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Norm inequalities for the generalised commutator in Banach algebras

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Abstract. In this paper, by utilising the Riesz functional calculus in a Banach algebra $\mathcal{B}$, we provide some norm inequalities for the generalized commutator

$$f(y)z - zf(x)$$

where x, y, $z \in \mathcal{B}$ and f is an analytic function for which the elements $f(y)$ and $f(x)$ exist. Some examples for the resolvent and exponential functions are also given.

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§1. Introduction

Let $\mathcal{B}$ be an algebra over $\mathbf{C}$. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any a, $b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}(\mathcal{B})$. If a, $b \in \mathrm{Inv}(\mathcal{B})$, then $ab \in \mathrm{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}(\mathcal{B})$;

(ii) $\{b \in \mathcal{B}: \ \|1-b\| < 1\} \subset \mathrm{Inv}(\mathcal{B})$;

(iii) $\mathrm{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;

(iv) The map $\mathrm{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \mathrm{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbf{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbf{C} : \ \lambda - a \in \mathrm{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbf{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \mathrm{Inv}(\mathcal{B})$,

$$R_a(\lambda) := (\lambda - a)^{-1}.$$

For each λ, $\mu \in \rho(a)$ we have the identity

$$R_a(\mu) - R_a(\lambda) = (\lambda - \mu) R_a(\lambda) R_a(\mu).$$

We also have that

$$\sigma(a) \subset \{\lambda \in \mathbf{C} : \ |\lambda| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Let $\mathcal{B}$ be a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbf{C}$;

(ii) For any *bounded linear functional* $\lambda : \mathcal{B} \to \mathbf{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;

(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbf{C}$;

(iv) We have

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0,R)$ given by the *power series*

$$f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j \ (|\lambda| < R).$$

If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly

$f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted by exp is defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a) \exp(b).$$

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [12] and [13].

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbf{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbf{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\gamma} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.1}$$

where $\gamma \subset G$ is taken to be a closed rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ.

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.2}$$

holds.

Let $\operatorname{Hol}(a)$ be the set of all the functions that are analytic in a neighborhood of $\sigma(a)$. Note that $\operatorname{Hol}(a)$ is an algebra where if f, $g \in \operatorname{Hol}(a)$ and f and g have domains $D(f)$ and $D(g)$, then fg and $f+g$ have domain $D(f) \cap D(g)$. $\operatorname{Hol}(a)$ is not, however a Banach algebra.

The following result is known as the *Riesz functional calculus Theorem* [4, p. 201-204]:

Theorem 1.1. *Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$.*

(a) The map $f \mapsto f(a)$ of $\operatorname{Hol}(a) \to \mathcal{B}$ is an algebra homomorphism.

(b) If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence $r > \nu(a)$, then $f \in \operatorname{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.

(c) If $f(z) \equiv 1$, then $f(a) = 1$.

(d) *If* $f(z) = z$ *for all* z, *then* $f(a) = a$.

(e) *If* f, $f_1, ..., f_n...$ *are analytic on* G, $\sigma(a) \subset G$ *and* $f_n(z) \to f(z)$ *uniformly on compact subsets of* G, *then* $\|f_n(a) - f(a)\| \to 0$ *as* $n \to \infty$.

(f) *The Riesz functional calculus is unique and if* a, b *are commuting elements in* $\mathcal{B}$ *and* $f \in \mathrm{Hol}(a)$, *then* $f(a)b = bf(a)$.

For some recent norm inequalities for functions on Banach algebras, see [2]-[3] and [5]-[11].

In this paper, by utilising the Riesz functional calculus in Banach algebra $\mathcal{B}$, we provide some norm inequalities for the *generalized commutator*

$$f(y)z - zf(x)$$

where x, y, $z \in \mathcal{B}$ and f is an analytic function for which the elements $f(y)$ and $f(x)$ exist. Some examples for the resolvent and exponential functions are also given.

§2. Main results

We have:

Lemma 2.1. *For any elements* a, b, c *in the Banach algebra* $\mathcal{B}$ *and for any* $n \geq 1$ *we have*

$$a^n c - cb^n = \sum_{i=0}^{n-1} a^{n-i-1}(ac - cb)b^i. \tag{2.1}$$

In particular, for $b = a$ *we have*

$$a^n c - ca^n = \sum_{i=0}^{n-1} a^{n-i-1}(ac - ca)a^i. \tag{2.2}$$

Proof. We prove it by induction over n. For $n = 1$ we obtain in both sides of (2.1) the same quantity $ac - cb$. Assume that for $k \geq 2$ we have that

$$a^k c - cb^k = \sum_{i=0}^{k-1} a^{k-i-1}(ac - cb)b^i$$

and let us prove that

$$a^{k+1} c - cb^{k+1} = \sum_{i=0}^{k} a^{k-i}(ac - cb)b^i.$$

We have

$$\begin{aligned}\sum_{i=0}^{k} a^{k-i}(ac-cb)b^{i} &= \sum_{i=0}^{k-1} a^{k-i}(ac-cb)b^{i} + a^{k-k}(ac-cb)b^{k}\\ &= a\sum_{i=0}^{k-1} a^{k-i-1}(ac-cb)b^{i} + (ac-cb)b^{k}\\ &= a(a^{k}c-cb^{k}) + (ac-cb)b^{k} \text{ (by induction hypothesis)}\\ &= a^{k+1}c - acb^{k} + acb^{k} - cb^{k+1} = a^{k+1}c - cb^{k+1}\end{aligned}$$

and the proof is completed. □

Remark. For $c=1$, we have from (2.1) that

$$a^{n}-b^{n} = \sum_{i=0}^{n-1} a^{n-i-1}(a-b)b^{i} \tag{2.3}$$

for all a, b in the Banach algebra $\mathcal{B}$, see [2] for details.

Corollary 2.2. *With the assumptions of Lemma 2.1 we have the inequality*

$$\|a^{n}c-cb^{n}\| \le \|ac-cb\| \times \begin{cases} \frac{\|a\|^{n}-\|b\|^{n}}{\|a\|-\|b\|}, \text{ if } \|b\| \ne \|a\|, \\ \\ n\|a\|^{n-1}, \text{ if } \|b\| = \|a\|. \end{cases} \tag{2.4}$$

In particular, for $b=a$, we have

$$\|a^{n}c-ca^{n}\| \le n\|a\|^{n-1}\|ac-ca\|. \tag{2.5}$$

Proof. By taking the norm and using its properties we have

$$\begin{aligned}\|a^{n}c-cb^{n}\| &\le \sum_{i=0}^{n-1}\left\|a^{n-i-1}(ac-cb)b^{i}\right\| \le \sum_{i=0}^{n-1}\left\|a^{n-i-1}\right\|\|ac-cb\|\left\|b^{i}\right\|\\ &\le \|ac-cb\|\sum_{i=0}^{n-1}\|a\|^{n-i-1}\|b\|^{i}\\ &= \|ac-cb\| \times \begin{cases} \frac{\|a\|^{n}-\|b\|^{n}}{\|a\|-\|b\|}, \text{ if } \|b\| \ne \|a\| \\ \\ n\|a\|^{n-1}, \text{ if } \|b\| = \|a\|, \end{cases}\end{aligned}$$

which proves (2.4). □

Now, by the help of power series $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ we can naturally construct another power series which will have, as coefficients, the absolute values of the coefficients of the original series, namely, $f_A(z):=\sum_{n=0}^{\infty}|\alpha_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n\geq 0$, then $f_A=f$.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbf{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbf{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1); \end{aligned} \tag{2.6}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_A(\lambda) &= \sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1); \\ g_A(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbf{C}; \\ h_A(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbf{C}; \\ l_A(\lambda) &= \sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1). \end{aligned} \tag{2.7}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned} \exp(\lambda) &= \sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n \qquad \lambda\in\mathbf{C}, \\ \frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda\in D(0,1); \end{aligned} \tag{2.8}$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is the *gamma function.*

We have:

Theorem 2.3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbf{C}$, $R > 0$. For any x, y, $z \in \mathcal{B}$ with $\|x\|$, $\|y\| < R$ we have*

$$\|f(y)z - zf(x)\| \le \|yz - zx\| \times \begin{cases} \frac{f_A(\|y\|)-f_A(\|x\|)}{\|y\|-\|x\|}, & \text{if } \|y\| \ne \|x\|, \\ f_A'(\|x\|), & \text{if } \|y\| = \|x\|. \end{cases} \tag{2.9}$$

In particular

$$\|f(x)z - zf(x)\| \le \|xz - zx\| f_A'(\|x\|) \tag{2.10}$$

and, see also [5] for details,

$$\|f(y) - f(x)\| \le \|y - x\| \times \begin{cases} \frac{f_A(\|y\|)-f_A(\|x\|)}{\|y\|-\|x\|}, & \text{if } \|y\| \ne \|x\|, \\ f_A'(\|x\|), & \text{if } \|y\| = \|x\|. \end{cases} \tag{2.11}$$

Proof. We have, for any $m \ge 1$, by making use of the inequality (2.4), that

$$\begin{aligned} &\left\| (\sum_{n=0}^{m} \alpha_n y^n) z - z(\sum_{n=0}^{m} \alpha_n x^n) \right\| \tag{2.12} \\ &= \left\| \sum_{n=1}^{m} \alpha_n (y^n z - z x^n) \right\| \le \sum_{n=1}^{m} |\alpha_n| \|y^n z - z x^n\| \\ &\le \|yz - zx\| \times \begin{cases} \sum_{n=1}^{m} |\alpha_n| \frac{\|y\|^n - \|x\|^n}{\|y\|-\|x\|}, \text{ if } \|y\| \ne \|x\|, \\ \sum_{n=1}^{m} n |\alpha_n| \|x\|^{n-1}, \text{ if } \|y\| = \|x\| \end{cases} \\ &= \|yz - zx\| \times \begin{cases} \frac{1}{\|y\|-\|x\|} (\sum_{n=0}^{m} |\alpha_n| \|y\|^n - \sum_{n=0}^{m} |\alpha_n| \|x\|^n), \\ \text{if } \|y\| \ne \|x\|, \\ \sum_{n=1}^{m} n |\alpha_n| \|x\|^{n-1}, \text{ if } \|y\| = \|x\|. \end{cases} \end{aligned}$$

Moreover, since $\|x\|, \|y\| < R$, then the series $\sum_{n=0}^{\infty} \alpha_n y^n$ and $\sum_{n=0}^{\infty} \alpha_n x^n$ are convergent in $\mathcal{B}$ and

$$\sum_{n=0}^{\infty} \alpha_n y^n = f(y), \sum_{n=0}^{\infty} \alpha_n x^n = f(x).$$

Also, the scalar series

$$\sum_{n=0}^{\infty} |\alpha_n| \|y\|^n, \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n \text{ and } \sum_{n=1}^{\infty} n |\alpha_n| \|x\|^{n-1}$$

are convergent

$$\sum_{n=0}^{\infty} |\alpha_n| \|y\|^n = f_A(\|y\|), \quad \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n = f_A(\|x\|)$$

and

$$\sum_{n=1}^{\infty} n |\alpha_n| \|x\|^{n-1} = f_A'(\|x\|).$$

Therefore, by taking $m \to \infty$ in the inequality (2.12) we get the desired result (2.9). $\square$

Corollary 2.4. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$, $g(z) = \sum_{n=0}^{\infty} \beta_n \lambda^n$ be two functions defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbf{C}$, $R > 0$. For any x, $y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have*

$$\|f(x)g(y) - g(y)f(x)\| \leq \|xy - yx\| f_A'(\|x\|)g_A'(\|y\|). \tag{2.13}$$

Proof. From (2.10) we get

$$\|f(x)z - zf(x)\| \leq \|xg(y) - g(y)x\| f_A'(\|x\|)$$

and

$$\|xg(y) - g(y)x\| \leq \|xy - yx\| g_A'(\|y\|),$$

which provide the desired result (2.13). $\square$

Remark. If we write the inequality (2.9) for the function $f(\lambda) = (1 \pm \lambda)^{-1}$ defined on the open disk $D(0, 1)$ we get for all x, y, $z \in \mathcal{B}$ with $\|x\|, \|y\| < 1$ that

$$\left\|(1 \pm y)^{-1}z - z(1 \pm x)^{-1}\right\| \leq \|yz - zx\| (1 - \|y\|)^{-1}(1 - \|x\|)^{-1}. \tag{2.14}$$

In particular,

$$\left\|(1 \pm x)^{-1}z - z(1 \pm x)^{-1}\right\| \leq \|xz - zx\| (1 - \|x\|)^{-2} \tag{2.15}$$

and, see [5] for details,

$$\left\|(1 \pm y)^{-1} - (1 \pm x)^{-1}\right\| \leq \|y - x\| (1 - \|y\|)^{-1}(1 - \|x\|)^{-1}. \tag{2.16}$$

We also have:

Theorem 2.5. *Let $f : D \subset \mathbf{C} \to \mathbf{C}$ be an analytic function on the domain D and x, y, $z \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D$ and γ a closed rectifiable path in D and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then we have*

$$\|f(y)z - zf(x)\| \leq \frac{1}{2\pi} \|yz - zx\| \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|y\|)(|\xi| - \|x\|)} |d\xi| . \tag{2.17}$$

In particular,

$$\|f(x)z - zf(x)\| \leq \frac{1}{2\pi} \|xz - zx\| \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|x\|)^2} |d\xi| \tag{2.18}$$

and

$$\|f(y) - f(x)\| \leq \frac{1}{2\pi} \|y - x\| \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|y\|)(|\xi| - \|x\|)} |d\xi| , \tag{2.19}$$

see also [7] for details.

Proof. Let $\lambda \in \mathbf{C}$, $\lambda \neq 0$ and a, $b \in \mathcal{B}$ such that $\lambda \in \rho(a) \cap \rho(b)$, then we have the following inequality for the resolvent function that is of interest in itself:

$$\|R_b(\lambda)z - zR_a(\lambda)\| \leq \|bz - za\| (|\lambda| - \|b\|)^{-1}(|\lambda| - \|a\|)^{-1}. \tag{2.20}$$

Indeed, by (2.14) we get for $\lambda \in \rho(a) \cap \rho(b)$, $\lambda \neq 0$ that

$$\begin{aligned}
&\left\|(\lambda - b)^{-1}z - z(\lambda - a)^{-1}\right\| \\
&= \left\|\lambda^{-1}(1 - \frac{b}{\lambda})^{-1}z - \lambda^{-1}z(1 - \frac{a}{\lambda})^{-1}\right\| \\
&= \frac{1}{|\lambda|}\left\|(1 - \frac{b}{\lambda})^{-1}z - z(1 - \frac{a}{\lambda})^{-1}\right\| \\
&\leq \frac{1}{|\lambda|}\left\|\frac{b}{\lambda}z - z\frac{a}{\lambda}\right\|(1 - \left\|\frac{b}{\lambda}\right\|)^{-1}(1 - \left\|\frac{a}{\lambda}\right\|)^{-1} \\
&= \frac{1}{|\lambda|^2}\|bz - za\| \, |\lambda|^2 \, (|\lambda| - \|b\|)^{-1}(|\lambda| - \|a\|)^{-1} \\
&= \|bz - za\| \, (|\lambda| - \|b\|)^{-1}(|\lambda| - \|a\|)^{-1}
\end{aligned}$$

and the inequality (2.20) is proved.

Let x, y, $z \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D$ and γ a closed rectifiable path in D and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Using the Riesz functional calculus we have

$$\begin{aligned}
f(y)z - zf(x) &= \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1}zd\xi - \int_\gamma f(\xi)z(\xi - x)^{-1}d\xi \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - y)^{-1}z - z(\xi - x)^{-1}\right]d\xi \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[R_y(\xi)z - zR_x(\xi)\right]d\xi.
\end{aligned}$$

By taking the norm in this equality and using the properties of integral, we get

$$\|f(y)z - zf(x)\| \le \frac{1}{2\pi}\int_\gamma |f(\xi)| \, \|R_y(\xi)z - zR_x(\xi)\| \, |d\xi| \, . \tag{2.21}$$

Using inequality (2.20) we have

$$\begin{aligned} &\frac{1}{2\pi}\int_\gamma |f(\xi)| \, \|R_y(\xi)z - zR_x(\xi)\| \, |d\xi| \\ &\le \|yz - zx\| \, \frac{1}{2\pi}\int_\gamma |f(\xi)| \, (|\xi| - \|y\|)^{-1}(|\xi| - \|x\|)^{-1} \, |d\xi| \, . \end{aligned} \tag{2.22}$$

By making use of (2.21) and (2.22) we get the desired result (2.17). □

Corollary 2.6. *With the assumptions of Theorem 2.5 and if*

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |f(\xi)| < \infty,$$

then

$$\|f(y)z - zf(x)\| \le \frac{1}{2\pi} \|yz - zx\| \, \|f\|_{\gamma,\infty} \int_\gamma \frac{|d\xi|}{(|\xi| - \|y\|)(|\xi| - \|x\|)}. \tag{2.23}$$

In particular,

$$\|f(x)z - zf(x)\| \le \frac{1}{2\pi} \|xz - zx\| \, \|f\|_{\gamma,\infty} \int_\gamma \frac{|d\xi|}{(|\xi| - \|x\|)^2} \tag{2.24}$$

and

$$\|f(y) - f(x)\| \le \frac{1}{2\pi} \|y - x\| \, \|f\|_{\gamma,\infty} \int_\gamma \frac{|d\xi|}{(|\xi| - \|y\|)(|\xi| - \|x\|)}, \tag{2.25}$$

see [7] for details.

Remark. If we assume that $f : D \subset \mathbf{C} \to \mathbf{C}$ is an analytic function on the domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$, then by taking γ parametrized by $\xi(t) = Re^{2\pi it}$ where $t \in [0,1]$, then $d\xi(t) = 2\pi i Re^{2\pi it}dt$, $|d\xi(t)| = 2\pi R dt$, $|\xi| = R$ and by (2.17) we get

$$\|f(y)z - zf(x)\| \le \frac{R\,\|yz - zx\|}{(R - \|y\|)(R - \|x\|)} \int_0^1 \left| f(Re^{2\pi it}) \right| dt. \tag{2.26}$$

In particular, we have

$$\|f(x)z - zf(x)\| \le \frac{R\,\|xz - zx\|}{(R - \|x\|)^2} \int_0^1 \left| f(Re^{2\pi it}) \right| dt \tag{2.27}$$

and

$$\|f(y)-f(x)\| \le \frac{R\,\|y-x\|}{(R-\|y\|)(R-\|x\|)} \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt, \tag{2.28}$$

see also [7] for details.

Moreover, if $\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f(Re^{2\pi it})\right| < \infty$, then we have the simpler inequality

$$\|f(y)z - zf(x)\| \le \frac{R\,\|yz-zx\|\,\|f\|_{R,\infty}}{(R-\|y\|)(R-\|x\|)} \tag{2.29}$$

and, in particular,

$$\|f(x)z - zf(x)\| \le \frac{R\,\|xz-zx\|\,\|f\|_{R,\infty}}{(R-\|x\|)^2} \tag{2.30}$$

and

$$\|f(y)-f(x)\| \le \frac{R\,\|y-x\|\,\|f\|_{R,\infty}}{(R-\|y\|)(R-\|x\|)}. \tag{2.31}$$

Corollary 2.7. *Let f, $g : D \subset \mathbf{C} \to \mathbf{C}$ be analytic functions on the domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D$ and γ a closed rectifiable path in D and such that $\sigma(x)$, $\sigma(y) \subset \mathrm{ins}(\gamma)$. Then we have*

$$\begin{aligned} &\|f(x)g(y)-g(y)f(x)\| \\ &\le \frac{1}{4\pi^2}\,\|xy-yx\| \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|x\|)^2}\,|d\xi| \int_\gamma \frac{|g(\xi)|}{(|\xi|-\|y\|)^2}\,|d\xi| \end{aligned} \tag{2.32}$$

and if

$$\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |f(\xi)| < \infty, \quad \|g\|_{\gamma,\infty} := \sup_{\xi\in\gamma} |g(\xi)| < \infty$$

then

$$\begin{aligned} &\|f(x)g(y)-g(y)f(x)\| \\ &\le \frac{1}{4\pi^2}\,\|xy-yx\|\,\|f\|_{\gamma,\infty}\,\|g\|_{\gamma,\infty} \int_\gamma \frac{|d\xi|}{(|\xi|-\|x\|)^2} \int_\gamma \frac{|d\xi|}{(|\xi|-\|y\|)^2}. \end{aligned} \tag{2.33}$$

Applying the inequality (2.18), the result follows and we omit the details.

Remark. If we assume that f, $g : D \subset \mathbf{C} \to \mathbf{C}$ are analytic functions on the domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$, then

$$\begin{aligned} &\|f(x)g(y)-g(y)f(x)\| \\ &\le \frac{R^2\,\|xy-yx\|}{(R-\|x\|)^2(R-\|y\|)^2} \int_0^1 \left|f(Re^{2\pi it})\right| dt \int_0^1 \left|g(Re^{2\pi it})\right| dt. \end{aligned} \tag{2.34}$$

Moreover, if

$$\|f\|_{R,\infty} := \sup_{t\in[0,1]} \left|f(Re^{2\pi it})\right| < \infty, \quad \|g\|_{R,\infty} := \sup_{t\in[0,1]} \left|g(Re^{2\pi it})\right| < \infty,$$

then

$$\|f(x)g(y) - g(y)f(x)\| \le \frac{R^2\,\|xy - yx\|\,\|f\|_{R,\infty}\,\|g\|_{R,\infty}}{(R-\|x\|)^2(R-\|y\|)^2}. \tag{2.35}$$

§3. Some Examples for Exponential Function

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. By using Theorem 2.3 for the exponential function, we get the inequalities

$$\|(\exp y)z - z(\exp x)\| \le \|yz - zx\| \times \begin{cases} \frac{\exp(\|y\|)-\exp(\|x\|)}{\|y\|-\|x\|}, \text{ if } \|y\| \ne \|x\|, \\ \\ \exp(\|x\|), \text{ if } \|y\| = \|x\|. \end{cases} \tag{3.1}$$

In particular

$$\|(\exp x)z - z(\exp x)\| \le \|xz - zx\| \exp(\|x\|) \tag{3.2}$$

and, see also [5] for details,

$$\|\exp y - \exp x\| \le \|y - x\| \times \begin{cases} \frac{\exp(\|y\|)-\exp(\|x\|)}{\|y\|-\|x\|}, \text{ if } \|y\| \ne \|x\|, \\ \\ \exp(\|x\|), \text{ if } \|y\| = \|x\|. \end{cases} \tag{3.3}$$

Now, assume that x, $y \in \mathcal{B}$ and $\|x\|$, $\|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp(Re^{2\pi it})\right| = \left|\exp\left[R(\cos(2\pi t) + i\sin(2\pi t))\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and then by (2.26) we get

$$\|(\exp y)z - z(\exp x)\| \le \frac{R\,\|yz - zx\|}{(R-\|y\|)(R-\|x\|)} \int_0^1 \exp\left[R\cos(2\pi t)\right] dt. \tag{3.4}$$

In particular, we have

$$\|(\exp x)z - z(\exp x)\| \le \frac{R\,\|xz - zx\|}{(R-\|x\|)^2} \int_0^1 \exp\left[R\cos(2\pi t)\right] dt \tag{3.5}$$

and

$$\|\exp y - \exp x\| \leq \frac{R\,\|y-x\|}{(R-\|y\|)\,(R-\|x\|)} \int_0^1 \exp\left[R\cos(2\pi t)\right] dt, \tag{3.6}$$

see also [7] for details.

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as, see [1, p. 376] for details,

$$I_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

If we change the variable $\theta = 2\pi t$, then $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned}\int_0^1 \exp\left[R\cos(2\pi t)\right] dt &= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\ &= \frac{1}{2}(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_\pi^{2\pi} \exp\left[R\cos\theta\right] d\theta) \\ &= \frac{1}{2}(I_0(R)+I_0(-R)) = I_0(R).\end{aligned}$$

From (3.4) we then get

$$\|(\exp y)z - z(\exp x)\| \leq \frac{R\,\|yz-zx\|\,I_0(R)}{(R-\|y\|)(R-\|x\|)}, \tag{3.7}$$

for x, y, $z \in \mathcal{B}$ with $\|x\|$, $\|y\| < R$.

In particular, we have

$$\|(\exp x)z - z(\exp x)\| \le \frac{R\,\|xz - zx\|}{(R - \|x\|)^2} I_0(R). \tag{3.8}$$

and

$$\|\exp y - \exp x\| \le \frac{R\,\|y - x\|}{(R - \|y\|)\,(R - \|x\|)} I_0(R), \tag{3.9}$$

for x, y, $z \in \mathcal{B}$ with $\|x\|$, $\|y\| < R$. For more details, see [7].

Since, in general $\exp u$ does not commute with $\exp v$, then from (3.2) we get

$$\|\exp u \exp v - \exp v \exp u\| \le \|uv - vu\| \exp(\|u\| + \|v\|) \tag{3.10}$$

for all $u, v \in \mathcal{B}$.

From (3.8) we also have

$$\|\exp u \exp v - \exp v \exp u\| \le \frac{R^2\,\|uv - vu\|}{(R - \|u\|)^2 (R - \|v\|)^2} I_0^2(R) \tag{3.11}$$

for u, $v \in \mathcal{B}$ with $\|u\|$, $\|v\| < R$.

By utilising the examples from (2.6), (2.7) and (2.8), the interested reader may obtain other similar inequalities for functions defined on the Banach algebra $\mathcal{B}$. We omit the details.

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