SUT Journal of Mathematics
Vol. 54, No. 2 (2018), 173–190

3+1-Moulton configuration

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(Received November 14, 2018; Revised November 30, 2018)

Dedicate to the memory of Professor Shuichi Takahasi

Abstract. We pose a new problem of collinear central configuration in Newtonian n-body problem. For a given three-body of collinear central configuration, we ask whether we can add another body in a way such that (a) the total four-body is also in a state of collinear central configuration and (b) the initial three-body keeps its motion without any change during the process. We find four solutions to the above problem having zero mass. We also discuss a similar but 'positive' problem by modifying the conditions such that (a') the four-body is in a state of collinear central configuration and the initial three bodies keep their position without any change, and (b') the masses of the two out of the three of initial bodies are invariant with the rest having a slight change and the mass of the added body is positive.
We also find explicit solutions to the second problem.

AMS 2010 *Mathematics Subject Classification.* 70F15, 70F10.

Key words and phrases. Collinear central configuration, four-body problem, celestial mechanics.

§1. Introduction

Euler found solutions of three-body problem on a line, collinear three-body problem [2] for the first time in history. In general, a solution of Newtonian n-body problem on a line, called a collinear n-body, forms *central configuration*, that is, the ratios of the distances of the bodies from the center of mass are constants [4]. F. R. Moulton [4] proved that for a fixed mass vector $\mathbf{m} = (m_1, \ldots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration $\mathbf{q} = (q_1, \ldots, q_n)$ with mass $\mathbf{m} = (m_1, \ldots, m_n)$ (up to translation and scaling), where q_i denotes the position of the i th-body on a line $i = 1, \ldots, n$. The configuration is also called a *Moulton configuration*, which will be sometimes abbreviated as M.c..

In this paper, we consider the following problem. We assume we are given a M.c. $\mathbf{q}_A = (q_{A_1}, q_{A_2}, q_{A_3})$ of three bodies A_1, A_2, A_3 such that $q_{A_1} < q_{A_2} < q_{A_3}$ with mass $\mathbf{m}_A = (m_{A_1}, m_{A_2}, m_{A_3})$ where each component of $\mathbf{m}_A$ is positive. We consider to add a body B of position q_B with mass m_B, to A_1, A_2, A_3 on the same line containing A_1, A_2 and A_3 so that

(a) the configuration of A_1, A_2, A_3 and B is M.c. with A_1, A_2, A_3 keeping the original positions, and

(b) the motion of A_1, A_2, A_3 are kept invariant during the process.

More precisely, let q_i denote one of the positions of A_1, A_2, A_3, B such that $q_1 < q_2 < q_3 < q_4$ and m_i denote its mass, respectively.

Definition 1 (3+1-Moulton configuration). *We call* $\mathbf{q} = (q_1, q_2, q_3, q_4)$ *with* $\mathbf{m} = (m_1, m_2, m_3, m_4)$ *a " 3+1-Moulton configuration" for the three bodies* A_1*,* A_2, A_3 *when it satisfies the following conditions.*

(i) A_1*,* A_2, A_3 *and* B *are in Moulton configuration and the configuration of* A_1*,* A_2, A_3 *is equal to the original one* $\mathbf{q}_A$ *with* $\mathbf{m}_A$*.*
(ii) *The center of mass of* A_1*,* A_2, A_3*,* B *is equal to that of* A_1*,* A_2, A_3*, and the motion of* A_1*,* A_2, A_3 *is the same as the original one.*

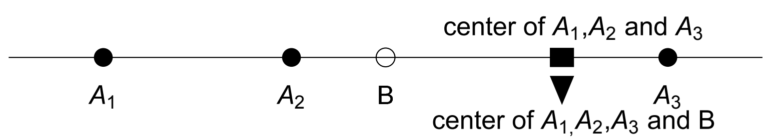

Figure 1: 3+1-Moulton configuration

Then we show in this paper

Theorem 1 (3+1-Moulton configuration). *For a given Moulton configuration* $\mathbf{q}_A = (q_{A_1}, q_{A_2}, q_{A_3})$ *with* $\mathbf{m}_A = (m_{A_1}, m_{A_2}, m_{A_3})$*,*

(i) *there exist four kinds of 3+1-Moulton configurations for* $\mathbf{q}_A$ *with* $\mathbf{m}_A$*,*

(ii) *the mass of the added body is zero.*

We remark here that the above result (ii) means that the added body has an " *infinitesimal zero* " mass. We also consider a configuration in which the added body has a positive mass, namely the configuration satisfies only a part of the condition (i) of Definition 1. That is A_1, A_2, A_3 and B are in a Moulton configuration and the positions of A_1, A_2, A_3 are the same as the original one, and further *the mass of the added body is positive* with a slight change of one of the masses of A_1, A_2, A_3. We make the following definition.

Definition 2 (Positive-3+1-Moulton configuration)**.** *We call* $\mathbf{q}$ *with* $\mathbf{m}$ *a "positive-3+1-Moulton configuration" for* $\mathbf{q}_A$ *with* $\mathbf{m}_A$ *when it satisfies the conditions:*

(i) *A_1, A_2, A_3 and B are in Moulton configuration and the positions of A_1, A_2, A_3 are equal to the original one.*
(ii) *The mass of the added body B is positive with a change of mass in one of A_1, A_2, A_3.*

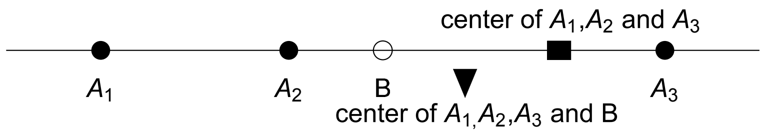

Figure 2: Positive-3+1-Moulton configuration

We also show

Theorem 2. *For a given Moulton configuration $\mathbf{q}_A = (q_{A_1}, q_{A_2}, q_{A_3})$ with $\mathbf{m}_A = (m_{A_1}, m_{A_2}, m_{A_3})$, there are intervals of q_B of position of added body such that every point of the interval yields a positive-3+1-Moulton configuration.*

In the previous paper [5], the first author considered 2+2-*Moulton configuration* for two bodies and obtained three solutions.

This paper is organized as follows. In Section 2, we define a Moulton manifold, which is regarded as a manifold of all Moulton configurations of n-body, and construct it for $n = 3$ and $n = 4$, and also give equations for its mass problem, (2.6) and (2.8) below. In Section 3 we prove Theorem 1 and Theorem 2 in Section 4, respectively.

§2. Manifold of Moulton configurations

2.1. Collinear central configuration

We consider the d-dimensional $(1 \leq d \leq 3)$ Newtonian n-body problem:

$$m_i\ddot{\mathbf{q}}_i(t) = \sum_{j=1\ i\neq j}^{n} \frac{m_i m_j(\mathbf{q}_j(t) - \mathbf{q}_i(t))}{\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\|^3} = \frac{\partial}{\partial \mathbf{q}_i} U(\mathbf{q}(t)), \quad (1 \leq i \leq n), \tag{2.1}$$

where $U(\mathbf{q})$ is the Newtonian potential function

$$U(\mathbf{q}) = \sum_{(i,j)\ i<j} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}, \quad (i, j = 1, \cdots, n),$$

$m_i \in \mathbb{R}^+ (i = 1, 2, \ldots, n)$ are masses of the bodies and
$\mathbf{q}(t) = (\mathbf{q}_1(t), \ldots, \mathbf{q}_n(t)) \in (\mathbb{R}^d)^n$ is their configuration. Here we except $\mathbf{q}_i(t) = \mathbf{q}_j(t)$ for some $i \neq j$.

It is well-known that the equation (2.1) is *scale and translation invariant.* That is, for a solution $\mathbf{q}(t) = (\mathbf{q}_1(t), \mathbf{q}_2(t), \ldots, \mathbf{q}_n(t))$ of (2.1), then

$$\kappa\, \mathbf{q}(\kappa^{-3/2} t) + \tilde{\mathbf{u}}\, t + \tilde{\mathbf{v}} = (\kappa\, \mathbf{q}_1(\kappa^{-3/2} t) + \mathbf{u}\, t + \mathbf{v}, \ldots, \kappa\, \mathbf{q}_n(\kappa^{-3/2} t) + \mathbf{u}\, t + \mathbf{v})$$

is also a solution, where κ is a positive constant and $\mathbf{u} = (u^1, \ldots, u^d)$, $\mathbf{v} = (v^1, \ldots, v^d)$ are constant d-vectors.

Let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in (\mathbb{R}^d)^n$ be a constant vector and $\mathbf{c}$ be the center of mass of the system $\mathbf{c} = \sum_{i=1}^n m_i \mathbf{q}_i / \sum_{i=1}^n m_i$. For a scalar-valued function $\phi(t)$, let us consider a vector-valued function

$$\mathbf{q}(t) = \tilde{\mathbf{c}} + \phi(t)(\mathbf{q} - \tilde{\mathbf{c}}),$$

where $\tilde{\mathbf{c}} = (\mathbf{c}, \ldots, \mathbf{c})$. It is easy to see if $\mathbf{q}(t)$ is a solution of (2.1) then $\mathbf{q}$ satisfies the equation (2.2) and $\phi(t)$ satisfies (2.3) below. Thus we naturally obtain the following concept (c.f. [3] Section 2.1.3, also [1]).

Definition 3 (Central Configuration)**.** *We call a configuration $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) \in (\mathbb{R}^d)^n$ with mass $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in (\mathbb{R}^+)^n$ a central configuration if $\mathbf{q}$ satisfies*

$$\sum_{j=1}^{n} \frac{m_j(\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} + \lambda(\mathbf{q}_i - \mathbf{c}) = \mathbf{0}, \quad i = 1, 2, \ldots, n \tag{2.2}$$

for some $\lambda \in \mathbb{R}^+$, where $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ is a distance of two bodies.

We easily see that the equations (2.2) yields $\lambda = U(\mathbf{q})/(2I)$, where $I = \sum_{i=1}^n m_i \|\mathbf{q}_i - \mathbf{c}\|^2/2$, then λ is positive.

Conversely, for a central configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ with mass $\mathbf{m} = (m_1, \ldots, m_n)$ and a real valued function $\phi(t)$ satisfying

$$\ddot{\phi} = -\lambda \phi / |\phi|^3, \tag{2.3}$$

then $\mathbf{q}(t) = \tilde{\mathbf{c}} + \phi(t)(\mathbf{q} - \tilde{\mathbf{c}})$ is a solution of the equation (2.1).

The invariance of the equation (2.1) naturally induces an equivariance of the equation (2.2) under the scaling and parallel transform. Let κ be a positive number and $\mathbf{u}$ be a vector in $\mathbb{R}^d$. For a solution $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ of the equation (2.2), we set

$$\hat{\mathbf{q}} = (\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, \ldots, \hat{\mathbf{q}}_n) = \kappa\, \mathbf{q} + \tilde{\mathbf{u}} = (\kappa\, \mathbf{q}_1 + \mathbf{u}, \ldots, \kappa\, \mathbf{q}_n + \mathbf{u}) \in (\mathbb{R}^d)^n.$$

Then $\hat{\mathbf{q}}$ satisfies

$$\sum_{j=1\ i\neq j}^{n} \frac{m_j(\hat{\mathbf{q}}_j - \hat{\mathbf{q}}_i)}{\|\hat{\mathbf{q}}_i - \hat{\mathbf{q}}_j\|^3} + \hat{\lambda}(\hat{\mathbf{q}}_i - \hat{\mathbf{c}}) = 0 \ (i = 1, 2, \ldots, n),$$

where $\hat{\lambda} = \kappa^{-3}\lambda$ and $\hat{\mathbf{c}} = (\kappa\mathbf{c} + \mathbf{u}, \ldots, \kappa\mathbf{c} + \mathbf{u})$.

Now we consider $d = 1$, which means that all bodies lie on a straight line. We call a solution $\mathbf{q}$ of (2.2) a *collinear central configuration*, or a *Moulton configuration*. Then the equation (2.2) is written in the form

$$A\ {}^t\mathbf{m} + \lambda^t(\mathbf{q} - \tilde{\mathbf{c}}) = \mathbf{0} \quad \text{for some } \lambda \in \mathbb{R}^+, \tag{2.4}$$

where $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n$ and A is a skew-symmetric matrix defined by $A = (a_{ij}),\ a_{ij} = (q_i - q_j)^{-2}$ for $i < j$, and $a_{ii} = 0$, $a_{ji} = -a_{ij}$ (c.f. [1]).

Following Moulton [4], we also consider the case where m_i $(i = 1, \ldots, n)$ are *infinitesimal zero*. Then in this paper we assume $m_i \geq 0$ $(i = 1, \ldots, n)$.

2.2. Moulton manifold

In this subsection, we consider the equation (2.4) in a geometric way.

Let us consider $2n+2$-dimensional Euclidean space $\mathbb{R}^{2n+2}$ with coordinates $(\mathbf{q}, \mathbf{m}, c, \lambda) = (q_1, \ldots, q_n, m_1, \ldots, m_n, c, \lambda)$ and consider an open domain

$$\mathcal{O}_n = \{(q_1, \ldots, q_n, m_1, \ldots, m_n, c, \lambda) \in \mathbb{R}^{2n+2} | \\ q_1 < q_2 < \cdots < q_n, m_1, \ldots, m_n \geq 0, q_1 < c < q_n, \lambda > 0\}.$$

Then the equation (2.4) gives a manifold

$$\mathcal{M}_n = \{(\mathbf{q}, \mathbf{m}, c, \lambda) \in \mathcal{O}_n | A\ {}^t\mathbf{m} + \lambda^t(\mathbf{q} - \tilde{\mathbf{c}}) = {}^t\mathbf{0}\},$$

called an *n-Moulton manifold*, which can be regarded as the set of all Moulton configurations of n bodies. The manifold $\mathcal{M}_n$ has a parametrization whose expression depends on the case where n is even or n is odd (cf.[1], [4]). In this paper we discuss the case $n = 3$ and $n = 4$, and the general cases for n are given in a similar way.

3-Moulton manifold. The equation (2.4) for $n = 3$ shows that 3-Moulton manifold is given by

$$\mathcal{M}_3 = \{(q_1, q_2, q_3, m_1, m_2, m_3, c, \lambda) \in \mathcal{O}_3 | \\ A\ {}^t\mathbf{m} + \lambda^t(\mathbf{q} - \tilde{\mathbf{c}}) = \mathbf{0} \quad \text{for some } \lambda \in \mathbb{R}^+\}.$$

The parametrization of 3-Moulton manifold $\mathcal{M}_3$ is given in the following way. For $n = 3$ the matrix A in the equation (2.4) is not invertible. Regarding (2.4) as an equation with respect to $\mathbf{m} = (m_1, m_2, m_3)$ we consider an augmented matrix of the equation

$$\begin{pmatrix} 0 & a_{12} & a_{13} & -\lambda(q_1 - c) \\ -a_{12} & 0 & a_{23} & -\lambda(q_2 - c) \\ -a_{13} & -a_{23} & 0 & -\lambda(q_3 - c) \end{pmatrix},$$

where $a_{ij} = (q_i - q_j)^{-2}$ $(i < j)$ and we obtain by the sweep-out method,

$$\begin{pmatrix} a_{12} & 0 & -a_{23} & \lambda(q_2 - c) \\ 0 & a_{12} & a_{13} & -\lambda(q_1 - c) \\ 0 & 0 & 0 & * \end{pmatrix},$$

where $* = \lambda(-a_{12}(q_3 - c) + a_{13}(q_2 - c) - a_{23}(q_1 - c))$. Then the equation $* = 0$ is the necessary and sufficient condition for (2.4) to have a solution, which is equivalent to

$$c = (a_{12}q_3 - a_{13}q_2 + a_{23}q_1)/P, \tag{2.5}$$

where $P = a_{12} - a_{13} + a_{23}$. Thus the equation (2.4) is reduced to

$$\begin{cases} a_{12}m_2 + a_{13}m_3 + \lambda(q_1 - c) = 0, \\ -a_{12}m_1 + a_{23}m_3 + \lambda(q_2 - c) = 0. \end{cases}$$

In order to parametrize the solutions (m_1, m_2, m_3), we introduce a parameter $M = m_1 + m_2 + m_3$ which represents the total mass. We consider an equation

$$\begin{cases} \phantom{-a_{12}m_1} \quad a_{12}m_2 + a_{13}m_3 + \lambda(q_1 - c) = 0, \\ -a_{12}m_1 \qquad\quad + a_{23}m_3 + \lambda(q_2 - c) = 0, \\ \quad m_1 \quad + m_2 \quad + m_3 \qquad\qquad\qquad = M. \end{cases}$$

Then the augmented matrix is the following

$$\begin{pmatrix} a_{12} & 0 & -a_{23} & \lambda(q_2 - c) \\ 0 & a_{12} & a_{13} & -\lambda(q_1 - c) \\ 1 & 1 & 1 & M \end{pmatrix}$$

and by the sweep-out method we obtain

$$\begin{pmatrix} 1 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & \alpha_2 \\ 0 & 0 & 1 & \alpha_3 \end{pmatrix},$$

where

$$\begin{aligned}\alpha_1 &= \frac{(a_{12}M + \lambda(q_1 - q_2))a_{23}}{(a_{12} - a_{13} + a_{23})a_{12}} - \frac{\lambda}{a_{12}}(c - q_2),\\ \alpha_2 &= \frac{-(a_{12}M + \lambda(q_1 - q_2))a_{13}}{(a_{12} - a_{13} + a_{23})a_{12}} - \frac{\lambda}{a_{12}}(q_1 - c),\\ \alpha_3 &= \frac{a_{12}M + \lambda(q_1 - q_2)}{a_{12} - a_{13} + a_{23}}.\end{aligned}$$

Then using (2.5) we obtain

$$\begin{aligned}m_1 &= (a_{23}M + \lambda(q_2 - q_3))/P,\\ m_2 &= -(a_{13}M + \lambda(q_1 - q_3))/P,\\ m_3 &= (a_{12}M + \lambda(q_1 - q_2))/P\end{aligned} \tag{2.6}$$

which are regarded as a parametrization of solutions of the equation (2.4). Using these functions we obtain a parametrization of the 3-Moulton manifold $\mathcal{M}_3$ in the following way. Remark $P = a_{12} - a_{13} + a_{23} > 0$ for $q_1 < q_2 < q_3$. Since m_i $(i = 1, 2, 3)$ are non-negative in $\mathcal{M}_3$, (2.6) yields $(q_2-q_1)^3, (q_3-q_2)^3 \leq M/\lambda \leq (q_3 - q_1)^3$, namely,

$$q_2 - q_1 \leq (M/\lambda)^{(1/3)}, q_3 - q_2 \leq (M/\lambda)^{(1/3)}, (M/\lambda)^{(1/3)} \leq q_3 - q_1. \tag{Q3}$$

Then we set an open set in $\mathbb{R}^5$ such that

$$\mathcal{D}_3 = \{(q_1, q_2, q_3, \lambda, M) \in \mathbb{R}^5 \mid q_1 < q_2 < q_3,\ \lambda > 0,\ M \geq 0 \,\text{and}\ (\text{Q3})\}$$

and we define a map $\widehat{\mathbf{m}} : \mathcal{D}_3 \to (\mathbb{R}^+)^3$ such that

$$\widehat{\mathbf{m}}(\mathbf{q}, \lambda, M) = (m_1(\mathbf{q}, \lambda, M), m_2(\mathbf{q}, \lambda, M), m_3(\mathbf{q}, \lambda, M)) \tag{2.7}$$

where $\mathbf{q} = (q_1, q_2, q_3)$ and $m_i(\mathbf{q}, \lambda, M)$, $i = 1, 2, 3$ are given by (2.6). The equation (2.5) gives a function $c(\mathbf{q})$. Then 3-Moulton manifold $\mathcal{M}_3$ is parametrized by $\mathcal{D}_3$ and is given as the graph of the map $\widehat{\mathbf{m}}$ and the function $c(\mathbf{q})$, i.e.,

$$\mathcal{M}_3 = \{(\mathbf{q},\ \widehat{\mathbf{m}}(\mathbf{q}, \lambda, M),\ c(\mathbf{q}),\ \lambda, M) \mid (\mathbf{q}, \lambda, M) \in \mathcal{D}_3\}.$$

4-Moulton manifold. The equation (2.4) for $n = 4$ gives the 4-Moulton manifold is given by

$$\begin{aligned}\mathcal{M}_4 = \{(q_1, q_2, q_3, q_4, m_1, m_2, m_3, m_4, c, \lambda) \in \mathcal{O}_4 \mid \\ A\ {}^t\mathbf{m} + \lambda^t(\mathbf{q} - \tilde{\mathbf{c}}) = \mathbf{0} \quad \text{for some } \lambda \in \mathbb{R}^+\},\end{aligned}$$

where $\tilde{\mathbf{c}} = (c, c, c, c)$. For $n = 4$ the coefficient matrix A in the equation (2.4) is invertible and then the equation is reduced to

$$(2.8) \qquad \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = -\frac{\lambda}{P} \begin{pmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{pmatrix} \begin{pmatrix} q_1 - c \\ q_2 - c \\ q_3 - c \\ q_4 - c \end{pmatrix},$$

where $P = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$. We can write the equation in the form

$$(2.9) \qquad m_i = -\lambda(\alpha_i - c\beta_i)/P, \quad i = 1, 2, 3, 4,$$

where

$$\begin{aligned} \alpha_1 &= -a_{34}q_2 + a_{24}q_3 - a_{23}q_4, \quad \alpha_2 = a_{34}q_1 - a_{14}q_3 + a_{13}q_4, \\ \alpha_3 &= -a_{24}q_1 + a_{14}q_2 - a_{12}q_4, \quad \alpha_4 = a_{23}q_1 - a_{13}q_2 + a_{12}q_3, \\ \beta_1 &= -a_{34} + a_{24} - a_{23}, \quad \beta_2 = a_{34} - a_{14} + a_{13}, \\ \beta_3 &= -a_{24} + a_{14} - a_{12}, \quad \beta_4 = a_{23} - a_{13} + a_{12}. \end{aligned}$$

We remark P is positive since $a_{12} > a_{13}$ and $a_{34} > a_{24}$ and similarly $\beta_1,\ \beta_3 < 0$, $\beta_2,\ \beta_4 > 0$. Thus the equation (2.9) defines functions $m_i(\mathbf{q}, c, \lambda) = -\lambda(\alpha_i - c\beta_i)/P$, $i = 1, 2, 3, 4$, where $\mathbf{q} = (q_1, q_2, q_3, q_4)$.

In order to consider $3 + 1$-M.c., we slightly extend the condition for mass in $\mathcal{M}_4$, namely, it is natural that the masses are non-negative $m_i \geq 0$, $i = 1, 2, 3, 4$. Hence from the equation (2.9) we see in $\mathcal{M}_4$, $\mathbf{q}$ and c satisfy

$$(\mathrm{Q4}) \qquad \alpha_i - c\beta_i \leq 0, \quad i = 1, 2, 3, 4$$

since P, λ are positive. Then we consider a set

$$\mathcal{D}_4 = \{(q_1, q_2, q_3, q_4, \lambda, c) \in \mathbb{R}^6 \mid q_1 < q_2 < q_3 < q_4, q_1 < c < q_4, \lambda > 0 \text{ and } (\mathrm{Q4})\}.$$

The equation (2.9) shows that the 4-Moulton manifold $\mathcal{M}_4$ is given as a graph of a map $\mathbf{m}(\mathbf{q}, c, \lambda) = (m_1(\mathbf{q}, c, \lambda), \cdots, m_4(\mathbf{q}, c, \lambda))$

$$\mathcal{M}_4 = \{(\mathbf{q}, \mathbf{m}(\mathbf{q}, c, \lambda),\ c, \lambda) \,|\, (\mathbf{q}, c, \lambda), \in \mathcal{D}_4\ \}.$$

§3. Proof of Theorem 1

Now suppose we are given a three-body A_1, A_2 and A_3 which is a Moulton configuration $\mathbf{q}_A = (q_{A_1}, q_{A_2}, q_{A_3})$ and positive mass $\mathbf{m}_A = (m_{A_1}, m_{A_2}, m_{A_3})$ such that $q_{A_1} < q_{A_2} < q_{A_3}$, $m_{A_1}, m_{A_2}, m_{A_3} > 0$. Hence we have certain $\lambda_A > 0$ and $M > 0$ such that $\mathbf{m}_A$ and $c = c_A$ are given as an image of the map $\widehat{\mathbf{m}}(\mathbf{q}_A, \lambda_A, M)$ in (2.7) and $c_A = c_A(\mathbf{q}_A)$ in (2.5), respectively. We consider to add a body B with a mass m_B in the same line so that A_1, A_2, A_3 and B form a 3+1-Moulton configuration for the three bodies A_1, A_2, A_3.

Problem. According to the condition (i) and (ii) of Definition 1, we consider the following, respectively.

(P-i) Since the four bodies A_1, A_2, A_3 and B are in a Moulton configuration, these satisfy the equation (2.8). This is equivalent to find a point $(\mathbf{q}, \lambda, c) \in \mathcal{D}_4$ where three components of $\mathbf{q} = (q_1, q_2, q_3, q_4)$ are given as $q_{A_1}, q_{A_2}, q_{A_3}$,

(P-ii) and $c = c_A$, $\lambda = \lambda_A$.

As a natural possibility we have the following four cases:
Case 1: $q_B < q_{A_1} < q_{A_2} < q_{A_3}$, Case 2: $q_{A_1} < q_B < q_{A_2} < q_{A_3}$,
Case 3: $q_{A_1} < q_{A_2} < q_B < q_{A_3}$, Case 4: $q_{A_1} < q_{A_2} < q_{A_3} < q_B$.
We will prove Theorem 1 for each case.

3.1. Case 1

We set $\mathbf{q} = (q_1, q_2, q_3, q_4) = (q_B, q_{A_1}, q_{A_2}, q_{A_3})$, $q_1 < q_2 < q_3 < q_4$ with $\mathbf{m} = (m_1, m_2, m_3, m_4) = (m_B, m_{A_1}, m_{A_2}, m_{A_3})$ (see Figure 3).

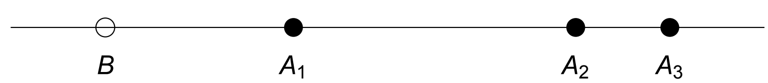

Figure 3: Case 1

We can rewrite the condition (2.5) and the equation (2.6) in this case as

$$c = c_A = (a_{23}q_4 - a_{24}q_3 + a_{34}q_2)/P_A \tag{3.1}$$

and

$$\begin{aligned} m_{A_1} &= (a_{34}M + \lambda_A(q_3 - q_4))/P_A, \\ m_{A_2} &= (-a_{24}M + \lambda_A(q_4 - q_2))/P_A, \\ m_{A_3} &= (a_{23}M + \lambda_A(q_2 - q_3))/P_A, \end{aligned}$$

where $P_A = a_{23} - a_{24} + a_{34}$, respectively.

The Problem (P-i) yields $m_{A_1} = m_2$, $m_{A_2} = m_3$, $m_{A_3} = m_4$, which is equivalent to the equation by means of (2.9)

$$\begin{pmatrix} m_2 \\ m_3 \\ m_4 \end{pmatrix} = \frac{\lambda}{P}\left(c\begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} - \begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \right) = \begin{pmatrix} m_{A_1} \\ m_{A_2} \\ m_{A_3} \end{pmatrix} = \frac{1}{P_A}\begin{pmatrix} F_2 \\ F_3 \\ F_4 \end{pmatrix}, \tag{3.2}$$

where F_i $(i = 2, 3, 4)$ are the numerator of m_{A_j} $(j = 1, 2, 3)$ in the equation (3.2), respectively. We remark here $F_i > 0$ since $m_{A_i} > 0$ and $P_A > 0$ in the 3-Moulton manifold $\mathcal{M}_3$. From the first and the second line of the equation (3.2) we obtain the equivalent relation

$$c = \frac{F_2\alpha_3 - F_3\alpha_2}{F_2\beta_3 - F_3\beta_2} \quad \text{and} \quad \lambda = \frac{F_2\beta_3 - F_3\beta_2}{\alpha_3\beta_2 - \alpha_2\beta_3}\frac{P}{P_A}, \tag{3.3}$$

where $F_2\beta_3 - F_3\beta_2 < 0$ because $\beta_3 < 0$ and $\beta_2 > 0$, then the positivity of λ, P_A and P yields $\alpha_3\beta_2 - \alpha_2\beta_3 < 0$.

Although we consider the third line of (3.2) later, here we consider Problem (P-ii), i.e., the equations $c = c_A$ and $\lambda = \lambda_A$.

Proposition 1. *The equations*

(c-1) $c = c_A$,
(l-1) $\lambda = \lambda_A$,
(f-1) $f_1 = \lambda_A(P_A q_1 - \bar{c}_A) + a_{12}F_2 + a_{13}F_3 + a_{14}F_4 = 0$

are mutually equivalent, where $\bar{c}_A$ is a numerator of c_A in (3.1) *such that* $\bar{c}_A = a_{23}q_4 - a_{24}q_3 + a_{34}q_2$.

Proof. Using (3.3), we calculate

$$\begin{aligned} &c_A - c = c_A - (F_2\alpha_3 - F_3\alpha_2)/(F_2\beta_3 - F_3\beta_2)\\ &= (a_{24}\delta_{34}+a_{34}\delta_{42})(\lambda_A(P_A q_1-\bar{c}_A)+a_{12}F_2+a_{13}F_3+a_{14}F_4)/P_A(F_2\beta_3-F_3\beta_2), \end{aligned}$$

where $\delta_{ij} = q_i - q_j$. Since $a_{24}\delta_{34} + a_{34}\delta_{42} > 0$, the equation $c - c_A = 0$ is equivalent to $f_1 = 0$.

We calculate also

$$\begin{aligned} &\lambda_A - \lambda = \lambda_A - P(F_2\beta_3 - F_3\beta_2)/P_A(\alpha_3\beta_2 - \alpha_2\beta_3)\\ &= -(a_{24}\beta_2+a_{34}\beta_3)(\lambda_A(P_A q_1-\bar{c}_A)+a_{12}F_2+a_{13}F_3+a_{14}F_4)/P_A(\alpha_3\beta_2-\alpha_2\beta_3). \end{aligned}$$

Since $a_{24}\beta_2+a_{34}\beta_3 = -a_{24}(a_{14}-a_{13})+a_{34}(a_{14}-a_{12}) < 0$, the equation $\lambda = \lambda_A$ and (f-1) are equivalent. □

Here we consider the third line of (3.2). We can easily see that the equation (f-1) of Proposition 1 also yields $m_4 = m_{A_3}$. In fact using the equation (3.2) we calculate as

$$m_{A_3} - m_4 = (a_{12}\delta_{34} + a_{13}\delta_{42} + a_{14}\delta_{23})f_1/P_A(\alpha_3\beta_2 - \alpha_2\beta_3).$$

Thus we see that a solution of the equation $f_1 = 0$ gives an existence of a 3+1-Moulton configuration.

Now we show the unique existence of the solution of $f_1 = 0$. We obtain easily $\lim_{q_1 \to -\infty} f_1 = -\infty$, $\lim_{q_1 \to q_2} f_1 = +\infty$. Moreover f_1 is monotone increasing since

$$\frac{df_1}{dq_1} = \lambda_A P_A + a'_{12} F_2 + a'_{13} F_3 + a'_{14} F_4 > 0$$

because $a'_{1i} = -2/(q_1 - q_i)^{-3} > 0$ and $F_i > 0$ $(i = 2, 3, 4)$, λ_A, $P_A > 0$. Then there exists uniquely $q_B = q_B^0 < q_{A_1}$ satisfying $f_1 = 0$, which gives a 3+1-Moulton configuration.

We see $m_B = 0$ for $q_B = q_B^0$. In fact the equation (2.9) yields

$$m_B = \lambda(c\beta_1 - \alpha_1)/P = \lambda_A(-c_A P_A + \bar{c}_A)/P = 0$$

because $\alpha_1 = -\bar{c}_A$, $\beta_1 = -P_A$. Thus we obtain Theorem 1 (ii) for case 1.

3.2. Case 2

We set $(q_1, \; q_2, \; q_3, \; q_4) = (q_{A_1}, q_B, q_{A_2}, q_{A_3})$, $\quad q_1 < q_2 < q_3 < q_4$ with $(m_1, m_2, m_3, m_4) = (m_{A_1}, m_B, m_{A_2}, m_{A_3})$ (see Figure 4).

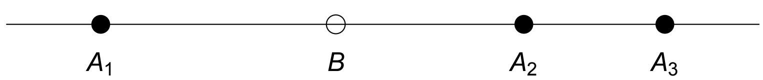

Figure 4: Case 2

We can rewrite the condition (2.5) and the equation (2.6) in this case as

$$c_A = (a_{13} q_4 - a_{14} q_3 + a_{34} q_1)/P_A$$

and

$$\begin{aligned} m_{A_1} &= (a_{34} M + \lambda_A (q_3 - q_4))/P_A, \\ m_{A_2} &= (-a_{14} M + \lambda_A (q_4 - q_1))/P_A, \\ m_{A_3} &= (a_{13} M + \lambda_A (q_1 - q_3))/P_A, \end{aligned} \tag{3.4}$$

where $P_A = a_{13} - a_{14} + a_{34}$, respectively.

The condition $m_{A_1} = m_1$, $m_{A_2} = m_3$, $m_{A_3} = m_4$ yields, by (2.9) and (3.4), the equation

$$\begin{pmatrix} m_1 \\ m_3 \\ m_4 \end{pmatrix} = \frac{\lambda}{P} \left(c \begin{pmatrix} \beta_1 \\ \beta_3 \\ \beta_4 \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \right) = \begin{pmatrix} m_{A_1} \\ m_{A_2} \\ m_{A_3} \end{pmatrix} = \frac{1}{P_A} \begin{pmatrix} F_1 \\ F_3 \\ F_4 \end{pmatrix},$$

where F_1, F_3, F_4 are the numerator of m_{A_1}, m_{A_2}, m_{A_3} in (3.4), respectively. From the second and the third line of the equation above, we obtain

$$c = \frac{F_3\alpha_4 - F_4\alpha_3}{F_3\beta_4 - F_4\beta_3}, \qquad \lambda = \frac{F_3\beta_4 - F_4\beta_3}{\alpha_4\beta_3 - \alpha_3\beta_4}\frac{P}{P_A}, \tag{3.5}$$

where $F_3\beta_4 - F_4\beta_3 > 0$, then $\alpha_4\beta_3 - \alpha_3\beta_4 > 0$.

Now let us consider the problem (P-ii). Using (3.5) we calculate

$$\begin{aligned}(3.6)\quad & c_A - c = c_A - (F_3\alpha_4 - F_4\alpha_3)/(F_3\beta_4 - F_4\beta_3) \\ &= (a_{13}\delta_{41}+a_{14}\delta_{13})(\lambda_A(P_A q_2-\bar{c}_A)-a_{12}F_1+a_{23}F_3+a_{24}F_4)/P_A(F_3\beta_4-F_4\beta_3)\end{aligned}$$

and

$$\begin{aligned}(3.7)\quad & \lambda_A - \lambda = \lambda_A - P(F_3\beta_4 - F_4\beta_3)/P_A(\alpha_4\beta_3 - \alpha_3\beta_4) \\ &= -(a_{13}\beta_3+a_{14}\beta_4)(\lambda_A(P_A q_2-\bar{c}_A)-a_{12}F_1+a_{23}F_3+a_{24}F_4)/P_A(\alpha_4\beta_3-\alpha_3\beta_4).\end{aligned}$$

Proposition 2. *The equation $c = c_A$ is equivalent to*

$$f_2 = \lambda_A(P_A q_2 - \bar{c}_A) - a_{12}F_1 + a_{23}F_3 + a_{24}F_4 = 0, \tag{f-2}$$

and the equation (f-2) *induces the condition $\lambda = \lambda_A$.*

Proof. Since $a_{13}\delta_{41} + a_{14}\delta_{13} > 0$, the identity (3.6) gives the first statement, and the equation (3.7) shows the second one. □

Furthermore the equation (f-2) in Proposition 2 induces $m_1 = m_{A_1}$. In fact we calculate as

$$m_{A_1} - m_1 = -(a_{12}\delta_{34} + a_{23}\delta_{41} + a_{24}\delta_{31})f_2/P_A(\alpha_4\beta_3 - \alpha_3\beta_4).$$

Thus a solution of $f_2 = 0$ gives a 3+1-Moulton configuration.

Now we show the unique existence of the solution $q_B = q_B^0$ of (f-2) in Proposition 2. It is easy to see $\lim_{q_2\to q_1} f_2 = -\infty$, $\lim_{q_2\to q_3} f_2 = +\infty$. Moreover f_2 is monotone increasing because

$$\frac{df_2}{dq_2} = \lambda_A P_A - a'_{12}F_1 + a'_{23}F_3 + a'_{24}F_4 > 0$$

since $a'_{12} = 2(q_1 - q_2)^{-3} < 0$, $a'_{2i} = -2(q_2 - q_i)^{-3} > 0$, $(i = 3, 4)$. Then there exists a unique $q_B = q_B^0$ such that $q_1 < q_B^0 < q_3$ satisfying $f_2 = 0$.

The equation (2.9) gives the mass of B satisfies

$$m_B = \lambda(c\beta_2 - \alpha_2)/P = \lambda_A(c_A P_A - \bar{c}_{A_1})/P = 0$$

since at $q_B = q_B^0$ it holds $\alpha_2 = \bar{c}_{A_1}$, $\beta_2 = P_A$ in this case. Then we obtain Theorem 1 (ii), and this completes the proof for case 2.

3.3. Case 3

We set $(q_1,\ q_2,\ q_3,\ q_4) = (q_{A_1}, q_{A_2}, q_B, q_{A_3}),\quad q_1 < q_2 < q_3 < q_4$ with $(m_1, m_2, m_3, m_4) = (m_{A_1}, m_{A_2}, m_B, m_{A_3})$ (see Figure 5).

Figure 5: Case 3

We can rewrite the condition (2.5) and the equation (2.6) in this case as

$$c_A = (a_{12}q_4 - a_{14}q_2 + a_{24}q_1)/P_A$$

$$\begin{aligned} m_{A_1} &= (a_{24}M + \lambda_A(q_2 - q_4))/P_A, \\ m_{A_2} &= (-a_{14}M + \lambda_A(q_4 - q_1))/P_A, \\ m_{A_3} &= (a_{12}M + \lambda_A(q_1 - q_2))/P_A, \end{aligned} \tag{3.8}$$

where $P_A = a_{12} - a_{14} + a_{24}$, respectively.

The condition $m_{A_1} = m_1$, $m_{A_2} = m_2$, $m_{A_3} = m_4$ together with (2.9) and (3.8) gives

$$\begin{pmatrix} m_1 \\ m_2 \\ m_4 \end{pmatrix} = \frac{\lambda}{P}\left(c \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_4 \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_4 \end{pmatrix} \right) = \begin{pmatrix} m_{A_1} \\ m_{A_2} \\ m_{A_3} \end{pmatrix} = \frac{1}{P_A}\begin{pmatrix} F_1 \\ F_2 \\ F_4 \end{pmatrix},$$

where F_i $(i = 1, 2, 4)$ are the numerator of m_{A_j} $(j = 1, 2, 3)$ in (3.8), respectively. The first and the second line of (3.3) are equivalent to

$$c = \frac{F_1\alpha_2 - F_2\alpha_1}{F_1\beta_2 - F_2\beta_1} \quad \text{and} \quad \lambda = \frac{F_1\beta_2 - F_2\beta_1}{\alpha_2\beta_1 - \alpha_1\beta_2}\frac{P}{P_A}, \tag{3.9}$$

where $F_1\beta_2 - F_2\beta_1 > 0$, then $\alpha_2\beta_1 - \alpha_1\beta_2 > 0$.

As to problem (P-ii) the equation (3.9) gives

Proposition 3. *The equation $c = c_A$ is equivalent to*

$$f_3 = \lambda_A(P_A q_3 - \bar{c}_A) - a_{13}F_1 - a_{23}F_2 + a_{34}F_3 = 0 \tag{f-3}$$

and the equation (f-3) *induces the condition $\lambda = \lambda_A$.*

Proof. We calculate

$$\begin{aligned} &c_A - c = c_A - (F_1\alpha_2 - F_2\alpha_1)/(F_1\beta_2 - F_2\beta_1) \\ &= (a_{14}\delta_{42} + a_{24}\delta_{14})(\lambda_A(P_A q_3 - \bar{c}_A) - a_{13}F_1 - a_{23}F_2 + a_{34}F_3)/P_A(F_1\beta_2 - F_2\beta_1), \end{aligned}$$

and

$$\lambda_A - \lambda = \lambda_A - P(F_1\beta_2 - F_2\beta_1)/P_A(\alpha_2\beta_1 - \alpha_1\beta_2)$$
$$= -(a_{14}\beta_1 + a_{24}\beta_2)(\lambda_A(P_A q_3 - \bar{c}_A) - a_{13}F_1 - a_{23}F_2 + a_{34}F_3)/P_A(\alpha_2\beta_1 - \alpha_1\beta_2).$$

Since the factor $a_{14}\delta_{42} + a_{24}\delta_{14} < 0$ in the first identity above, then we obtain the desired result. □

Now we see the condition (f-3) yields $m_4 = m_{A_3}$. We calculate

$$m_{A_3} - m_4 = (a_{13}\delta_{42} + a_{23}\delta_{14} + a_{34}\delta_{12})f_3/P_A(\alpha_2\beta_1 - \alpha_1\beta_2).$$

Thus a solution of $f_3 = 0$ gives a 3+1-Moulton configuration.

We now show there exists uniquely a solution $q_B = q_B^0$ of $f_3 = 0$. We have $\lim_{q_3 \to q_2} f_3 = -\infty$, $\lim_{q_3 \to q_4} f_3 = +\infty$. Moreover f_3 is monotone increasing because

$$\frac{df_3}{dq_3} = \lambda_A P_A - a'_{13}F_1 - a'_{23}F_2 + a'_{34}F_4 > 0$$

since $a'_{i3} = 2(q_i - q_3)^{-3} < 0$, $(i = 1, 2)$, $a'_{34} = -2(q_3 - q_4)^{-3} > 0$. Then there exists a unique solution $q_B = q_B^0$ of the equation $f_3 = 0$.

By means of the equation (2.9), we have

$$m_B = \lambda(c\beta_3 - \alpha_3)/P = \lambda_A(-c_A P_A + \bar{c}_{A_1})/P = 0$$

at $q_B = q_B^0$ because $\alpha_3 = -\bar{c}_{A_1}$, $\beta_3 = -P_A$ hold. Thus we have proved Theorem 1 (ii) in this case.

3.4. Case 4

We set $(q_1,\ q_2,\ q_3,\ q_4) = (q_{A_1}, q_{A_2}, q_{A_3}, q_B)$, $q_1 < q_2 < q_3 < q_4$ with $(m_1, m_2, m_3, m_4) = (m_{A_1}, m_{A_2}, m_{A_3}, m_B)$ (see Figure 6).

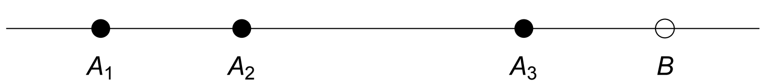

Figure 6: Case 4

We can rewrite the condition (2.5) and the equation (2.6) in this case as

$$c_A = (a_{12}q_3 - a_{13}q_2 + a_{23}q_1)/P_A$$

and

$$
\begin{aligned}
m_{A_1} &= (a_{23}M + \lambda_{A}(q_2 - q_3))/P_{A},\\
m_{A_2} &= (-a_{13}M + \lambda_{A}(q_3 - q_1))/P_{A},\\
m_{A_3} &= (a_{12}M + \lambda_{A}(q_1 - q_2))/P_{A},
\end{aligned}
\tag{3.10}
$$

where $P_{A} = a_{12} - a_{13} + a_{23}$, respectively.

The condition $m_{A_1} = m_1$, $m_{A_2} = m_2$, $m_{A_3} = m_3$ and (2.9), (3.10) give an equation

$$
\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \frac{\lambda}{P}\left(c \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \right) = \begin{pmatrix} m_{A_1} \\ m_{A_2} \\ m_{A_3} \end{pmatrix} = \frac{1}{P_{A}} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix},
$$

where F_i $(i = 1, 2, 3)$ are the numerator of m_{A_i}, respectively.

From the first and the second line of this equation we obtain

$$
c = \frac{F_1\alpha_2 - F_2\alpha_1}{F_1\beta_2 - F_2\beta_1} \quad \text{and} \quad \lambda = \frac{F_1\beta_2 - F_2\beta_1}{\alpha_2\beta_1 - \alpha_1\beta_2}\frac{P}{P_{A}},
\tag{3.11}
$$

where $F_1\beta_2 - F_2\beta_1 > 0$, then $\alpha_2\beta_1 - \alpha_1\beta_2 > 0$.

Using (3.11) we calculate

$$
\begin{aligned}
&c_{A} - c = c_{A} - (F_1\alpha_2 - F_2\alpha_1)/(F_1\beta_2 - F_2\beta_1) \\
&= (a_{13}\delta_{23} + a_{23}\delta_{31})(\lambda_{A}(\bar{c}_{A} - P_{A}q_4) + a_{14}F_1 + a_{24}F_2 + a_{34}F_3)/P_{A}(F_1\beta_2 - F_2\beta_1)
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&\lambda_{A} - \lambda = \lambda_{A} - P(F_1\beta_2 - F_2\beta_1)/P_{A}(\alpha_2\beta_1 - \alpha_1\beta_2) \\
&= -(a_{13}\beta_1 + a_{23}\beta_2)(\lambda_{A}(\bar{c}_{A} - P_{A}q_4) + a_{14}F_1 + a_{24}F_2 + a_{34}F_3)/P_{A}(\alpha_2\beta_1 - \alpha_1\beta_2).
\end{aligned}
\tag{3.13}
$$

Then we have

Proposition 4. *The conditions*

(c-4) $c = c_{A}$,

(l-4) $\lambda = \lambda_{A}$,

(f-4) $f_4 = \lambda_{A}(\bar{c}_{A} - P_{A}q_4) + a_{14}F_1 + a_{24}F_2 + a_{34}F_3 = 0$

are mutually equivalent.

Proof. Since the factors $a_{13}\delta_{23} + a_{23}\delta_{31} > 0$ in (3.12), $a_{13}\beta_1 + a_{23}\beta_2 = a_{23}(a_{34} - a_{14}) - a_{13}(a_{34} - a_{24}) > 0$ in (3.13), we obtain the proposition. □

Furthermore the condition (f-4) yields $m_3 = m_{A_3}$. In fact we calculate

$$m_{A_3} - m_3 = -(a_{14}\delta_{23} + a_{24}\delta_{31} + a_{34}\delta_{12})f_4/P_A(\alpha_2\beta_1 - \alpha_1\beta_2).$$

Now we show the existence of a unique solution $q_B = q_B^0$ of $f_4 = 0$. We remark a solution $q_B = q_B^0$ of $f_4 = 0$ satisfies (c-4) and (l-4) and hence satisfies the condition (ii) of Definition 1. We see easily $\lim_{q_4 \to q_3} f_4 = +\infty$, $\lim_{q_4 \to +\infty} f_4 = -\infty$. Moreover f_4 is monotone decreasing because

$$\frac{df_4}{dq_4} = -\lambda_A P_A + a'_{14}F_1 + a'_{24}F_2 + a'_{34}F_3 < 0$$

since $a'_{i4} = 2/(q_i - q_4)^{-3} < 0$, $i = 1, 2, 3$. Then there exists a unique solution $q_B = q_B^0$ such that $q_{A_3} < q_B^0$.

By means of the equation (2.9) similarly as the previous cases we see

$$m_B = \lambda(c\beta_4 - \alpha_4)/P = \lambda_A(c_A P_A - \bar{c}_A)/P = 0$$

at $q_B = q_B^0$, since $\alpha_4 = \bar{c}_A$, $\beta_4 = P_A$ in this case. Thus we obtain Theorem 1 in case 4.

§4. Proof of Theorem 2

We show there exists an interval of q_B for each case, where m_B is positive and each point q_B belonging to the interval gives a positive-3+1-Moulton configuration for the three bodies A_1, A_2, A_3.

We consider case 1. We note that the equation (3.3) determines $c = c(q_B)$ and $\lambda = \lambda(q_B)$ as a function of q_B, and each $(q_B, c(q_B), \lambda(q_B))$ gives a solution of (3.2). Further substituting $(q_B, c(q_B), \lambda(q_B))$ into $m_B = \lambda(c\beta_1 - \alpha_1)/P$ gives a function $m_B(q_B)$ which represents the mass of B for the configuration given by $(q_B, c(q_B), \lambda(q_B))$.

Now we consider the positivity of $m_B(q_B)$. The inequality

$$m_B(q_B) = \lambda(c\beta_1 - \alpha_1)/P > 0$$

is equivalent $c\beta_1 - \alpha_1 > 0$. We remark $\alpha_1 = -\bar{c}_A$ and $\beta_1 = -P_A$, then

$$c\beta_1 - \alpha_1 = -cP_A + \bar{c}_A.$$

Thus $-cP_A + \bar{c}_A > 0$ is equivalent $c < c_A$ because $P_A > 0$. Then for the interval of q_B where $c(q_B) < c_A$ we have $m_B(q_B) > 0$ (see figure 7).

For the other cases we can consider in the similar way and we have

Proposition 5. *An inequality $m_B(q_B) > 0$ holds if and only if $c(q_B) < c_A$ for the cases 1 and 3, and $c(q_B) > c_A$ for the other cases, respectively.*

We remark the function m_4 of the third line of (3.2) gives the mass of A_3 in case 1. When $c(q_B) = c_A$, that is, $q_B = q_B^0$, the function m_4 is equal to the initial mass of A_3, i.e., $m_4 = m_{A_3} > 0$, thus for q_B being sufficiently close to q_B^0 in the interval of Propostion 5, the mass $m_B(q_B)$ and m_4 of A_3 are positive. We can show similarly for the other cases. Thus, for each case we obtain an interval of q_B such that each point belonging to this interval gives a positive-3+1-Moulton configuration. This completes the proof of Theorem 2 (see Figure 8, 9, 10.)

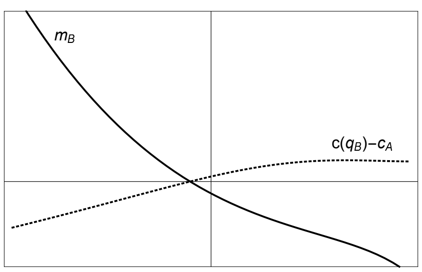

Figure 7: The curves of $m_B(q_B)$ and $c(q_B) - c_A$ in case 1.

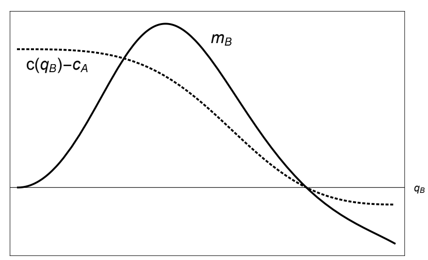

Figure 8: The curves of $m_B(q_B)$ and $c(q_B) - c_A$ in case 2.

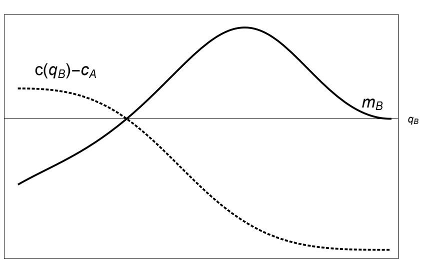

Figure 9: The curves of $m_B(q_B)$ and $c(q_B) - c_A$ in case 3.

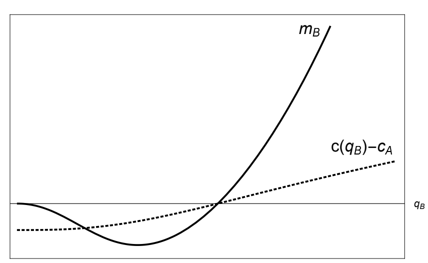

Figure 10: The curves of $m_B(q_B)$ and $c(q_B) - c_A$ in case 4.

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