$\begin{aligned} I^\beta(\gamma, \mu, s) = \ &\beta^{4/(4-N(p-1))}(S_\infty(\gamma) + S_\infty(\mu) + S_\infty(s)) \\ &+ \frac{1}{2}\left(\min_{x\in\mathbb{R}^N} V_1(x)\gamma + \min_{x\in\mathbb{R}^N} V_2(x)\mu + \min_{x\in\mathbb{R}^N} V_3(x)s\right) + o(1), \\ &\text{as } \beta \to \infty. \end{aligned}$$

Moreover, for the rescaled function $\vec{v}_n$ defined by (1.2), up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^{\infty} \subset \mathbb{R}^N$ $(j = 1, 2, 3)$, and $\theta_j \in \mathbb{R}$ $(j = 1, 2, 3)$ and $z_{j,0} \in \mathbb{R}^N$ $(j = 1, 2, 3)$ such that

$$\begin{aligned} &\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi_j\|_{H^1} \to 0, \quad \text{as } n \to \infty, \\ &y_n^{(j)}/\beta_n^{2/(4-N(p-1))} \to z_{j,0} \text{ in } \mathbb{R}^N, \\ &\min_{x\in\mathbb{R}^N} V_j(x) = V_j(z_{j,0}), \end{aligned}$$

where $\Psi_1 = \Psi_\gamma$, $\Psi_2 = \Psi_\mu$, $\Psi_3 = \Psi_s$.

Remark 1.4. By Theorem 1.3, we can say that the effect of the three wave interaction appears in the first order term in the case $\kappa \geq \kappa_1$ and in the second order term in the case $\kappa_2 \leq \kappa < \kappa_1$, but disappears in the case $\kappa < \kappa_2$. We also emphasize that we use the different rescaled functions in the case (ii)–(v) and in the case (i), respectively, to obtain the asymptotic behavior of minimizing sequences precisely.

This paper is organized as follows: In Section 2, we prepare the characterization of $S^1(\gamma, \mu, s)$ to prove Theorem 1.3 in the cases (iii) and (iv). In Section 3, we prove Theorem 1.3 concerning the asymptotic expansion of $I^\beta(\gamma, \mu, s)$ and the asymptotic behavior of a minimizing sequence for the cases (i)–(v). In appendix, we note that the asymptotic expansion of $I_\alpha^{\alpha^\tau}$ as $\alpha \to \infty$ for a given $\tau \leq 0$ and the asymptotic behavior of a minimizing sequence for $I_{\alpha_n}^{\alpha_n^\tau}$ where $\alpha_n \to \infty$.

§2. Preliminaries

For simplicity, we prove Theorem 1.3 as $\gamma = \mu = s = 1$. So for simplicity, we write $I^\beta(\gamma, \mu, s)$, $S_\infty(\gamma)$, $S^1(\gamma, \mu, s)$, $I_\infty(\gamma, \mu, s)$, $\Sigma_0(\gamma, \mu, s)$ and $\Sigma_1(\gamma, \mu, s)$ as I^β, S_∞, S^1, I_∞, Σ_0 and Σ_1. Moreover, when $\gamma = 1$, Ψ_γ in Remark 1.2 is abbreviated as Ψ.

As stated in Remark 1.2, the following compactness of the minimizing sequence for S_∞ is known (see Lions [12, 13]).

Lemma 2.1. *Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence for S_∞. Then up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\theta \in \mathbb{R}$ such that*

$$\|u_n(\cdot + y_n) - e^{i\theta}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

Here, we note that the fact on rearrangements (see [4]).

Lemma 2.2. *We assume that $N \in \mathbb{N}$ and let $f, g, h \in C(\mathbb{R}^N)$ be functions such that positive, radialy symmetric and strictly decreasing and*

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} g(x) = \lim_{|x|\to\infty} h(x) = 0,$$
$$\int_{\mathbb{R}^N} f(x)g(x)h(x)\,dx < \infty.$$

For $y_0, y_1 \in \mathbb{R}^N$, if $y_0 \neq 0$ or $y_1 \neq 0$, then

$$\int_{\mathbb{R}^N} f(x)g(x - y_0)h(x - y_1)\,dx < \int_{\mathbb{R}^N} f(x)g(x)h(x)\,dx$$

holds.

Lemma 2.3 (characterization of maximizer for S^1)**.** *Let $\vec{u}$ be a maximizer for S^1. Then there exist $y \in \mathbb{R}^N$ and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ with $\theta_1 + \theta_2 = \theta_3$ such that*

$$\vec{u} = (e^{i\theta_1}\Psi(\cdot + y), e^{i\theta_2}\Psi(\cdot + y), e^{i\theta_3}\Psi(\cdot + y)),$$
$$S^1 = \int_{\mathbb{R}^N} \Psi^3 \, dx \ (> 0).$$

Proof. By the definition of S^1,

$$\begin{aligned} S^1 &= \sup_{\theta_1,\theta_2,\theta_3 \in \mathbb{R}} \operatorname{Re}(e^{i(\theta_1+\theta_2-\theta_3)}) \sup_{z_1,z_2 \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Psi(x)\Psi(x+z_1)\Psi(x+z_2)\, dx \\ &= \sup_{z_1,z_2 \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Psi(x)\Psi(x+z_1)\Psi(x+z_2)\, dx \end{aligned}$$

with $\theta_1 + \theta_2 = \theta_3 + 2k\pi$ $(k \in \mathbb{Z})$. From Lemma 2.2, we have

$$\sup_{z_1,z_2 \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Psi(x)\Psi(x+z_1)\Psi(x+z_2)\, dx = \int_{\mathbb{R}^N} \Psi(x)\Psi(x)\Psi(x)\, dx$$

and the supremum is attained only for the case $z_1 = z_2 = 0$. Thus

$$S^1 = \int_{\mathbb{R}^N} \Psi(x)^3 \, dx \ (> 0).$$

□

We note the following compactness of minimizing sequence for I_∞.

Lemma 2.4 ([10])**.** *Let $N \leq 3$, $1 < p < 1+4/N$. Let $\{\vec{u}_n\}_{n=1}^\infty$ be a minimizing sequence for I_∞. Then up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and a minimizer $\vec{u}$ for I_∞ such that*

$$\|u_{j,n}(\cdot + y_n) - u_j\|_{H^1} \to 0, \quad \text{as } n \to \infty.$$

§3. Proof of Theorem 1.3

Throughout this section, we assume that $N \leq 3$, $1 < p < 1 + 4/N$, $\beta > 0$, $\alpha = \beta^\kappa$ with $\kappa \in \mathbb{R}$ and $\gamma = \mu = s = 1$. First, we give the proof of the cases (ii)–(v) of Theorem 1.3. Finally, we give the proof of the case (i) of Theorem 1.3.

To show the results in the cases (ii)–(v), we rescale the function $\vec{u}$ as (1.2), the functional E^β and its energy I^β as follows:

Let $\vec{u}$ be a function such that

$$\|u_1\|_2^2 = \|u_2\|_2^2 = \|u_3\|_2^2 = 1.$$

We rescale the function $\vec{u}$ as follows:

$$\vec{v}(x) := \beta^{-N/(4-N(p-1))}\vec{u}(\beta^{-2/(4-N(p-1))}x).$$

Then it follows that

$$\|v_1\|_2^2 = \|v_2\|_2^2 = \|v_3\|_2^2 = 1$$

and

$$\begin{aligned} E^\beta(\vec{u}) &= \beta^{4/(4-N(p-1))}\tilde{E}^\beta(\vec{v}), \\ I^\beta &= \beta^{4/(4-N(p-1))}\tilde{I}^\beta, \end{aligned}$$

where

$$\begin{aligned} \tilde{E}^\beta(\vec{v}) :=& \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |\nabla v_j|^2\,dx - \frac{1}{p+1}\sum_{j=1}^{3}\int_{\mathbb{R}^N} |v_j|^{p+1}\,dx \\ &- \beta^{(N-4)/(4-N(p-1))+\kappa}\mathrm{Re}\int_{\mathbb{R}^N} v_1 v_2 \overline{v}_3\,dx \\ &+ \frac{1}{\beta^{4/(4-N(p-1))}}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx, \\ \tilde{I}^\beta :=& \inf\{\tilde{E}^\beta(\vec{v}) \mid \vec{v} \in H^1(\mathbb{R}^N;\mathbb{C}^3),\ \|v_j\|_2^2 = 1\ (j=1,2,3)\}. \end{aligned}$$

So it is sufficient to prove the energy expansion of $\tilde{I}^\beta$ and the asymptotic behavior of $\vec{v}_n$ to prove the cases (ii)–(v) in Theorem 1.3.

3.1. Proof of Theorem 1.3 (ii)

For the case $\kappa = \kappa_1$, we have

$$\tilde{E}^\beta(\vec{v}) = E_\infty(\vec{v}) + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx.$$

3.1.1. Upper bound

Lemma 3.1 (upper bound for $\tilde{I}^\beta$)**.** *Under the assumptions in the case (ii), it follows that*

$$\begin{aligned} \tilde{I}^\beta \le I_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} \\ + o(\beta^{-4/(4-N(p-1))}), \quad \textit{as } \beta \to \infty. \end{aligned}$$

Proof. From Lemma 2.4, there exists a minimizer $\vec{v}$ for I_∞. Let $x_0 \in \mathbb{R}^N$ be a point which attains

$$\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}.$$

For $\beta > 0$, we set

$$\vec{\varphi}_\beta(x) := \vec{v}(x-\beta^{2/(4-N(p-1))}x_0).$$

Then it holds that

$$\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|\varphi_{j,\beta}(x)|^2\,dx = \int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}+x_0\right)|v_j(x)|^2\,dx.$$

From (V2), it follows that

$$\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}+x_0\right)|v_j(x)|^2\,dx \to \int_{\mathbb{R}^N} V_j(x_0)|v_j(x)|^2\,dx, \quad \text{as } \beta\to\infty.$$

Then we have

$$\begin{aligned}
\tilde{I}^\beta &\le \tilde{E}^\beta(\vec{\varphi}_\beta)\\
&= I_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|\varphi_{j,\beta}(x)|^2\,dx\\
&= I_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}\\
&\qquad + o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.
\end{aligned}$$

□

3.1.2. Lower bound and the completion of the proof of Theorem 1.3 (ii)

Theorem 1.3 (ii) with $\gamma=\mu=s=1$ is reduced to the following lemma.

Lemma 3.2. *Under the assumptions in the case (ii), it follows that*

$$\begin{aligned}
\tilde{I}^\beta &= I_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}\\
&\qquad + o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.
\end{aligned}$$

Moreover, for the rescaled function $\vec{v}_n$ defined by (1.2), up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$, a minimizer $\vec{v}$ for I_∞ and $z_0\in\mathbb{R}^N$ such that

$$\begin{aligned}
&\|\vec{v}_n(\cdot+y_n)-\vec{v}\|_{H^1}\to 0, \quad y_n/\beta_n^{2/(4-N(p-1))}\to z_0 \text{ in } \mathbb{R}^N, \quad \text{as } n\to\infty,\\
&\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\} = V_1(z_0)+V_2(z_0)+V_3(z_0).
\end{aligned}$$

Proof. Note that $\vec{v}_n$ satisfies

$$\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1,$$
$$\tilde{E}^{\beta_n}(\vec{v}_n) = \tilde{I}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}),$$

where $\beta_n \to \infty$. From Lemma 3.1, it follows that

$$\begin{aligned}
&I_\infty + o(1)\\
&\geq \tilde{I}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}) = \tilde{E}^{\beta_n}(\vec{v}_n)\\
&= E_\infty(\vec{v}_n) + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}|^2\,dx \qquad (3.1)\\
&\geq I_\infty + o(1).
\end{aligned}$$

Therefore $\{\vec{v}_n\}_{n=1}^\infty$ is a minimizing sequence for I_∞. From Lemma 2.4, up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\vec{v} \in H^1(\mathbb{R}^N;\mathbb{C}^3)$ such that

$$\|\vec{v}_n(\cdot + y_n) - \vec{v}\|_{H^1} \to 0, \quad \text{as } n \to \infty$$
$$\vec{v} \text{ is a minimizer for } I_\infty.$$

Since $\|v_{j,n}(\cdot + y_n) - v_j\|_2 \to 0$ (as $n \to \infty$), up to a subsequence, there exists $g_j \in L^2(\mathbb{R}^N)$ such that

$$v_{j,n}(x + y_n) \to v_j(x), \quad \text{as } n \to \infty, \text{ a.e. } x \in \mathbb{R}^N,$$
$$|v_{j,n}(x + y_n)| \leq g_j(x), \quad \text{for all } n \in \mathbb{N}, \text{ a.e. } x \in \mathbb{R}^N.$$

Claim. $\{y_n/\beta_n^{2/(4-N(p-1))}\}_{n=1}^\infty$ is bounded.

If not, up to a subsequence, $|y_n|/\beta_n^{2/(4-N(p-1))} \to \infty$ (as $n \to \infty$). From (V2),

$$\int_{\mathbb{R}^N} V_j\left(\frac{x + y_n}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}(x + y_n)|^2\,dx \to 0, \quad \text{as } n \to \infty.$$

From Lemma 3.1, we have

$$\begin{aligned}
&I_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x) + V_2(x) + V_3(x)\} + o(\beta_n^{-4/(4-N(p-1))})\\
&\geq \tilde{I}^{\beta_n} = \tilde{E}^{\beta_n}(\vec{v}_n) + o(\beta_n^{-4/(4-N(p-1))})\\
&\geq I_\infty + o(\beta_n^{-4/(4-N(p-1))}), \quad \text{as } n \to \infty.
\end{aligned}$$

Then we have

$$\min_{x\in\mathbb{R}^N}\{V_1(x) + V_2(x) + V_3(x)\} \geq 0.$$

On the other hand, since $V_j(x) \le 0$ (for all $x \in \mathbb{R}^N$) and $V_1 \not\equiv 0$ or $V_2 \not\equiv 0$ or $V_3 \not\equiv 0$, it follows that

$$\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}<0.$$

This is a contradiction. Thus the claim holds. Therefore, up to a subsequence, there exists $z_0 \in \mathbb{R}^N$ such that

$$y_n/\beta_n^{2/(4-N(p-1))} \to z_0, \quad \text{as } n\to\infty.$$

From (V2), we have

$$\begin{aligned}(3.2)\qquad &\int_{\mathbb{R}^N} V_j\left(\frac{x+y_n}{\beta_n^{2/(4-N(p-1))}}\right)|v_{j,n}(x+y_n)|^2\,dx\\ &\to \int_{\mathbb{R}^N} V_j(z_0)|v_j(x)|^2\,dx, \quad \text{as } n\to\infty.\end{aligned}$$

From (3.1)–(3.2), we have

$$\begin{aligned}&I_\infty+\beta_n^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}+o(\beta_n^{-4/(4-N(p-1))})\\ &\ge \tilde{I}^{\beta_n}=\tilde{E}^{\beta_n}(\vec{v}_n)+o(\beta_n^{-4/(4-N(p-1))})\\ &\ge I_\infty+\beta_n^{-4/(4-N(p-1))}\frac{1}{2}(V_1(z_0)+V_2(z_0)+V_3(z_0))+o(\beta_n^{-4/(4-N(p-1))})\\ &\ge I_\infty+\beta_n^{-4/(4-N(p-1))}\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}+o(\beta_n^{-4/(4-N(p-1))}),\\ &\hspace{30em}\text{as } n\to\infty.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}=V_1(z_0)+V_2(z_0)+V_3(z_0),\\ &\lim_{n\to\infty}\beta_n^{4/(4-N(p-1))}(\tilde{I}^{\beta_n}-I_\infty)=\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}.\end{aligned}$$

Since $\{\beta_n\}_{n=1}^\infty$ is arbitrary sequence satisfying $\beta_n\to\infty$ (as $n\to\infty$), we have

$$\begin{aligned}\tilde{I}^{\beta}=I_\infty+\beta^{-4/(4-N(p-1))}&\frac{1}{2}\min_{x\in\mathbb{R}^N}\{V_1(x)+V_2(x)+V_3(x)\}\\ &+o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.\end{aligned}$$

□

Remark 3.3. The result of Theorem 1.3 (ii) indicates that $\vec{u}_n$ concentrates at z_0. Indeed, $\vec{u}_n$ behaves like

$$\begin{aligned}\vec{u}_n(x) &= \beta_n^{N/(4-N(p-1))}\vec{v}_n(\beta_n^{2/(4-N(p-1))}x)\\ &\sim \beta_n^{N/(4-N(p-1))}\vec{v}(\beta_n^{2/(4-N(p-1))}x - y_n)\\ &\sim \beta_n^{N/(4-N(p-1))}\vec{v}(\beta_n^{2/(4-N(p-1))}(x - z_0)), \quad \text{as } \beta_n \to \infty.\end{aligned}$$

3.2. Proof of Theorem 1.3 (iii)

Note that for the case (iii)

$$-4/(4-N(p-1)) < (N-4)/(4-N(p-1)) + \kappa < 0$$

and

$$\begin{aligned}\tilde{E}^\beta(\vec{v}) =& \sum_{j=1}^{3} J_\infty(v_j) - \beta^{(N-4)/(4-N(p-1))+\kappa} J^1(\vec{v})\\ &+ \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx.\end{aligned}$$

First, we prove the upper bound for $\tilde{I}^\beta$. Taking $\vec{\Psi} = (\Psi, \Psi, \Psi)$, where Ψ is the function Ψ_γ defined in Remark 1.2 with $\gamma = 1$, under the assumption in the case (iii), from Lemma 2.3, it is easy to obtain

$$\tilde{I}^\beta \le \tilde{E}^\beta(\vec{\Psi}) \le 3S_\infty - \beta^{(N-4)/(4-N(p-1))+\kappa} S^1, \quad \text{as } \beta \to \infty.$$

Theorem 1.3 (iii) with $\gamma = \mu = s = 1$ is reduced to the following lemma.

Lemma 3.4. *Under the assumption in the case (iii), it holds that*

$$\tilde{I}^\beta = 3S_\infty - \beta^{(N-4)/(4-N(p-1))+\kappa} S^1 + o(\beta^{(N-4)/(4-N(p-1))+\kappa}), \quad \text{as } \beta \to \infty.$$

Moreover, for the rescaled function $\vec{v}_n$ defined by (1.2)*, up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$, and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that*

$$\begin{aligned}&\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty,\ j = 1, 2, 3,\\ &\theta_1 + \theta_2 = \theta_3.\end{aligned}$$

Proof. **(Step 1)** Note that $\vec{v}_n$ satisfies

$$\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1, \tag{3.3}$$

$$\tilde{E}^{\beta_n}(\vec{v}_n) = \tilde{I}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}). \tag{3.4}$$

From the upper bound for $\tilde{I}^\beta$, it holds that

$$\begin{aligned}
&3S_\infty + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq \tilde{I}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}) = \tilde{E}^{\beta_n}(\vec{v}_n) \\
&\geq J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) \\
&\quad + O(1/\beta_n^{4/(4-N(p-1))}) - \beta_n^{(N-4)/(4-N(p-1))+\kappa} \mathrm{Re} \int_{\mathbb{R}^N} v_{1,n} v_{2,n} \overline{v}_{3,n}\, dx \\
&\geq \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |\nabla v_{j,n}|^2\, dx - \frac{1}{p+1} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |v_{j,n}|^{p+1}\, dx \\
&\quad + O(1/\beta_n^{4/(4-N(p-1))}) - \frac{\beta_n^{(N-4)/(4-N(p-1))+\kappa}}{3} \sum_{j=1}^{3} \int_{\mathbb{R}^N} |v_{j,n}|^3\, dx.
\end{aligned} \tag{3.5}$$

Here we note that $N \leq 3$, $1 < p < 1+4/N$ and $(N-4)/(4-N(p-1)) + \kappa < 0$. Then for sufficiently large n, it follows that $\beta_n^{(N-4)/(4-N(p-1))+\kappa} \leq 1$. From Gagliardo-Nirenberg's inequality (see Adams [1]) and (3.3), for $q = p+1$ and $q = 3$, we have

$$\begin{aligned}
\|v_{j,n}\|_q^q &\leq C(N,q) \|\nabla v_{j,n}\|_2^{N(q-2)/2} \|v_{j,n}\|_2^{q-N(q-2)/2} \\
&\leq \varepsilon \|\nabla v_{j,n}\|_2^2 + C(\varepsilon, N, q), \quad \text{for all } \varepsilon > 0.
\end{aligned} \tag{3.6}$$

Here $C(N,q), C(\varepsilon, N, q) > 0$ is a constant. From (3.5),(3.6), we have

$$3S_\infty + O(1) \geq \left(\frac{1}{2} - \frac{1}{p+1}\varepsilon - \frac{1}{3}\varepsilon \right) \sum_{j=1}^{3} \|\nabla v_{j,n}\|_2^2.$$

Fix $\varepsilon > 0$ such that $1/2 - \varepsilon/(p+1) - \varepsilon/3 > 0$. Combining with (3.3), we find that there exists a positive constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\sum_{j=1}^{3} \|v_{j,n}\|_{H^1}^2 \leq C. \tag{3.7}$$

(Step 2) From the upper bound for $\tilde{I}^\beta$, we have

$$\begin{aligned}
3S_\infty \geq \tilde{I}^{\beta_n} &= \tilde{E}^{\beta_n}(\vec{v}_n) + o(\beta_n^{-4/(4-N(p-1))}) \\
&\geq J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) \\
&\quad + O(1/\beta_n^{4/(4-N(p-1))}) \\
&\quad - \beta_n^{(N-4)/(4-N(p-1))+\kappa} \mathrm{Re} \int_{\mathbb{R}^N} v_{1,n} v_{2,n} \overline{v}_{3,n}\, dx.
\end{aligned} \tag{3.8}$$

From (3.7) and $N \le 3$, $1 < p < 1 + 4/N$ and $(N-4)/(4-N(p-1)) + \kappa < 0$, we deduce that

$$\beta_n^{(N-4)/(4-N(p-1))+\kappa} \mathrm{Re} \int_{\mathbb{R}^N} v_{1,n} v_{2,n} \overline{v}_{3,n}\, dx = o(1), \quad \text{as } n \to \infty.$$

From (3.3),(3.8) and the definition of S_∞, we have

$$\begin{aligned} 3S_\infty \ge \tilde{I}^{\beta_n} &\ge J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) + o(1) \\ &\ge 3S_\infty + o(1), \quad \text{as } n \to \infty. \end{aligned}$$

Thus we have

$$\lim_{n\to\infty} J_\infty(v_{j,n}) = S_\infty, \quad j = 1, 2, 3.$$

Thus $\{v_{1,n}\}_{n=1}^\infty$,$\{v_{2,n}\}_{n=1}^\infty$,$\{v_{3,n}\}_{n=1}^\infty$ are minimizing sequences for S_∞. From Lemma 2.1, up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\theta_j \in \mathbb{R}$ such that

$$\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty,\ j = 1, 2, 3 \tag{3.9}$$

(Step 3) Set

$$\begin{aligned} &\Psi_{j,n} := e^{i\theta_j}\Psi(\cdot - y_n^{(j)}),\ j = 1, 2, 3 \\ &\vec{\Psi}_n := (\Psi_{1,n}, \Psi_{2,n}, \Psi_{3,n}). \end{aligned}$$

From (3.9) and $\{\vec{v}_n\}_{n=1}^\infty$ and $\{\vec{\Psi}_n\}_{n=1}^\infty$ are bounded in $H^1(\mathbb{R}^N; \mathbb{C}^3)$, we have

$$\begin{aligned} &|J^1(\vec{v}_n) - J^1(\vec{\Psi}_n)| \\ &\le \int_{\mathbb{R}^N} |v_{1,n}||v_{2,n}||v_{3,n} - \Psi_{3,n}|\, dx + \int_{\mathbb{R}^N} |v_{1,n}||v_{2,n} - \Psi_{2,n}||\Psi_{3,n}|\, dx \\ &\quad + \int_{\mathbb{R}^N} |v_{1,n} - \Psi_{1,n}||\Psi_{2,n}||\Psi_{3,n}|\, dx \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.10}$$

Moreover, since $\Psi_{j,n}$ is a minimizer for S_∞, it follows that

$$J^1(\vec{\Psi}_n) \le S^1.$$

From the upper bound for $\tilde{I}^\beta$, it follows that

$$\begin{aligned} &3S_\infty - \beta_n^{(N-4)/(4-N(p-1))+\kappa} S^1 \\ &\ge \tilde{I}^{\beta_n} = \tilde{E}^{\beta_n}(\vec{v}_n) + o(\beta_n^{-4/(4-N(p-1))}) \\ &= J_\infty(v_{1,n}) + J_\infty(v_{2,n}) + J_\infty(v_{3,n}) - \beta_n^{(N-4)/(4-N(p-1))+\kappa} J^1(\vec{v}_n) \\ &\quad + o(\beta_n^{(N-4)/(4-N(p-1))+\kappa}) \end{aligned}$$

$$\geq 3S_\infty - \beta_n^{(N-4)/(4-N(p-1))+\kappa} J^1(\vec{\Psi}_n) + o(\beta_n^{(N-4)/(4-N(p-1))+\kappa})$$
$$\geq 3S_\infty - \beta_n^{(N-4)/(4-N(p-1))+\kappa} S^1 + o(\beta_n^{(N-4)/(4-N(p-1))+\kappa}), \quad \text{as } n \to \infty.$$

Thus we have

$$\lim_{n\to\infty} \frac{\tilde{I}^{\beta_n} - 3S_\infty}{\beta_n^{(N-4)/(4-N(p-1))+\kappa}} = -S^1, \quad \lim_{n\to\infty} J^1(\vec{\Psi}_n) = S^1. \tag{3.11}$$

Since $\{\beta_n\}_{n=1}^\infty$ is arbitrary sequence satisfying $\beta_n \to \infty$, we have

$$\tilde{I}^\beta = 3S_\infty - \beta^{(N-4)/(4-N(p-1))+\kappa} S^1 + o(\beta^{(N-4)/(4-N(p-1))+\kappa}), \quad \text{as } \beta \to \infty.$$

(Step 4) From (3.11), it follows that

(3.12)
$$\mathrm{Re}(e^{i(\theta_1+\theta_2-\theta_3)}) \lim_{n\to\infty} \int_{\mathbb{R}^N} \Psi(x)\Psi(x+y_n^{(1)}-y_n^{(2)})\Psi(x+y_n^{(1)}-y_n^{(3)})\,dx = S^1.$$

We prove $\{y_n^{(1)} - y_n^{(2)}\}_{n=1}^\infty$ and $\{y_n^{(1)} - y_n^{(3)}\}_{n=1}^\infty$ are bounded in $\mathbb{R}^N$. If not, for example, if $\{y_n^{(1)} - y_n^{(2)}\}_{n=1}^\infty$ is not bounded, up to a subsequence, then it holds that

$$|y_n^{(1)} - y_n^{(2)}| \to \infty, \quad \text{as } n \to \infty.$$

From Remark 1.2, $\Psi \in L^2(\mathbb{R}^N)$ is radially symmetric and decreasing, it holds that

$$\lim_{|x|\to\infty} \Psi(x) = 0.$$

Thus for all $\varepsilon > 0$, there exists $R > 0$ such that

$$|x| \geq R \Longrightarrow \Psi(x) < \varepsilon.$$

In addition, since $|y_n^{(1)} - y_n^{(2)}| \to \infty$ (as $n \to \infty$), for n sufficiently large, we have

$$\Psi(x + y_n^{(1)} - y_n^{(2)}) < \varepsilon, \quad \text{for all } |x| < R.$$

Thus for n sufficiently large, it follows that

$$\begin{aligned}|J^1(\vec{\Psi}_n)| &= |J^1(\Psi(\cdot - y_n^{(1)}), \Psi(\cdot - y_n^{(2)}), \Psi(\cdot - y_n^{(3)}))|\\
&\leq \int_{\mathbb{R}^N} \Psi(x)\Psi(x+y_n^{(1)}-y_n^{(2)})\Psi(x+y_n^{(1)}-y_n^{(3)})\,dx\\
&\leq \varepsilon \int_{|x|<R} \Psi(x)\Psi(x+y_n^{(1)}-y_n^{(3)})\,dx\\
&\quad + \varepsilon \int_{|x|\geq R} \Psi(x+y_n^{(1)}-y_n^{(2)})\Psi(x+y_n^{(1)}-y_n^{(3)})\,dx\end{aligned}$$

$$\leq \varepsilon\|\Psi\|_2^2 + \varepsilon\|\Psi\|_2^2 = 2\varepsilon.$$

Thus we have

$$\lim_{n\to\infty} J^1(\vec{\Psi}_n) = 0.$$

Although

$$\lim_{n\to\infty} J^1(\vec{\Psi}_n) = S^1,$$

this is a contradiction to $S^1 > 0$ from Lemma 2.3. Therefore $\{y_n^{(1)} - y_n^{(2)}\}_{n=1}^\infty$ is bounded. We can prove that $\{y_n^{(1)} - y_n^{(3)}\}_{n=1}^\infty$ is bounded in the same way. Hence up to a subsequence, there exist $y^{(2)}, y^{(3)} \in \mathbb{R}^N$ such that

$$\begin{aligned} y_n^{(1)} - y_n^{(2)} &\to y^{(2)}, \quad \text{as } n \to \infty, \\ y_n^{(1)} - y_n^{(3)} &\to y^{(3)}, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \Psi(x)\Psi(x + y_n^{(1)} - y_n^{(2)})\Psi(x + y_n^{(1)} - y_n^{(3)})\, dx \\ &\to \int_{\mathbb{R}^N} \Psi(x)\Psi(x + y^{(2)})\Psi(x + y^{(3)})\, dx, \quad \text{as } n \to \infty. \end{aligned}$$

From (3.12), it holds that

$$\mathrm{Re}(e^{i(\theta_1+\theta_2-\theta_3)}) \int_{\mathbb{R}^N} \Psi(x)\Psi(x + y^{(2)})\Psi(x + y^{(3)})\, dx = S^1.$$

Therefore $(e^{i\theta_1}\Psi, e^{i\theta_2}\Psi(\cdot + y^{(2)}), e^{i\theta_3}\Psi(\cdot + y^{(3)}))$ is a maximizer for S^1. From Lemma 2.3, $y^{(2)} = y^{(3)} = 0$ and we may assume that $\theta_1 + \theta_2 = \theta_3$.

Moreover we have

$$\|v_{j,n}(\cdot + y_n^{(1)}) - e^{i\theta_j}\Psi\|_{H^1} \to 0, \quad \text{as } n \to \infty,\ j = 2, 3.$$

Indeed, setting $z_n^{(j)} := y_n^{(1)} - y_n^{(j)}$ $(j = 2, 3)$, we have

$$\begin{aligned} &\|v_{j,n}(\cdot + y_n^{(1)}) - e^{i\theta_j}\Psi\|_{H^1} = \|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi(\cdot - z_n^{(j)})\|_{H^1} \\ &\leq \|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi\|_{H^1} + \|e^{i\theta_j}\Psi - e^{i\theta_j}\Psi(\cdot - z_n^{(j)})\|_{H^1} \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

□

3.3. Proof of Theorem 1.3 (iv)

For the case $\kappa = \kappa_2$, we have

$$\tilde{E}^{\beta}(\vec{v}) = \sum_{j=1}^{3} J_{\infty}(v_j) - \beta^{-4/(4-N(p-1))} \times$$
$$\times \left(J^1(\vec{v}) - \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} V_j \left(\frac{x}{\beta^{2/(4-N(p-1))}} \right) |v_j|^2 \, dx \right).$$

For the proof of the upper bound, we use the following test function:

$$\vec{\varphi}_{\beta}(x) := \vec{\Psi}(x - \beta^{2/(4-N(p-1))} z_0),$$

where z_0 is unique minimum point of V_j. By using the arguments used in Theorem 1.3 (ii), we can prove the upper bound:

$$\tilde{I}^{\beta} \le \tilde{E}^{\beta}(\vec{\varphi}_{\beta}) = 3S_{\infty} - \beta^{-4/(4-N(p-1))} \times$$
$$\times \left(S^1 - \frac{1}{2} \min_{x \in \mathbb{R}^N} \{ V_1(x) + V_2(x) + V_3(x) \} \right)$$
$$+ o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta \to \infty.$$

For the proof of the lower bound, note that the rescaled function $\vec{v}_n$ defined by (1.2) satisfies

$$\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1,$$
$$\tilde{E}^{\beta_n}(\vec{v}_n) = \tilde{I}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}),$$

where $\beta_n \to \infty$. By the similar argument as in the proof of Theorem 1.3 (iii), it holds that $\{\vec{v}_n\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N; \mathbb{C}^3)$ and each $\{v_{j,n}\}_{n=1}^{\infty}$ is a minimizing sequence for S_{∞}. Therefore up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^{\infty} \subset \mathbb{R}^N$ and $\theta_j \in \mathbb{R}$ such that

$$\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j} \Psi\|_{H^1} \to 0.$$

From the upper bound for $\tilde{I}^{\beta_n}$, we have

$$3S_{\infty} - \beta_n^{-4/(4-N(p-1))} \times$$
$$\times \left(S^1 - \frac{1}{2} \min_{x \in \mathbb{R}^N} \{ V_1(x) + V_2(x) + V_3(x) \} \right) + o(\beta_n^{-4/(4-N(p-1))})$$
$$\ge \tilde{I}^{\beta_n}$$
$$\ge 3S_{\infty} - \beta_n^{-4/(4-N(p-1))} \times$$

$$\times \left(J^1(\vec{v}_n) - \frac{1}{2} \sum_{j=1}^{3} \int_{\mathbb{R}^N} V_j \left(\frac{x}{\beta_n^{2/(4-N(p-1))}} \right) |v_{j,n}|^2 \, dx \right)$$
$$+ o(\beta_n^{-4/(4-N(p-1))}).$$

Since

$$J^1(\vec{v}_n) = J^1(e^{i\theta_1}\Psi(\cdot - y_n^{(1)}), e^{i\theta_2}\Psi(\cdot - y_n^{(2)}), e^{i\theta_3}\Psi(\cdot - y_n^{(3)})) + o(1)$$
$$\le S^1 + o(1),$$

by the same argument as in Theorem 1.3 (ii) and (iii), we have

$$y_n^{(j)}/\beta_n^{2/(4-N(p-1))} \to z_{j,0} = z_0,$$
$$\int_{\mathbb{R}^N} V_j \left(\frac{x}{\beta_n^{2/(4-N(p-1))}} \right) |v_{j,n}|^2 \, dx \to V_j(z_{j,0}) = V_j(z_0).$$

Thus we have

$$3S_\infty - \beta_n^{-4/(4-N(p-1))}(S^1 - \frac{1}{2} \min_{x \in \mathbb{R}^N} \{V_1(x) + V_2(x) + V_3(x)\})$$
$$+ o(\beta_n^{-4/(4-N(p-1))})$$
$$\ge \tilde{I}^{\beta_n}$$
$$\ge 3S_\infty - \beta_n^{-4/(4-N(p-1))} \times$$
$$\times \Big(J^1(e^{i\theta_1}\Psi(\cdot - y_n^{(1)}), e^{i\theta_2}\Psi(\cdot - y_n^{(2)}), e^{i\theta_3}\Psi(\cdot - y_n^{(3)}))$$
$$- \frac{1}{2}\{V_1(z_0) + V_2(z_0) + V_3(z_0)\} \Big) + o(\beta_n^{-4/(4-N(p-1))})$$
$$\ge 3S_\infty - \beta_n^{-4/(4-N(p-1))}(S^1 - \frac{1}{2} \min_{x \in \mathbb{R}^N} \{V_1(x) + V_2(x) + V_3(x)\})$$
$$+ o(\beta_n^{-4/(4-N(p-1))}).$$

By the same argument as in Theorem 1.3 (iii), we have

$$\tilde{I}^{\beta} = 3S_\infty - \beta^{-4/(4-N(p-1))}(S^1 - \frac{1}{2} \min_{x \in \mathbb{R}^N} \{V_1(x) + V_2(x) + V_3(x)\})$$
$$+ o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta \to \infty,$$
$$\theta_1 + \theta_2 = \theta_3 + 2k\pi, \quad k \in \mathbb{Z},$$
$$y_n^{(1)} - y_n^{(2)} \to 0, \quad y_n^{(1)} - y_n^{(3)} \to 0,$$
$$\|v_{j,n}(\cdot + y_n) - e^{i\theta_j}\Psi\|_{H^1} \to 0,$$
$$y_n/\beta_n^{2/(4-N(p-1))} \to z_0.$$

3.4. Proof of Theorem 1.3 (v)

For the case (v) $\kappa < \kappa_2$, note that

$$(N-4)/(4-N(p-1)) + \kappa < -4/(4-N(p-1))$$

and

$$\begin{aligned}\tilde{E}^\beta(\vec{v}) = \sum_{j=1}^{3} J_\infty(v_j) + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j\left(\frac{x}{\beta^{2/(4-N(p-1))}}\right)|v_j|^2\,dx \\ - \beta^{(N-4)/(4-N(p-1))+\kappa} J^1(\vec{v}).\end{aligned}$$

First we prove the upper bound for $\tilde{I}^\beta$. Let $x_{j,0} \in \mathbb{R}^N$ such that $\min_{x\in\mathbb{R}^N} V_j(x) = V_j(x_{j,0})$ for all $j = 1, 2, 3$.
Set $v_j(x) = \Psi(x - \beta^{2/(4-N(p-1))}x_{j,0})$, $\vec{v} = (v_1, v_2, v_3)$. Then we have

$$\begin{aligned}\tilde{I}^\beta &\le \tilde{E}^\beta(\vec{v}) \\ &= 3S_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(\beta^{-2/(4-N(p-1))}x + x_{j,0})|\Psi|^2\,dx \\ &+ o(\beta^{-4/(4-N(p-1))}) \\ &= 3S_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\} \\ &+ o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta\to\infty.\end{aligned}$$

Next, we prove the lower bound for $\tilde{I}^\beta$. Recall that the rescaled function $\vec{v}_n$ defined by (1.2) satisfies

$$\begin{aligned}&\|v_{1,n}\|_2^2 = \|v_{2,n}\|_2^2 = \|v_{3,n}\|_2^2 = 1, \\ &\tilde{E}^{\beta_n}(\vec{v}_n) = \tilde{I}^{\beta_n} + o(\beta_n^{-4/(4-N(p-1))}),\end{aligned}$$

where $\beta_n \to \infty$. Since $\{v_{j,n}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, by the same argument as in the proof of Theorem 1.3 (iii) and (iv), $\{v_{j,n}\}_{n=1}^\infty$ is a minimizing sequence for S_∞. Thus up to a subsequence, there exist $\{y_n^{(j)}\}_{n=1}^\infty \subset \mathbb{R}^N$ and $\theta_j \in \mathbb{R}$ such that

$$\|v_{j,n}(\cdot + y_n^{(j)}) - e^{i\theta_j}\Psi\|_{H^1} \to 0.$$

By the same argument as in the proof of Theorem 1.3 (ii), since $\{y_n^{(j)}/\beta_n^{2/(4-N(p-1))}\}_{n=1}^\infty$ is bounded, up to a subsequence, there exists $z_{j,0} \in \mathbb{R}^N$ such that

$$y_n^{(j)}/\beta_n^{2/(4-N(p-1))} \to z_{j,0}.$$

Moreover we have

$$\int_{\mathbb{R}^N} V_j(\beta_n^{-2/(4-N(p-1))}x)|v_{j,n}|^2\,dx \to V_j(z_{j,0}).$$

From the upper bound for $\tilde{I}^\beta$, it follows that

$$\begin{aligned}
&3S_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\}\\
&\quad + o(\beta_n^{-4/(4-N(p-1))})\\
&\geq \tilde{I}^{\beta_n}\\
&\geq 3S_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\{V_1(z_{1,0}) + V_2(z_{2,0}) + V_3(z_{3,0})\} + o(\beta_n^{-4/(4-N(p-1))})\\
&\geq 3S_\infty + \beta_n^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\}\\
&\quad + o(\beta_n^{-4/(4-N(p-1))}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\tilde{I}^\beta = 3S_\infty + \beta^{-4/(4-N(p-1))}\frac{1}{2}\{\min_{x\in\mathbb{R}^N} V_1(x) + \min_{x\in\mathbb{R}^N} V_2(x) + \min_{x\in\mathbb{R}^N} V_3(x)\}\\
+o(\beta^{-4/(4-N(p-1))}), \quad \text{as } \beta \to \infty
\end{aligned}$$

and $\min_{x\in\mathbb{R}^N} V_j(x) = V_j(z_{j,0})$.

3.5. Proof of Theorem 1.3 (i)

Let $\vec{u}$ be a function such that

$$\|u_1\|_2^2 = \|u_2\|_2^2 = \|u_3\|_2^2 = 1.$$

We consider the rescaled function $\vec{w}$ as (1.1) such that

$$\vec{w}(x) := \beta^{-\kappa N/(4-N)}\vec{u}(\beta^{-2\kappa/(4-N)}x).$$

Then it follows that

$$\|w_1\|_2^2 = \|w_2\|_2^2 = \|w_3\|_2^2 = 1$$

and

$$\begin{aligned}
E^\beta(\vec{u}) &= \beta^{4\kappa/(4-N)}\tilde{F}^\beta(\vec{w}),\\
I^\beta &= \beta^{4\kappa/(4-N)}\tilde{K}^\beta
\end{aligned}$$

where

$$\begin{aligned}\tilde{F}^\beta(\vec{w}) &:= E^0(\vec{w}) - \beta^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\vec{w}) \\ &\quad + \beta^{-4\kappa/(4-N)}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(\beta^{-2\kappa/(4-N)}x)|w_j|^2\,dx, \\ \tilde{K}^\beta &:= \inf\{\tilde{F}^\beta(\vec{w}) \mid \vec{w}\in H^1(\mathbb{R}^N;\mathbb{C}^3), \quad \|w_j\|_2^2 = 1\ (j=1,2,3)\}.\end{aligned}$$

For the case (i) $\kappa > \kappa_1$, note that

$$-4\kappa/(4-N) < \kappa(N(p-1)-4)/(4-N) + 1 < 0.$$

We first prove the upper bound for $\tilde{K}^\beta$. Let $\vec{W}_n$ be a maximizing sequence for Σ_1, that is, $\vec{W}_n$ satisfies

$$\begin{aligned}&\vec{W}_n \text{ is a minimizer for } \Sigma_0, \\ &E^1(\vec{W}_n) \to \Sigma_1, \quad \text{as } n\to\infty.\end{aligned}$$

Then we have

$$\begin{aligned}\tilde{K}^\beta \le \tilde{F}^\beta(\vec{W}_n) &= E^0(\vec{W}_n) - \beta^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\vec{W}_n) \\ &\quad + \beta^{-4\kappa/(4-N)}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(\beta^{-2\kappa/(4-N)}x)|W_{j,n}|^2\,dx \\ &\le \Sigma_0 - \beta^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\vec{W}_n).\end{aligned}$$

Then letting $n\to\infty$, we have

$$\tilde{K}^\beta \le \Sigma_0 - \beta^{\kappa(N(p-1)-4)/(4-N)+1}\Sigma_1.$$

Next we prove the lower bound for $\tilde{K}^\beta$. Note that the rescaled function $\vec{w}_n$ defined by (1.1) satisfies

$$\begin{aligned}&\|w_{1,n}\|_2^2 = \|w_{2,n}\|_2^2 = \|w_{3,n}\|_2^2 = 1, \\ &\tilde{F}^{\beta_n}(\vec{w}_n) = \tilde{K}^{\beta_n} + o(\beta_n^{-4\kappa/(4-N)}),\end{aligned}$$

where $\beta_n\to\infty$ as $n\to\infty$. Since $\{w_{j,n}\}_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, by the same argument as in Theorem 1.3 (iii), $\{\vec{w}_n\}_{n=1}^\infty$ is a minimizing sequence for Σ_0. From the compactness of minimizing sequence for Σ_0 (see Kurata-Osada [10]), up to a subsequence, there exist $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$ and a minimizer $\vec{w}$ for Σ_0 such that

$$\|\vec{w}_n(\cdot + y_n) - \vec{w}\|_{H^1} \to 0, \quad \text{as } n\to\infty.$$

From the upper bound for $\tilde{K}^{\beta}$, we have

$$\begin{aligned}
&\Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}\Sigma_1 \\
&\geq \tilde{K}^{\beta_n} = \tilde{F}^{\beta_n}(\vec{w}_n) + o(\beta_n^{-4\kappa/(4-N)}) \\
&\geq \Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\vec{w}_n) \\
&\quad + \beta_n^{-4\kappa/(4-N)}\frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^N} V_j(\beta_n^{-2\kappa/(4-N)}x)|w_{j,n}|^2\,dx \\
&= \Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}E^1(\vec{w}) + o(\beta_n^{\kappa(N(p-1)-4)/(4-N)+1}) \\
&\geq \Sigma_0 - \beta_n^{\kappa(N(p-1)-4)/(4-N)+1}\Sigma_1 + o(\beta_n^{\kappa(N(p-1)-4)/(4-N)+1}).
\end{aligned}$$

Thus we have

$$\tilde{K}^{\beta} = \Sigma_0 - \beta^{\kappa(N(p-1)-4)/(4-N)+1}\Sigma_1 + o(\beta^{\kappa(N(p-1)-4)/(4-N)+1}), \quad \text{as } \beta \to \infty$$

and $\vec{w}$ is a maximizer for Σ_1.

§A. Appendix: Asymptotic expansion of I_α as $\alpha \to \infty$

We remark the another asymptotic expansion of the energy $I_\alpha^\beta(\gamma, \mu, s)$ as $\alpha \to \infty$ with $\beta = \alpha^\tau$ for a given $\tau \in \mathbb{R}$. For $\tau > 0$, the result of asymptotic expansion of $I_\alpha^\beta(\gamma, \mu, s)$ as $\alpha \to \infty$ with $\beta = \alpha^\tau$ is included in Theorem 1.3. So we consider the case $\tau \leq 0$. For a given $\tau \leq 0$, as $\beta = \alpha^\tau$ define

$$\begin{aligned}
E_\alpha(\vec{u}) &:= E_\alpha^{\alpha^\tau}(\vec{u}), \\
I_\alpha(\gamma, \mu, s) &:= I_\alpha^{\alpha^\tau}(\gamma, \mu, s).
\end{aligned}$$

Let $\{\alpha_n\}_{n=1}^\infty$ be a positive number sequence such that $\alpha_n \to \infty$ as $n \to \infty$. We say that $\{\vec{u}_n\}_{n=1}^\infty$ is a minimizing sequence for $I_{\alpha_n}(\gamma, \mu, s)$ if

$$\begin{aligned}
&\|u_{1,n}\|_2^2 = \gamma, \quad \|u_{2,n}\|_2^2 = \mu, \quad \|u_{3,n}\|_2^2 = s, \\
&E_{\alpha_n}(\vec{u}_n) = I_{\alpha_n}(\gamma, \mu, s) + o(1), \quad \text{as } n \to \infty.
\end{aligned}$$

We use the rescaled function $\vec{w}_n$ defined by (1.1) to analyse the asymptotic expansion for $I_\alpha(\gamma, \mu, s)$ as $\alpha \to \infty$. The asymptotic expansion up to the first term for $I_\alpha(\gamma, \mu, s)$ for the case $\tau = 0$ is treated in Kurata-Osada [10].

Proposition A.1. *(I)* $-N(p-1)/(4-N) < \tau \leq 0$

Assume that $N \leq 2$. *Then it holds that*

$$\begin{aligned}
I_\alpha(\gamma, \mu, s) = \alpha^{4/(4-N)}\Sigma_0(\gamma, \mu, s) - \alpha^{N(p-1)/(4-N)+\tau}\Sigma_1(\gamma, \mu, s) \\
+ o(\alpha^{N(p-1)/(4-N)+\tau}), \quad \textit{as } \alpha \to \infty.
\end{aligned}$$

Moreover let $\vec{u}_n$ *be a minimizing sequence for* $I_{\alpha_n}(\gamma, \mu, s)$ *where* $\alpha_n \to \infty$. *For the rescaled function* $\vec{w}_n$ *defined by* (1.1), *up to a subsequence, there exist a maximizer* $\vec{w}$ *for* $\Sigma_1(\gamma, \mu, s)$ *and* $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ *such that*

$$\|\vec{w}_n(\cdot + y_n) - \vec{w}\|_{H^1} \to 0.$$

(II) $\tau = -N(p-1)/(4-N)$
Assume that $N \leq 2$, *(V2) and* $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. *Then it holds that*

$$\begin{aligned} &I_\alpha(\gamma, \mu, s) \\ &= \alpha^{4/(4-N)}\Sigma_0(\gamma, \mu, s) - \Sigma_1(\gamma, \mu, s) \\ &\quad + \frac{1}{2}\min_{x \in \mathbb{R}^N}\{V_1(x)\gamma + V_2(x)\mu + V_3(x)s\} + o(1), \quad \text{as } \alpha \to \infty. \end{aligned}$$

Moreover let $\vec{u}_n$ *be a minimizing sequence for* $I_{\alpha_n}(\gamma, \mu, s)$ *where* $\alpha_n \to \infty$. *For the rescaled function* $\vec{w}_n$ *defined by* (1.1), *up to a subsequence, there exist a maximizer* $\vec{w}$ *for* $\Sigma_1(\gamma, \mu, s)$, $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ *and* $z_0 \in \mathbb{R}^N$ *such that*

$$\begin{aligned} &\|w_{j,n}(\cdot + y_n) - w_j\|_{H^1} \to 0, \\ &y_n/\alpha_n^{2/(4-N)} \to z_0, \\ &\min_{x \in \mathbb{R}^N}\{V_1(x)\gamma + V_2(x)\mu + V_3(x)s\} = V_1(z_0)\gamma + V_2(z_0)\mu + V_3(z_0)s. \end{aligned}$$

(III) $\tau < -N(p-1)/(4-N)$
Assume that (V2) and $(V_1, V_2, V_3) \not\equiv (0, 0, 0)$. *Then it holds that*

$$\begin{aligned} I_\alpha(\gamma, \mu, s) = \alpha^{4/(4-N)}\Sigma_0(\gamma, \mu, s) + \frac{1}{2}\min_{x \in \mathbb{R}^N}\{V_1(x)\gamma + V_2(x)\mu + V_3(x)s\} \\ + o(1), \quad \text{as } \alpha \to \infty. \end{aligned}$$

Moreover let $\vec{u}_n$ *be a minimizing sequence for* $I_{\alpha_n}(\gamma, \mu, s)$ *where* $\alpha_n \to \infty$. *For the rescaled function* $\vec{w}_n$ *defined by* (1.1), *up to a subsequence, there exist a minimizer* $\vec{w}$ *for* $\Sigma_0(\gamma, \mu, s)$, $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ *and* $z_0 \in \mathbb{R}^N$ *such that*

$$\begin{aligned} &\|w_{j,n}(\cdot + y_n) - w_j\|_{H^1} \to 0, \\ &y_n/\alpha_n^{2/(4-N)} \to z_0, \\ &\min_{x \in \mathbb{R}^N}\{V_1(x)\gamma + V_2(x)\mu + V_3(x)s\} = V_1(z_0)\gamma + V_2(z_0)\mu + V_3(z_0)s. \end{aligned}$$

Since we can prove Proposition A.1 in a similar way as in the proof of Theorem 1.3, we omit the details. We note that we assume an additional condition for the bottom of the potentials in the case (iv) in Theorem 1.3. But we do not need the additional condition in Proposition A.1 since the compactness of the minimizing sequence of a minimization problem for appearing in the first term of the asymptotic expansion of I_α aligns the translations for each component.

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Yuki Osada
Department of Mathematical Sciences, Tokyo Metropolitan University
1-1 Minami Osawa, Hachioji, Tokyo 192-0397, Japan
E-mail: `osada-yuki@ed.tmu.ac.jp`