

Pseudo-projective curvature tensor on warped product manifolds and its applications in space-times

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Abstract. In this paper we study the pseudo-projective curvature tensor on warped product manifolds. We obtain some significant results of the pseudo-projective curvature tensor on warped product manifolds in terms of its base and fiber manifolds. Moreover, we derive some interesting results which describe the geometry of base and fiber manifolds for a pseudo-projectively flat warped product manifold. Lastly, we study the pseudo-projective curvature tensor on generalized Robertson-Walker space-times and standard static space-times.

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§1. Introduction

Bishop and O'Neill [6] had given the idea of warped product in Riemannian manifolds. They introduced the notion of warped product for making a large class of complete manifolds having negative curvature. The main idea of this warped product actually appeared on account of a surface of revolution. Later, Nölker [13] also developed the concept of multiply warped product as a generalization of warped product. The warped product plays a very significant role in differential geometry, especially in mathematical physics and general relativity. Schwarzschild solution, Robertson-walker model, static model and Kruscal model etc. are the examples of warped products. There are so many exact solutions of Einstein field equations and modified field equations. These solutions can be written in terms of warped products.

The pseudo-projective curvature tensor had been defined by Prasad [15]. The pseudo-projective curvature tensor includes the projective curvature tensor. Many authors [8, 10, 11, 12] studied the pseudo-projective curvature

tensor in different ways. The pseudo-projective curvature tensor has been studied in mathematics as well as physics as a research topic. Shenawy and Ünal [19] studied on the W_2 -curvature tensor on warped product manifolds. In view of the above interesting works, we wish to study the pseudo-projective curvature tensor on warped product manifolds and space-times.

The aim of this paper is to study the geometry of pseudo-projective curvature tensor on warped product manifolds. Besides this we discuss its applications to Robertson-Walker space-times and standard static space-times. Hence this paper connects the pseudo-projective curvature tensor to warped product manifold, Robertson-Walker space-times and standard static space-times.

This paper has been arranged in the following way. In section 2, we state the concept of pseudo-projective curvature tensor and warped product manifolds. In section 3, we discuss some interesting results of pseudo-projective curvature tensor on warped product manifolds in terms of its base and fiber manifolds. In section 4, we study pseudo-projective curvature tensor on generalized Robertson-Walker space-times. The last section is devoted to the study of standard static space-times admitting the pseudo-projective curvature tensor.

§2. Preliminaries

In this part, we just recall some basic ideas on warped product and pseudo-projective curvature tensor.

Let (B, g_B) and (F, g_F) be two Riemannian manifolds with $\dim(B) > 0$ and $\dim(F) > 0$. Let $f : B \rightarrow (0, \infty)$ be a positive smooth function on B . Suppose the natural projections of the product manifold $B \times F$ are $\pi : B \times F \rightarrow B$ and $\eta : B \times F \rightarrow F$. The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the Riemannian structure such that

$$\langle X, X \rangle = \langle \pi^*(X), \pi^*(X) \rangle + f^2(\pi(X)) \langle \eta^*(X), \eta^*(X) \rangle,$$

for each tangent vector $X \in \mathfrak{X}(M)$. Therefore, we obtain the metric relation $g_M = g_B \oplus f^2 g_F$. B and F are respectively the base and fiber of this warped product manifold. The function f is known as the warping function of this warped product.

Proposition 2.1 ([14]). *Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor R . If $X, Y, Z \in \mathfrak{X}(B)$ and $U, V, W \in \mathfrak{X}(F)$, then*

- (1) $R(X, Y)Z = R^B(X, Y)Z$,
- (2) $R(V, X)Y = \frac{H^f(X, Y)}{f}V$,
- (3) $R(X, Y)V = R(V, W)X = 0$,

$$(4) \quad R(X, V)W = \frac{g(V, W)}{f} D_X^1(\nabla f),$$

$$(5) \quad R(V, W)U = R^F(V, W)U + \frac{\|\nabla f\|^2}{f^2} [g(W, U)V - g(V, U)W].$$

Proposition 2.2 ([14]). *On the warped product $M = B \times_f F$ with $\dim(F) = d > 1$, let $X, Y \in \mathfrak{X}(B)$ and $V, W \in \mathfrak{X}(F)$. Then the Ricci tensor S_M of M are given by*

$$(1) \quad S_M(X, Y) = S_B(X, Y) - \frac{d}{f} H^f(X, Y),$$

$$(2) \quad S_M(X, V) = 0,$$

$$(3) \quad S_M(V, W) = S_F(V, W) - g(V, W)f^\#, \quad f^\# = \frac{\Delta f}{f} + \frac{d-1}{f^2} \|\nabla f\|^2,$$

where $\Delta f = \text{tr}(H^f)$ and H^f are respectively the Laplacian and the Hessian of f on B .

Proposition 2.3 ([7]). *Let $M = B \times_f F$ be a semi-Riemannian warped product furnished with the metric $g_M = g_B \oplus f^2 g_F$. Then the scalar curvature τ of M admits the following relation*

$$\tau = \tau_B + \frac{\tau_F}{f^2} - 2s \frac{\Delta_B(f)}{f} - s(s-1) \frac{\|\text{grad}_B f\|_B^2}{f^2},$$

where $r = \dim(B)$ and $s = \dim(F)$.

The pseudo-projective curvature tensor \bar{P}^* on a pseudo-Riemannian manifold is defined by

$$(2.1) \quad \begin{aligned} \bar{P}^*(X, Y, Z, W) &= a_1 \bar{R}(X, Y, Z, W) + a_2 [S(Y, Z)g(X, W) \\ &\quad - S(X, Z)g(Y, W)] - \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \\ &\quad \times [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \end{aligned}$$

where a_1 and a_2 ($\neq 0$) are two constants, S is the Ricci tensor of $(0, 2)$ -type, the scalar curvature of the manifold is τ , $\bar{P}^*(X, Y, Z, W) = g(P^*(X, Y)Z, W)$, $\bar{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$, where R is the Riemannian curvature tensor.

If $a_1 = 1$ and $a_2 = -\frac{1}{n-1}$, then Eq. (2.1) reduces to the projective curvature tensor. Moreover, if $P^* = 0$ for $n > 3$, then a pseudo-Riemannian manifold is called pseudo-projectively flat.

It clearly follows from Eq. (2.1) that

$$(2.2) \quad \begin{aligned} P^*(X, Y)Z &= a_1 R(X, Y)Z + a_2 [S(Y, Z)X - S(X, Z)Y] \\ &\quad - \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Remark. Suppose M is a semi-Riemannian manifold. Then

$$P^*(X, Y)Z + P^*(Y, Z)X + P^*(Z, X)Y = 0,$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Proposition 2.4. *Suppose M is a semi-Riemannian manifold. Then the pseudo-projective curvature tensor vanishes if and only if the tensor P^* vanishes.*

A Riemannian metric g is said to be of Hessian type metric if $H^{f_1} = f_2g$ for any two smooth functions f_1 and f_2 , where H^{f_1} denotes the Hessian of the function f_1 .

§3. Pseudo-projective curvature tensor on warped product manifolds

Here we study the pseudo-projective curvature tensor on warped product manifolds. We consider the warped product $M = M_1 \times_f M_2$ where $\dim(M) = n$, $\dim(M_1) = n_1$ and $\dim(M_2) = n_2$ such that $n = n_1 + n_2$, $n_i \neq 1$ for $i = 1, 2$. We denote R, R^i as the curvature tensor and S, S^i as the Ricci tensor on M, M_i respectively. On the other hand, $\nabla f, \Delta f$ and H^f are respectively the gradient, Laplacian and Hessian of f on M_1 . D, D^i indicate the Levi-Civita connection with respect to the metric g, g_i for $i = 1, 2$ respectively. Throughout our entire study we use the relation $f^\# = \frac{\Delta f}{f} + \frac{n_2-1}{f^2} \|\nabla f\|^2$. Last of all, we denote the pseudo-projective curvature tensor and the tensor P^* on M and M_i by \bar{P}^*, P^* and \bar{P}_i^*, P_i^* respectively.

Now we obtain the following theorems for the pseudo-projective curvature tensor on warped product manifolds. These theorems describe the warped geometry in terms of its base and fiber manifolds.

Theorem 3.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2g_2$. If $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ for $i = 1, 2$, then*

$$\begin{aligned} P^*(X_1, Y_1)Z_1 &= P_1^*(X_1, Y_1)Z_1 + \tau \left[\frac{n_2(n + n_1 - 1)}{nn_1(n - 1)(n_1 - 1)}a_1 + \frac{n_2}{nn_1}a_2 \right] \\ &\quad \times [g_1(Y_1, Z_1)X_1 - g_1(X_1, Z_1)Y_1] \\ &\quad + \frac{a_2n_2}{f} \left[H^f(X_1, Z_1)Y_1 - H^f(Y_1, Z_1)X_1 \right], \\ P^*(X_1, Y_1)Z_2 &= P^*(X_2, Y_2)Z_1 = 0, \\ P^*(X_1, Y_2)Z_1 &= \left(\frac{a_2n_2 - a_1}{f} \right) H^f(X_1, Z_1)Y_2 - a_2S^1(X_1, Z_1)Y_2 \\ &\quad + \frac{\tau}{n} \left(\frac{a_1}{n - 1} + a_2 \right) g_1(X_1, Z_1)Y_2, \end{aligned}$$

$$\begin{aligned}
P^*(X_1, Y_2)Z_2 &= a_1 f g_2(Y_2, Z_2) D_{X_1}^1 \nabla f + a_2 S^2(Y_2, Z_2) X_1 \\
&\quad - f^2 \left[a_2 f^\# + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] g_2(Y_2, Z_2) X_1, \\
P^*(X_2, Y_2)Z_2 &= P_2^*(X_2, Y_2)Z_2 + \left[\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{n n_2 (n-1)(n_2-1)} \right) a_1 \tau \right. \\
&\quad \left. + \left(\frac{n - n_2 f^2}{n n_2} \right) \tau a_2 - a_2 f^2 f^\# + a_1 \|\nabla f\|^2 \right] \\
&\quad \times [g_2(Y_2, Z_2) X_2 - g_2(X_2, Z_2) Y_2].
\end{aligned}$$

Proof. Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Let $\dim(M) = n$, $\dim(M_i) = n_i$ for $i = 1, 2$ and $n = n_1 + n_2$. If $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ for $i = 1, 2$. Then, we obtain

$$\begin{aligned}
P^*(X_1, Y_1)Z_1 &= a_1 R(X_1, Y_1)Z_1 + a_2 [S(Y_1, Z_1)X_1 - S(X_1, Z_1)Y_1] \\
&\quad - \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1] \\
&= a_1 R^1(X_1, Y_1)Z_1 + a_2 \left[\{S^1(Y_1, Z_1) - \frac{n_2}{f} H^f(Y_1, Z_1)\} X_1 \right. \\
&\quad \left. - \{S^1(X_1, Z_1) - \frac{n_2}{f} H^f(X_1, Z_1)\} Y_1 \right] \\
&\quad - \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g_1(Y_1, Z_1)X_1 - g_1(X_1, Z_1)Y_1] \\
&= a_1 R^1(X_1, Y_1)Z_1 + a_2 [S^1(Y_1, Z_1)X_1 - S^1(X_1, Z_1)Y_1] \\
&\quad - \frac{\tau}{n_1} \left(\frac{a_1}{n_1-1} + a_2 \right) [g_1(Y_1, Z_1)X_1 - g_1(X_1, Z_1)Y_1] \\
&\quad + \left[\frac{\tau}{n_1} \left(\frac{a_1}{n_1-1} + a_2 \right) - \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] \\
&\quad \times [g_1(Y_1, Z_1)X_1 - g_1(X_1, Z_1)Y_1] \\
&\quad + \frac{a_2 n_2}{f} [H^f(X_1, Z_1)Y_1 - H^f(Y_1, Z_1)X_1] \\
&= P_1^*(X_1, Y_1)Z_1 + \tau \left[\frac{n_2(n + n_1 - 1)}{n n_1 (n-1)(n_1-1)} a_1 + \frac{n_2}{n n_1} a_2 \right] \\
&\quad \times [g_1(Y_1, Z_1)X_1 - g_1(X_1, Z_1)Y_1] \\
&\quad + \frac{a_2 n_2}{f} [H^f(X_1, Z_1)Y_1 - H^f(Y_1, Z_1)X_1], \\
P^*(X_1, Y_1)Z_2 &= a_1 R(X_1, Y_1)Z_2 + a_2 [S(Y_1, Z_2)X_1 - S(X_1, Z_2)Y_1]
\end{aligned}$$

$$\begin{aligned}
& -\frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g(Y_1, Z_2)X_1 - g(X_1, Z_2)Y_1] \\
& = 0, \\
P^*(X_1, Y_2)Z_1 &= a_1 R(X_1, Y_2)Z_1 + a_2 [S(Y_2, Z_1)X_1 - S(X_1, Z_1)Y_2] \\
& -\frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g(Y_2, Z_1)X_1 - g(X_1, Z_1)Y_2] \\
& = -\left(\frac{a_1}{f} \right) H^f(X_1, Z_1)Y_2 - a_2 \left[S^1(X_1, Z_1)Y_2 \right. \\
& \left. - \frac{n_2}{f} H^f(X_1, Z_1)Y_2 \right] + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) g_1(X_1, Z_1)Y_2 \\
& = \left(\frac{a_2 n_2 - a_1}{f} \right) H^f(X_1, Z_1)Y_2 - a_2 S^1(X_1, Z_1)Y_2 \\
& + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) g_1(X_1, Z_1)Y_2, \\
P^*(X_1, Y_2)Z_2 &= a_1 R(X_1, Y_2)Z_2 + a_2 [S(Y_2, Z_2)X_1 - S(X_1, Z_2)Y_2] \\
& -\frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g(Y_2, Z_2)X_1 - g(X_1, Z_2)Y_2] \\
& = \left(\frac{a_1}{f} \right) g(Y_2, Z_2) D_{X_1}^1 \nabla f + a_2 [S^2(Y_2, Z_2)X_1 \\
& - f^\# g(Y_2, Z_2)X_1] - \frac{\tau f^2}{n} \left(\frac{a_1}{n-1} + a_2 \right) g_2(Y_2, Z_2)X_1 \\
& = a_1 f g_2(Y_2, Z_2) D_{X_1}^1 \nabla f + a_2 S^2(Y_2, Z_2)X_1 \\
& - f^2 \left[a_2 f^\# + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] g_2(Y_2, Z_2)X_1, \\
P^*(X_2, Y_2)Z_1 &= a_1 R(X_2, Y_2)Z_1 + a_2 [S(Y_2, Z_1)X_2 - S(X_2, Z_1)Y_2] \\
& -\frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g(Y_2, Z_1)X_2 - g(X_2, Z_1)Y_2] \\
& = 0, \\
P^*(X_2, Y_2)Z_2 &= a_1 R(X_2, Y_2)Z_2 + a_2 [S(Y_2, Z_2)X_2 - S(X_2, Z_2)Y_2] \\
& -\frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g(Y_2, Z_2)X_2 - g(X_2, Z_2)Y_2] \\
& = a_1 \left[R^2(X_2, Y_2)Z_2 + \frac{\|\nabla f\|^2}{f^2} \{g(Y_2, Z_2)X_2 - g(X_2, Z_2)Y_2\} \right] \\
& + a_2 [\{S^2(Y_2, Z_2)X_2 - f^\# g(Y_2, Z_2)X_2\} \\
& - \{S^2(X_2, Z_2)Y_2 - f^\# g(X_2, Z_2)Y_2\}] \\
& - \frac{\tau f^2}{n} \left(\frac{a_1}{n-1} + a_2 \right) [g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2] \\
& = a_1 R^2(X_2, Y_2)Z_2 + a_2 [S^2(Y_2, Z_2)X_2 - S^2(X_2, Z_2)Y_2] \\
& - \frac{\tau}{n_2} \left(\frac{a_1}{n_2-1} + a_2 \right) [g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2]
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{\tau}{n_2} \left(\frac{a_1}{n_2 - 1} + a_2 \right) - \frac{\tau f^2}{n} \left(\frac{a_1}{n - 1} + a_2 \right) \right. \\
& \left. - a_2 f^2 f^\# + a_1 \|\nabla f\|^2 \right] [g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2] \\
& = P_2^*(X_2, Y_2)Z_2 + \left[\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)} \right) a_1 \tau \right. \\
& \left. + \left(\frac{n - n_2 f^2}{nn_2} \right) \tau a_2 - a_2 f^2 f^\# + a_1 \|\nabla f\|^2 \right] \\
& \times [g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2].
\end{aligned}$$

This completes the proof. \square

Theorem 3.2. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then*

$$\begin{aligned}
\bar{P}_1^*(X_1, Y_1, Z_1, W_1) & = \tau \left[\frac{n_2(n + n_1 - 1)}{nn_1(n-1)(n_1-1)} a_1 + \frac{n_2}{nn_1} a_2 \right] \\
& \times [g_1(X_1, Z_1)g_1(Y_1, W_1) - g_1(Y_1, Z_1)g_1(X_1, W_1)] \\
& + \frac{a_2 n_2}{f} [H^f(Y_1, Z_1)g_1(X_1, W_1) - H^f(X_1, Z_1)g_1(Y_1, W_1)],
\end{aligned}$$

for $X_1, Y_1, Z_1, W_1 \in \mathfrak{X}(M_1)$.

Proof. Let us assume that $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold. Therefore, in view of Theorem 3.1, we obtain

$$\begin{aligned}
P_1^*(X_1, Y_1)Z_1 & = \tau \left[\frac{n_2(n + n_1 - 1)}{nn_1(n-1)(n_1-1)} a_1 + \frac{n_2}{nn_1} a_2 \right] \\
& \times [g_1(X_1, Z_1)Y_1 - g_1(Y_1, Z_1)X_1] \\
& + \frac{a_2 n_2}{f} [H^f(Y_1, Z_1)X_1 - H^f(X_1, Z_1)Y_1].
\end{aligned}$$

Therefore, we derive

$$\begin{aligned}
\bar{P}_1^*(X_1, Y_1, Z_1, W_1) & = g_1(P_1^*(X_1, Y_1)Z_1, W_1) \\
& = \tau \left[\frac{n_2(n + n_1 - 1)}{nn_1(n-1)(n_1-1)} a_1 + \frac{n_2}{nn_1} a_2 \right] \\
& \times [g_1(X_1, Z_1)g_1(Y_1, W_1) - g_1(Y_1, Z_1)g_1(X_1, W_1)] \\
& + \frac{a_2 n_2}{f} [H^f(Y_1, Z_1)g_1(X_1, W_1) \\
& - H^f(X_1, Z_1)g_1(Y_1, W_1)].
\end{aligned}$$

This completes the proof. \square

Theorem 3.3. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then the base manifold M_1 is pseudo-projectively flat if and only if*

$$\begin{aligned} & \tau \left[\frac{n_2(n + n_1 - 1)}{nn_1(n - 1)(n_1 - 1)} a_1 + \frac{n_2}{nn_1} a_2 \right] \\ & \times [g_1(X_1, Z_1)g_1(Y_1, W_1) - g_1(Y_1, Z_1)g_1(X_1, W_1)] \\ & + \frac{a_2 n_2}{f} \left[H^f(Y_1, Z_1)g_1(X_1, W_1) - H^f(X_1, Z_1)g_1(Y_1, W_1) \right] = 0, \end{aligned}$$

for $X_1, Y_1, Z_1, W_1 \in \mathfrak{X}(M_1)$.

Proof. Let the base manifold M_1 be pseudo-projectively flat. Then

$$\bar{P}_1^*(X_1, Y_1, Z_1, W_1) = 0.$$

Clearly, the proof follows from Theorem 3.2. \square

Theorem 3.4. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then the scalar curvature τ_1 of M_1 is given by*

$$\tau_1 = \frac{1}{a_2} \left[\left(\frac{a_2 n_2 - a_1}{f} \right) \Delta f + \frac{\tau n_1}{n} \left(\frac{a_1}{n - 1} + a_2 \right) \right].$$

Proof. Let us assume that $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold. Then Theorem 3.1 implies that

$$S^1(X_1, Z_1) = \frac{1}{a_2} \left[\left(\frac{a_2 n_2 - a_1}{f} \right) H^f(X_1, Z_1) + \frac{\tau}{n} \left(\frac{a_1}{n - 1} + a_2 \right) g_1(X_1, Z_1) \right].$$

Taking contraction over X_1 and Z_1 , we gain

$$\tau_1 = \frac{1}{a_2} \left[\left(\frac{a_2 n_2 - a_1}{f} \right) \Delta f + \frac{\tau n_1}{n} \left(\frac{a_1}{n - 1} + a_2 \right) \right].$$

This completes the proof. \square

Remark. Proposition 2.3 [7] and Theorem 3.4 jointly imply that the scalar curvature τ_2 of (M_2, g_2) is a constant since the left hand side of the equation in Theorem 3.4 depends only on the base manifold (M_1, g_1) .

Theorem 3.5. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. Then the pseudo-projective*

curvature tensor of M_2 is given by

$$\begin{aligned} \bar{P}_2^*(X_2, Y_2, Z_2, W_2) = & \left[\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)} \right) a_1 \tau + \left(\frac{n - n_2 f^2}{nn_2} \right) \tau a_2 \right. \\ & - a_2 f^2 f^\# + a_1 \|\nabla f\|^2 \left. \right] [g_2(X_2, Z_2)g_2(Y_2, W_2) \\ & - g_2(Y_2, Z_2)g_2(X_2, W_2)], \end{aligned}$$

for $X_2, Y_2, Z_2, W_2 \in \mathfrak{X}(M_2)$.

Proof. Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold. From Theorem 3.1, it follows that

$$\begin{aligned} 0 = P_2^*(X_2, Y_2)Z_2 + & \left[\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)} \right) a_1 \tau + \left(\frac{n - n_2 f^2}{nn_2} \right) \tau a_2 \right. \\ & \left. - a_2 f^2 f^\# + a_1 \|\nabla f\|^2 \right] [g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2]. \end{aligned}$$

Therefore,

$$\begin{aligned} \bar{P}_2^*(X_2, Y_2, Z_2, W_2) = & g_2(P_2^*(X_2, Y_2)Z_2, W_2) \\ = & \left[\left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)} \right) a_1 \tau + \left(\frac{n - n_2 f^2}{nn_2} \right) \tau a_2 \right. \\ & \left. - a_2 f^2 f^\# + a_1 \|\nabla f\|^2 \right] [g_2(X_2, Z_2)g_2(Y_2, W_2) \\ & - g_2(Y_2, Z_2)g_2(X_2, W_2)]. \end{aligned}$$

This completes the proof. \square

Theorem 3.6. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. If the fiber manifold M_2 is Ricci flat, then the base manifold M_1 is of Hessian type.*

Proof. Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold. Then from Theorem 3.1, we derive

$$\begin{aligned} 0 = & a_1 f g_2(Y_2, Z_2) D_{X_1}^1 \nabla f + a_2 S^2(Y_2, Z_2) X_1 \\ & - f^2 \left[a_2 f^\# + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] g_2(Y_2, Z_2) X_1. \end{aligned}$$

Suppose that M_2 is Ricci flat. Then $S^2(X_2, Y_2) = 0$ for any $X_2, Y_2 \in \mathfrak{X}(M_2)$. Hence, we obtain from the above relation

$$D_{X_1}^1 \nabla f = \frac{f}{a_1} \left[a_2 f^\# + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] X_1.$$

This implies that

$$H^f = \frac{f}{a_1} \left[a_2 f^\# + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] g_1.$$

Hence, M_1 is of Hessian type. This completes the proof. \square

Theorem 3.7. *Let $M = M_1 \times_f M_2$ be a pseudo-projectively flat warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. If the fiber manifold M_2 is Ricci flat, then the pointwise constant sectional curvature τ_2 of M_2 is given by*

$$\begin{aligned} \tau_2 = \frac{1}{a_1} \left[- \left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)} \right) a_1 \tau - \left(\frac{n - n_2 f^2}{nn_2} \right) \tau a_2 + a_2 f^2 f^\# \right. \\ \left. - a_1 \|\nabla f\|^2 + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right]. \end{aligned}$$

Proof. Let M_2 be Ricci flat. Therefore, from Eq. (2.1), we have

$$\begin{aligned} \bar{R}^2(X_2, Y_2, Z_2, W_2) = \frac{1}{a_1} \left[\bar{P}_2^*(X_2, Y_2, Z_2, W_2) + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right. \\ \left. \times \{g_2(Y_2, Z_2)g_2(X_2, W_2) - g_2(X_2, Z_2)g_2(Y_2, W_2)\} \right]. \end{aligned}$$

In view of Theorem 3.1, we derive from the above relation that

$$\begin{aligned} \bar{R}^2(X_2, Y_2, Z_2, W_2) = \frac{1}{a_1} \left[- \left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)} \right) a_1 \tau - \left(\frac{n - n_2 f^2}{nn_2} \right) \tau a_2 \right. \\ \left. + a_2 f^2 f^\# - a_1 \|\nabla f\|^2 + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] \\ \times \{g_2(Y_2, Z_2)g_2(X_2, W_2) - g_2(X_2, Z_2)g_2(Y_2, W_2)\}. \end{aligned}$$

This implies that M_2 has a pointwise constant sectional curvature and this curvature is given by

$$\begin{aligned} \tau_2 = \frac{1}{a_1} \left[- \left(\frac{n^2 - n - n_2^2 f^2 + n_2 f^2}{nn_2(n-1)(n_2-1)} \right) a_1 \tau - \left(\frac{n - n_2 f^2}{nn_2} \right) \tau a_2 + a_2 f^2 f^\# \right. \\ \left. - a_1 \|\nabla f\|^2 + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right]. \end{aligned}$$

This completes the proof. \square

Theorem 3.8. *Let $M = M_1 \times_f M_2$ be a warped product manifold furnished with the metric $g = g_1 \oplus f^2 g_2$. If $H^f = 0$, $\Delta f = 0$ and M is pseudo-projectively flat, then M_2 is an Einstein manifold.*

Proof. Let M be pseudo-projectively flat. Therefore, M_1 is flat in view of Theorem 3.2. Furthermore, from Theorem 3.1, we obtain

$$(3.1) \quad 0 = a_1 f g_2(Y_2, Z_2) D_{X_1}^1 \nabla f + a_2 S^2(Y_2, Z_2) X_1 \\ - f^2 \left[a_2 f^\# + \frac{\tau}{n} \left(\frac{a_1}{n-1} + a_2 \right) \right] g_2(Y_2, Z_2) X_1.$$

Since $H^f(X_1, Y_1) = 0$ and $\Delta f = 0$. Therefore, we derive from Eq. (3.1) that

$$S^2(Y_2, Z_2) = \left[(n_2 - 1) \|\nabla f\|^2 + \frac{\tau f^2}{a_2 n} \left(\frac{a_1}{n-1} + a_2 \right) \right] g_2(Y_2, Z_2).$$

This implies that M_2 is an Einstein manifold. This completes the proof. \square

§4. Pseudo-projective curvature tensor on generalized Robertson-Walker space-times

Let (M, g) be a Riemannian manifold of dimension n . The function $f : I \rightarrow (0, \infty)$ is a smooth function where I is a connected and open subinterval of \mathbb{R} . Then the warped product manifold $\check{M} = I \times_f M$ of dimension $(n + 1)$ equipped with the metric $\check{g} = -dt^2 \oplus f^2 g$ is known as generalized Robertson-Walker space-time. Here dt^2 is the Euclidean metric on I . This structure is the generalization of Robertson-Walker space-times [9, 16, 17, 18]. We use ∂_t instead of $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ for simplicity in the following results.

With the help of Proposition 2.1, Proposition 2.2 and Eq. (2.2), we obtain the following theorem after some elementary calculations.

Theorem 4.1. *Let $\check{M} = I \times_f M$ be a generalized Robertson-Walker space-time furnished with the metric $\check{g} = -dt^2 \oplus f^2 g$. Then the curvature tensor \check{P}^* on \check{M} is given by*

$$\check{P}^*(\partial_t, \partial_t) \partial_t = \check{P}^*(\partial_t, \partial_t) X = \check{P}^*(X, Y) \partial_t = 0, \\ \check{P}^*(\partial_t, X) \partial_t = \left[\left(\frac{na_2 - a_1}{f} \right) \ddot{f} - \frac{\tau}{n+1} \left(\frac{a_1}{n} + a_2 \right) \right] X, \\ \check{P}^*(X, \partial_t) Y = \left[\left\{ - (a_1 + a_2) f \ddot{f} - (n-1) a_2 \dot{f}^2 \right. \right. \\ \left. \left. + \frac{\tau f^2}{n+1} \left(\frac{a_1}{n} + a_2 \right) \right\} g(X, Y) - a_2 S(X, Y) \right] \partial_t, \\ \check{P}^*(X, Y) Z = a_1 R(X, Y) Z + a_2 [S(Y, Z) X - S(X, Z) Y] \\ + \left[- a_1 \dot{f}^2 + a_2 f \ddot{f} + a_2 (n-1) \dot{f}^2 - \frac{\tau f^2}{n+1} \left(\frac{a_1}{n} + a_2 \right) \right] \\ \times [g(Y, Z) X - g(X, Z) Y],$$

for $X, Y, Z \in \mathfrak{X}(M)$ and $\partial_t \in \mathfrak{X}(I)$.

Theorem 4.2. *Let $\check{M} = I \times_f M$ be a generalized Robertson-Walker space-time furnished with the metric $\check{g} = -dt^2 \oplus f^2g$. If \check{M} is pseudo-projectively flat, then the warping function f is given by*

$$f = \begin{cases} c_1 e^{\mu t} + c_2 e^{-\mu t}, & \text{if } \mu^2 \text{ is positive} \\ c_1 + c_2 t, & \text{if } \mu^2 = 0 \\ c_1 \cos \mu t + c_2 \sin \mu t, & \text{if } \mu^2 \text{ is negative} \end{cases}$$

where $\mu^2 = \frac{\tau(a_1 + na_2)}{n(n+1)(na_2 - a_1)}$ and c_1, c_2 are two arbitrary constants.

Proof. Let \check{M} be pseudo-projectively flat. Then from the second relation of Theorem 4.1, we have

$$\ddot{f} - \mu^2 f = 0.$$

Hence, by solving the above differential equation the warping function f is obtained and it is given by

$$f = \begin{cases} c_1 e^{\mu t} + c_2 e^{-\mu t}, & \text{if } \mu^2 \text{ is positive} \\ c_1 + c_2 t, & \text{if } \mu^2 = 0 \\ c_1 \cos \mu t + c_2 \sin \mu t, & \text{if } \mu^2 \text{ is negative} \end{cases}$$

where c_1, c_2 are two arbitrary constants. This completes the proof. \square

Theorem 4.3. *Let $\check{M} = I \times_f M$ be a generalized Robertson-Walker space-time furnished with the metric $\check{g} = -dt^2 \oplus f^2g$. If \check{M} is pseudo-projectively flat, then M is an Einstein manifold.*

Proof. Let \check{M} be pseudo-projectively flat. Then from the third relation of Theorem 4.1, we have

$$S(X, Y) = \frac{1}{a_2} \left[-(a_1 + a_2) f \ddot{f} - (n-1) a_2 \dot{f}^2 + \frac{\tau f^2}{n+1} \left(\frac{a_1}{n} + a_2 \right) \right] g(X, Y).$$

Hence, M is an Einstein manifold. This completes the proof. \square

§5. Pseudo-projective curvature tensor on standard static space-times

Let (M, g) be a Riemannian manifold of dimension n . The function $f : M \rightarrow (0, \infty)$ is a smooth function. Then the warped product manifold $\check{M} = I \times_f M$

of dimension $(n + 1)$ equipped with the metric $\check{g} = -f^2 dt^2 \oplus g$ is known as standard static space-time. Here I is the connected, open subinterval of \mathbb{R} and dt^2 is the Euclidean metric on I . This structure is the generalization of Einstein static universe [1, 2, 3, 4, 5]. We write ∂_t instead of $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ for expressing the following results in simpler way.

In view of Proposition 2.1, Proposition 2.2 and Eq. (2.2), we obtain the following theorem after some elementary calculations.

Theorem 5.1. *Let $\check{M} = I \times_f M$ be a standard static space-time furnished with the metric $\check{g} = -f^2 dt^2 \oplus g$. Then the curvature tensor \check{P}^* on \check{M} is given by*

$$\begin{aligned} \check{P}^*(\partial_t, \partial_t)\partial_t &= \check{P}^*(\partial_t, \partial_t)X = \check{P}^*(X, Y)\partial_t = 0, \\ \check{P}^*(\partial_t, X)\partial_t &= f \left[a_1 D_X^1 \nabla f - a_2 \Delta f X - \frac{\tau f}{n+1} \left(\frac{a_1}{n} + a_2 \right) X \right], \\ \check{P}^*(\partial_t, X)Y &= \left[\left(\frac{a_1 - a_2}{f} \right) H^f(X, Y) + a_2 S(X, Y) \right. \\ &\quad \left. - \frac{\tau}{n+1} \left(\frac{a_1}{n} + a_2 \right) g(X, Y) \right] \partial_t, \\ \check{P}^*(X, Y)Z &= a_1 R(X, Y)Z + a_2 [S(Y, Z)X - S(X, Z)Y] \\ &\quad - \frac{a_2}{f} [H^f(Y, Z)X - H^f(X, Z)Y] \\ &\quad - \frac{\tau}{n+1} \left(\frac{a_1}{n} + a_2 \right) [g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$ and $\partial_t \in \mathfrak{X}(I)$.

Theorem 5.2. *Let $\check{M} = I \times_f M$ be a standard static space-time furnished with the metric $\check{g} = -f^2 dt^2 \oplus g$. If \check{M} is pseudo-projectively flat, then $H^f = \frac{\Delta f}{n} g$.*

Proof. Let $\check{M} = I \times_f M$ be pseudo-projectively flat. Then from the second relation of Theorem 5.1, we have

$$\begin{aligned} D_X^1 \nabla f &= \frac{1}{a_1} \left[a_2 \Delta f + \frac{\tau f}{n+1} \left(\frac{a_1}{n} + a_2 \right) \right] X \\ (5.1) \quad \text{i.e., } H^f &= \frac{1}{a_1} \left[a_2 \Delta f + \frac{\tau f}{n+1} \left(\frac{a_1}{n} + a_2 \right) \right] g. \end{aligned}$$

Taking trace on both sides, we obtain

$$(5.2) \quad \Delta f = \frac{nf\tau}{(n+1)(a_1 - na_2)} \left(\frac{a_1}{n} + a_2 \right).$$

Using Eq. (5.2) in Eq. (5.1), we derive $H^f = \frac{\Delta f}{n} g$. This completes the proof. \square

Theorem 5.3. *Let $\check{M} = I \times_f M$ be a standard static space-time furnished with the metric $\check{g} = -f^2 dt^2 \oplus g$. If \check{M} is pseudo-projectively flat, then M is an Einstein manifold.*

Proof. Let $\check{M} = I \times_f M$ be pseudo-projectively flat. We derive from the third relation of Theorem 5.1 by using Theorem 5.2 and Eq. (5.2) that

$$S(X, Y) = \frac{(1-n)\Delta f}{nf}g(X, Y).$$

This implies that M is an Einstein manifold. This completes the proof. \square

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References

- [1] D. Allison, Energy conditions in standard static spacetimes, *General Relativity and Gravitation* **20** (1988), 115–122.
- [2] D. Allison, Geodesic completeness in static space-times, *Geometriae Dedicata* **26** (1988), 85–97.
- [3] D. E. Allison and B. Ünal, Geodesic structure of standard static space-times, *Journal of Geometry and Physics* **46** (2003), 193–200.
- [4] A. L. Besse, *Einstein manifolds* (Classics in Mathematics), Springer-Verlag, Berlin, (2008).
- [5] N. Bhunia, S. Pahan and A. Bhattacharyya, Application of hyper-generalized quasi-Einstein spacetimes in general relativity, *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences* **91** (2021), 297–307.
- [6] R. L. Bishop and B. O’Neill, Geometry of slant submanifolds, *Transactions of the American Mathematical Society* **145** (1969), 1–49.
- [7] F. Dobarro and B. Ünal, Curvature of multiply warped products, *Journal of Geometry and Physics* **55** (2005), 75–106.
- [8] Y. Dogru, Hypersurfaces satisfying some curvature conditions on pseudo projective curvature tensor in the semi-Euclidean space, *Mathematical Sciences and Applications E-Notes* **2** (2014), 99–105.
- [9] J. L. Flores and M. Sánchez, Geodesic connectedness and conjugate points in GRW space-times, *Journal of Geometry and Physics* **36**, (2000), 285–314.

- [10] J. P. Jaiswal and R. H. Ojha, On weakly pseudo-projectively symmetric manifolds, *Differential Geometry-Dynamical Systems* **12** (2010), 83–94.
- [11] H. G. Nagaraaja and G. Somashekhara, On pseudo projective curvature tensor in Sasakian manifolds, *Int. J. Contemp. Math. Sciences* **6** (2011), 1319–1328.
- [12] D. Narain, A. Prakash and B. Prasad, A pseudo projective curvature tensor on a Lorentzian para-Sasakian manifold, *Analele Stiintifice ale Universitatii Al I Cuza din Iasi - Matematica* **55** (2009), 275–284.
- [13] S. Nölker, Isometric immersions of warped products, *Differential Geometry and its Applications* **6** (1996), 1–30.
- [14] B. O’Neill, *Semi-Riemannian geometry with applications to relativity*, Pure and Applied Mathematics, Academic Press. Inc., New York, (1983), 336–341.
- [15] B. Prasad, A pseudo-projective curvature tensor on a Riemannian manifold, *Bulletin of Calcutta Mathematical Society* **94** (2002), 163–166.
- [16] M. Sánchez, On the geometry of generalized Robertson-Walker space-times: curvature and Killing fields, *Journal of Geometry and Physics* **31** (1999), 1–15.
- [17] M. Sánchez, On the geometry of generalized Robertson-Walker space-times: Geodesics, *General Relativity and Gravitation* **30** (1998), 915–932.
- [18] S. Shenawy and B. Ünal, 2-Killing vector fields on warped product manifolds, *International Journal of Mathematics* **26** (2015), 1550065(1)–1550065(17).
- [19] S. Shenawy and B. Ünal, The W2-curvature tensor on warped product manifolds and applications, *International Journal of Geometric Methods in Modern Physics* **13** (2016), 1650099(1)–1650099(14).

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