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Super geometric mean graphs

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Abstract. Let G be a graph and $f : V(G) \to \{1, 2, 3, \ldots, p + q\}$ be an injection. For each edge uv, the induced edge labeling f^* is defined as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$. Then f is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv); uv \in E(G)\} = \{1, 2, 3, \ldots, p + q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph. In this paper, we discuss the super geometric meanness of union of any paths, union of any cycles of order ≥ 5, the graph $P_n \odot S_m$ for $m \leq 3$, square graph, total graph, the H-graph, the graph $G \odot S_1$ and $G \odot S_2$ for any H-graph G, subdivision of $K_{1,3}$ and some chain graphs.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [5]. For a detailed survey on graph labeling, we refer to [4].

A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. A star graph S_n is the complete bipartite graph $K_{1,n}$. $G \odot S_m$ is the graph obtained from G by attaching m pendant vertices to each vertex of G. A square of a graph G, denoted by G^2, has the vertex set as in G and two vertices are adjacent in G^2 if they are at a distance either 1 or 2 apart in G. The total graph $T(G)$ of graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if either they are adjacent vertices of G or adjacent edges of G or one is a vertex of G and the other one is an edge incident on it. The H-graph is obtained from two paths $u_1, u_2, \ldots, u_n$

and $v_1, v_2, \ldots, v_n$ of equal length by joining an edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ when n is odd and $u_{\frac{n+2}{2}}v_{\frac{n}{2}}$ when n is even. A subdivision of a graph G, denoted by $S(G)$, is a graph obtained from G by a sequence of elementary subdivisions forming edges into paths through new vertices of degree 2.

Barrientos [1] defines a chain graph as one with blocks $B_1, B_2, B_3, \ldots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block cut point graph is a path. The chain graph $\widehat{G}(p_1, k_1, p_2, k_2, \ldots, k_{n-1}, p_n)$ is obtained from n cycles of length $p_1, p_2, p_3, \ldots, p_n$ and $(n-1)$ paths on $k_1, k_2, k_3, \ldots, k_{n-1}$ vertices respectively by identifying a cycle and a path at a vertex alternatively as follows. If the i^{th} cycle is of odd length, then its $\left(\frac{p_i+3}{2}\right)^{th}$ vertex is identified with a pendant vertex of the i^{th} path and if the i^{th} cycle is of even length, then its $\left(\frac{p_i+2}{2}\right)^{th}$ vertex is identified with a pendant vertex of the i^{th} path while the other pendant vertex of the i^{th} path is identified with the first vertex of the $(i+1)^{th}$ cycle. The chain graph $G^*(p_1, p_2, \ldots, p_n)$ is obtained from n cycles of length $p_1, p_2, \ldots, p_n$ by identifying consecutive cycles at a vertex as follows. If the i^{th} cycle is of odd length, then its $\left(\frac{p_i+3}{2}\right)^{th}$ vertex is identified with the first vertex of $(i+1)^{th}$ cycle and if the i^{th} cycle is of even length, then its $\left(\frac{p_i+2}{2}\right)^{th}$ vertex is identified with the first vertex of $(i+1)^{th}$ cycle.

If every edge of the path P_n is replaced by a triangle C_3, then the resulting graph is called a triangular snake T_n. The graph Tadpoles $T(n,k)$ is obtained by identifying a vertex of the cycle C_n to an end vertex of the path P_{k+1}.

The geometric mean labeling was introduced in [2] and the geometric meanness property for some standard graphs was studied in [3].

The concept of super mean labeling was first introduced by R. Ponraj and D. Ramya and studied the super mean labeling of some standard graphs [6]. In [7, 8, 9], R. Vasuki et al. discussed the super mean labeling of the H-graph, corona of the H-graph and some special classes of graphs.

Motivated by the works on super mean labeling, we introduced a new type of labeling called super geometric mean labeling.

The geometric mean of any two numbers need not be an integer. To assign the edge label as an integer based on the geometric mean, we may use either flooring function or ceiling function. In this paper, we consider the ceiling function of our discussion.

A vertex labeling of G is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For a vertex labeling f, the induced edge labeling f^* is defined as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$. Then f is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv); uv \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph.

The graph shown in Figure 1 is a super geometric mean graph.

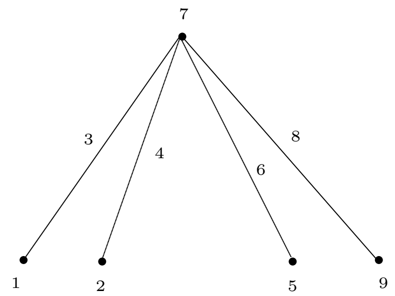

Figure 1.

In this paper, we have established the super geometric meanness of union of any paths, union of any cycles of order ≥ 5, the graph $P_n \odot S_m$ for $m \leq 3$, square graph, total graph, the H-graph, the graph $G \odot S_1$ and $G \odot S_2$ for any H-graph G, subdivision of $K_{1,3}$ and the chain graphs $\widehat{G}(p_1, k_1, p_2, k_2, \ldots, k_{n-1}, p_n)$ and $G^*(p_1, p_2, \ldots, p_n)$.

§2. Main Results

Theorem 2.1. *Union of any path P_n is a super geometric mean graph, for $n \geq 2$.*

Proof. Let the graph G be the union of k paths. Let $\{v_j^{(i)}; 1 \leq j \leq p_i\}$ be the vertices of the i^{th} path P_{p_i} with $p_i \geq 2$ and $1 \leq i \leq k$.

We define $f : V(G) \to \left\{1, 2, 3, \ldots, \sum\limits_{i=1}^{k} 2p_i - k\right\}$ as follows:

$f\left(v_j^{(1)}\right) = 2j - 1$, for $1 \leq j \leq p_1$ and

$f\left(v_j^{(i)}\right) = f\left(v_{p_{i-1}}^{(i-1)}\right) + 2j - 1$, for $2 \leq i \leq k$ and $1 \leq j \leq p_i$.

The induced edge labeling is as follows:

$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = 2j$, for $1 \leq j \leq p_1 - 1$ and

$f^*\left(v_j^{(i)} v_{j+1}^{(i)}\right) = f\left(v_{p_{i-1}}^{(i-1)}\right) + 2j$, for $2 \leq i \leq k$ and $1 \leq j \leq p_i - 1$.

Hence, f is a super geometric mean labeling of G. Thus the graph G is a super geometric mean graph. □

A super geometric mean labeling of $P_5 \cup P_3 \cup P_4$ is shown in Figure 2.

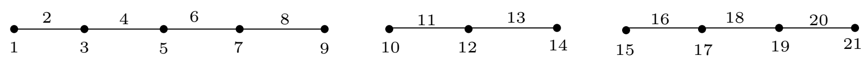

Figure 2.

Corollary 2.2. *Every path $P_n, n \geq 1$ is a super geometric mean graph.*

Theorem 2.3. *Union of any cycles C_n is a super geometric mean graph, for $n \geq 5$.*

Proof. Let the graph G be union of k cycles. Let $\{v_j^{(i)}; 1 \leq j \leq p_i\}$ be the vertices of the i^{th} cycle C_{p_i} with $p_i \geq 5$ and $1 \leq i \leq k$.

We define $f : V(G) \to \left\{1, 2, 3, \ldots, \sum_{i=1}^{k} 2p_i\right\}$ as follows:

When p_1 is odd,

$$f\left(v_j^{(1)}\right) = \begin{cases} 1 & j = 1 \\ 4j-4 & 2 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4j-5 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4j-6 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \\ 4p_1+5-4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \leq j \leq p_1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \begin{cases} 4j-2 & 1 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4j-3 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4j-8 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \\ 4p_1+3-4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \leq j \leq p_1 - 1 \text{ and} \end{cases}$$

$$f^*\left(v_1^{(1)} v_{p_1}^{(1)}\right) = 3.$$

When p_1 is even,

$$f\left(v_j^{(1)}\right) = \begin{cases} 1 & j = 1 \\ 4j-4 & 2 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4p_1+5-4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \leq j \leq p_1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \begin{cases} 4j-2 & 1 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4p_1+3-4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq j \leq p_1 - 1 \text{ and} \end{cases}$$

$$f^*\left(v_1^{(1)} v_{p_1}^{(1)}\right) = 3.$$

Case (i) For $2 \leq i \leq k$, p_i is odd and p_{i-1} is odd.

$$f\left(v_j^{(i)}\right) = \begin{cases} f\left(v^{(i-1)}_{\left\lfloor \frac{p_{i-1}}{2} \right\rfloor + 2}\right) + 1 & j = 1 \\ f\left(v^{(i-1)}_{\left\lfloor \frac{p_{i-1}}{2} \right\rfloor + 2}\right) + 4j - 5 & 2 \leq j \leq \left\lfloor \frac{p_i}{2} \right\rfloor + 1 \\ f\left(v^{(i-1)}_{\left\lfloor \frac{p_{i-1}}{2} \right\rfloor + 2}\right) + 4pi + 6 - 4j & \left\lfloor \frac{p_i}{2} \right\rfloor + 2 \leq j \leq p_i. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right)=\begin{cases} f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+2 & j=1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4j-3 & 2\le j\le\lfloor\frac{p_i}{2}\rfloor+1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4pi+4-4j & \lfloor\frac{p_i}{2}\rfloor+2\le j\le p_i-1 \text{ and}\end{cases}$$

$$f^*\left(v_1^{(i)}v_{p_i}^{(i)}\right)=f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4.$$

Case (ii) For $2\le i\le k$, p_i is odd and p_{i-1} is even.

$$f\left(v_j^{(i)}\right)=\begin{cases} f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+1 & j=1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-5 & 2\le j\le\lfloor\frac{p_i}{2}\rfloor+1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4pi+6-4j & \lfloor\frac{p_i}{2}\rfloor+2\le j\le p_i.\end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right)=\begin{cases} f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+2 & j=1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-3 & 2\le j\le\lfloor\frac{p_i}{2}\rfloor+1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4pi+4-4j & \lfloor\frac{p_i}{2}\rfloor+2\le j\le p_i-1 \text{ and}\end{cases}$$

$$f^*\left(v_1^{(i)}v_{p_i}^{(i)}\right)=f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4.$$

Case (iii) For $2\le i\le k$, p_i is even and p_{i-1} is odd.

$$f\left(v_j^{(i)}\right)=\begin{cases} f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+1 & j=1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4j-5 & 2\le j\le\lfloor\frac{p_i}{2}\rfloor\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4j-4 & j=\lfloor\frac{p_i}{2}\rfloor+1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4j-11 & j=\lfloor\frac{p_i}{2}\rfloor+2\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4p_i+6-4j & \lfloor\frac{p_i}{2}\rfloor+3\le j\le p_i.\end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right)=\begin{cases} f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+2 & j=1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4j-3 & 2\le j\le\lfloor\frac{p_i}{2}\rfloor-1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4j-2 & j=\lfloor\frac{p_i}{2}\rfloor\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4j-5 & j=\lfloor\frac{p_i}{2}\rfloor+1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4p_i+4-4j & \lfloor\frac{p_i}{2}\rfloor+2\le j\le p_i-1 \text{ and} \end{cases}$$

$$f^*\left(v_1^{(i)}v_{p_i}^{(i)}\right)=f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+2}\right)+4.$$

Case (iv) For $2\le i\le k$, p_i is even and p_{i-1} is even.

$$f\left(v_j^{(i)}\right)=\begin{cases} f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+1 & j=1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-5 & 2\le j\le\lfloor\frac{p_i}{2}\rfloor\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-4 & j=\lfloor\frac{p_i}{2}\rfloor+1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-11 & j=\lfloor\frac{p_i}{2}\rfloor+2\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4pi+6-4j & \lfloor\frac{p_i}{2}\rfloor+3\le j\le p_i. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right)=\begin{cases} f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+2 & j=1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-3 & 2\le j\le\lfloor\frac{p_i}{2}\rfloor-1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-2 & j=\lfloor\frac{p_i}{2}\rfloor\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4j-5 & j=\lfloor\frac{p_i}{2}\rfloor+1\\ f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4pi+4-4j & \lfloor\frac{p_i}{2}\rfloor+2\le j\le p_i-1 \text{ and} \end{cases}$$

$$f^*\left(v_1^{(i)}v_{p_i}^{(i)}\right)=f\left(v^{(i-1)}_{\lfloor\frac{p_{i-1}}{2}\rfloor+1}\right)+4.$$

Hence, f is a super geometric mean labeling of G. Thus the graph G is a super geometric mean graph. □

A super geometric mean labeling of $C_6 \cup C_8 \cup C_5 \cup C_7$ is shown in Figure 3.

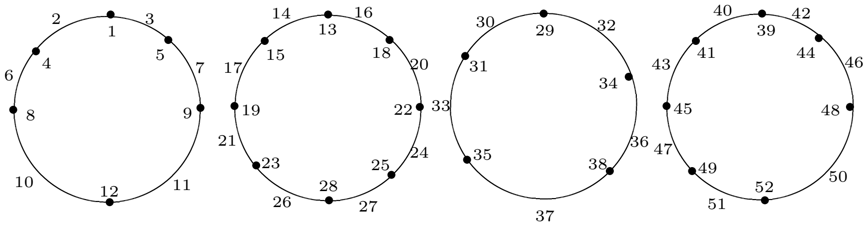

Figure 3.

Corollary 2.4. *Any cycle C_n is a super geometric mean graph, for $n \geq 3$.*

Proof. By Theorem 2.3, the result holds for $n \geq 5$. □

The Super geometric mean labeling of C_3 and C_4 are shown in Figure 4.

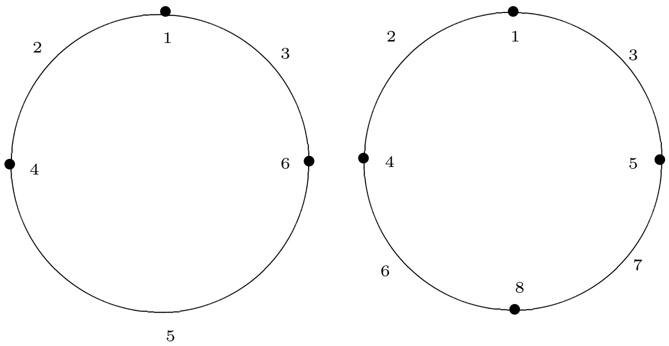

Figure 4.

Theorem 2.5. *The graph $P_n \odot S_m$ is a super geometric mean graph, for $n \geq 1$ and $m \leq 3$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}, \ldots, v_m^{(i)}$ be the pendant vertices at each vertex u_i of the path P_n, for $1 \leq i \leq n$.
Case (i) $m = 1$.

We define $f : V(P_n \odot S_1) \to \{1, 2, 3, \ldots, 4n-1\}$ as follows:
$f(u_i) = 4i - 1$, for $1 \leq i \leq n$ and

$$f\left(v_1^{(i)}\right) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \leq i \leq n. \end{cases}$$

The induced edge labeling is as follows:
$f^*(u_i u_{i+1}) = 4i + 1$, for $1 \leq i \leq n-1$ and
$f^*\left(v_1^{(i)} u_i\right) = 4i - 2$, for $1 \leq i \leq n$.

Case (ii) $m = 2$.

We define $f : V(P_n \odot S_2) \to \{1, 2, 3, \ldots, 6n - 1\}$ as follows:

$f(u_i) = 6i - 3$, for $1 \leq i \leq n$,

$f(v_1^{(i)}) = 6i - 5$, for $1 \leq i \leq n$ and

$f\left(v_2^{(i)}\right) = 6i - 1$, for $1 \leq i \leq n$.

The induced edge labeling is as follows:

$f^*(u_i u_{i+1}) = 6i$, for $1 \leq i \leq n - 1$,

$f^*(v_1^{(i)} u_i) = 6i - 4$, for $1 \leq i \leq n$ and

$f^*\left(v_2^{(i)} u_i\right) = 6i - 2$, for $1 \leq i \leq n$.

Case (iii) $m = 3$.

We define $f : V(P_n \odot S_3) \to \{1, 2, 3, \ldots, 8n - 1\}$ as follows:

$f(u_i) = 8i - 3$, for $1 \leq i \leq n$,

$$f(v_1^{(i)}) = \begin{cases} 1 & i = 1 \\ 8i - 8 & 2 \leq i \leq n \end{cases}$$

$f(v_2^{(i)}) = 8i - 6$, for $1 \leq i \leq n$ and

$f\left(v_3^{(i)}\right) = 8i - 1$, for $1 \leq i \leq n$.

The induced edge labeling is as follows:

$f^*(u_i u_{i+1}) = 8i + 1$, for $1 \leq i \leq n - 1$,

$f^*(v_1^{(i)} u_i) = 8i - 5$, for $1 \leq i \leq n$,

$f^*(v_2^{(i)} u_i) = 8i - 4$, for $1 \leq i \leq n$ and

$f^*\left(v_3^{(i)} u_i\right) = 8i - 2$, for $1 \leq i \leq n$.

Hence, f is a super mean geometric mean labeling of $P_n \odot S_m$. Thus the graph $P_n \odot S_m$ is a super geometric mean graph, for $n \geq 1$ and $m \leq 3$. □

The Super geometric mean labeling of $P_6 \odot S_1, P_5 \odot S_2$ and $P_4 \odot S_3$ are shown in Figure 5.

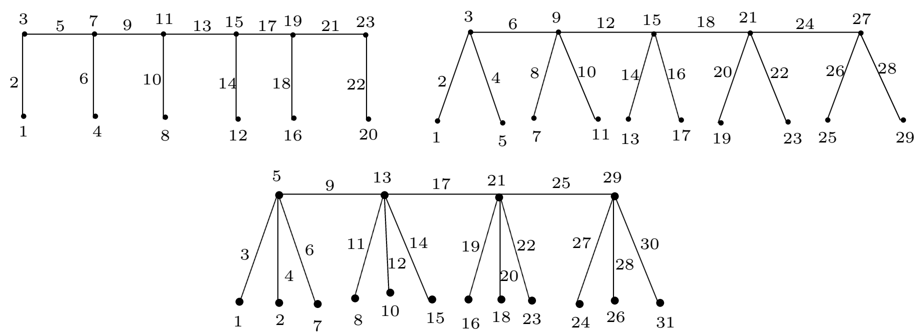

Figure 5.

Theorem 2.6. *P_n^2 is a super geometric mean graph, for $n \geq 3$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n.

We define $f : V(P_n^2) \to \{1, 2, 3, \ldots, 3n-3\}$ as follows:

$f(v_1) = 1$,

$$f(v_i) = \begin{cases} 3i-3 & 3 \le i \le n-1 \text{ and } i \text{ is odd} \\ 3i-2 & 2 \le i \le n-1 \text{ and } i \text{ is even and} \end{cases}$$

$f(v_n) = 3n-3$.

The induced edge labeling is as follows:

$f^*(v_i v_{i+1}) = 3i-1$, for $1 \le i \le n-1$ and

$$f^*(v_i v_{i+2}) = \begin{cases} 3i & 1 \le i \le n-2 \text{ and } i \text{ is odd} \\ 3i+1 & 2 \le i \le n-2 \text{ and } i \text{ is even.} \end{cases}$$

Hence, f is a super geometric mean labeling of P_n^2. Thus the graph P_n^2 is a super geometric mean graph, for $n \ge 3$. □

A super geometric mean labeling of P_7^2 is shown in Figure 6.

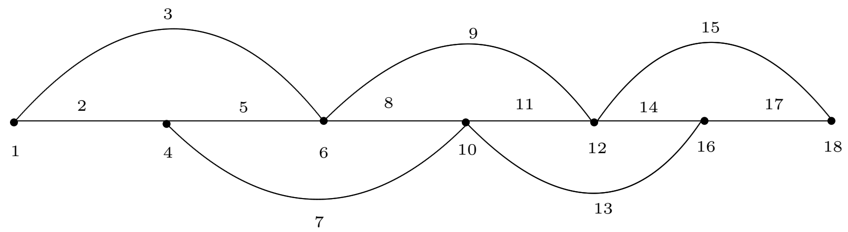

Figure 6.

Theorem 2.7. *The total graph $T(P_n)$ is a super geometric mean graph, for $n \ge 2$.*

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1}; 1 \le i \le n-1\}$ be the vertex set and edge set of the path P_n. Then

$V(T(P_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\}$ and

$E(T(P_n)) = \{v_i, v_{i+1}, e_i v_i, e_i v_{i+1}; 1 \le i \le n-1\} \cup \{e_i e_{i+1}; 1 \le i \le n-2\}$.

We define $f : V(T(P_n)) \to \{1, 2, 3, \ldots, 6n-6\}$ as follows:

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 6i-6 & 2 \le i \le n \text{ and} \end{cases}$$

$f(e_i) = 6i-2, \quad$ for $1 \le i \le n-1$.

The induced edge labeling is as follows:

$f^*(v_i v_{i+1}) = 6i-3$, for $1 \le i \le n-1$,

$f^*(e_i v_i) = 6i-4$, for $1 \le i \le n-1$,

$f^*(e_i v_{i+1}) = 6i-1$, for $1 \le i \le n-1$ and

$f^*(e_i e_{i+1}) = 6i+1$, for $1 \le i \le n-2$.

Hence, f is a super geometric mean labeling of $T(P_n)$. Thus the graph $T(P_n)$ is a super geometric mean graph, for $n \geq 2$. □

A super geometric mean labeling of $T(P_6)$ is shown in Figure 7.

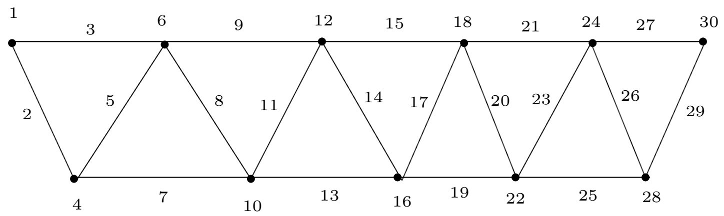

Figure 7.

Theorem 2.8. *Any $H-$ graph G is a super geometric mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths of equal length in G.

Case (i) n is odd.

We define $f : V(G) \to \{1, 2, 3, \ldots, 4n-1\}$ as follows:

$f(u_i) = 2i - 1$, for $1 \leq i \leq n$ and

$$f(v_i) = \begin{cases} 2n - 3 + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 6n + 2 - 4i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n. \end{cases}$$

The induced edge labeling is as follows:

$f^*(u_iu_{i+1}) = 2i$, for $1 \leq i \leq n-1$, $f^*(u_iv_i) = 2n$, for $i = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and

$$f^*(v_iv_{i+1}) = \begin{cases} 2n - 1 + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 6n - 4i & \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1. \end{cases}$$

Case (ii) n is even.

We define $f : V(G) \to \{1, 2, 3, \ldots, 4n-1\}$ as follows:

$f(u_i) = 2i - 1$, for $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} 2n + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 6n - 1 - 4i & \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n-1 \text{ and} \end{cases}$$

$f(v_n) = 2n$.

The induced edge labeling is as follows:

$f^*(u_iu_{i+1}) = 2i$, for $1 \leq i \leq n-1$, $f^*(u_{i+1}v_i) = 2n+1$, for $i = \left\lfloor \frac{n}{2} \right\rfloor$,

$$f^*(v_iv_{i+1}) = \begin{cases} 2n + 2 + 4i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 6n - 3 - 4i & \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n-2 \text{ and} \end{cases}$$

$f^*(v_{n-1}v_n) = 2n + 2$.

Hence, f is a super geometric mean labeling of G. Thus the graph the $H-$graph G is a super geometric mean graph. □

The super geometric mean labeling of G_1 and G_2 are shown in Figure 8.

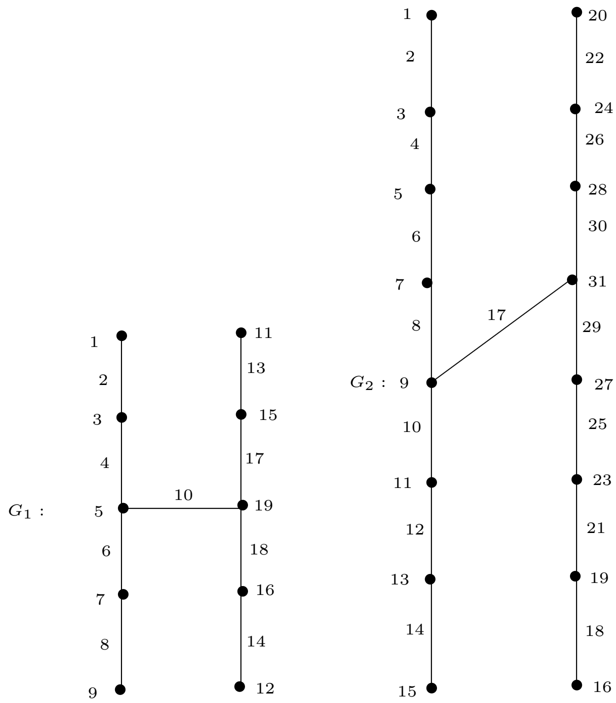

Figure 8.

Theorem 2.9. *For a $H-$graph G, $G \odot S_1$ is a super geometric mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of G. Then
$V(G \odot S_1) = V(G) \cup \{u_1', u_2', \ldots, u_n'\} \cup \{v_1', v_2', \ldots, v_n'\}$ and
$E(G \odot S_1) = E(G) \cup \{u_i u_i', v_i v_i',; 1 \leq i \leq n\}$.

Case (i) $n \equiv 0 (mod\ 4)$.

We define $f : V(G \odot S_1) \to \{1, 2, 3, \ldots, 8n-1\}$ as follows:

$f(u_i) = 4i - 1$, for $1 \leq i \leq n$, $f(u_i') = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \leq i \leq n, \end{cases}$

$$f(v_i) = \begin{cases} 4n + 2 + 8i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is odd} \\ 4n + 8i & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_{n+1-i}) = \begin{cases} 4n + 2 & i = 1 \\ 4n - 9 + 8i & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \end{cases}$$

$$f(v_i') = \begin{cases} 4n + 8i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is odd} \\ 4n + 2 + 8i & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \text{ and } i \text{ is even and} \end{cases}$$

$$f(v_{n+1-i}') = \begin{cases} 4n & i = 1 \\ 4n - 12 + 8i & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

The induced edge labeling is as follows:

$f^*(u_i u_{i+1}) = 4i + 1$, for $1 \leq i \leq n - 1$, $f^*(u_i u_i') = 4i - 2$, for $1 \leq i \leq n$,
$f^*(u_{i+1} v_i) = 4n+3$, for $i = \left\lfloor \frac{n}{2} \right\rfloor$, $f^*(v_i v_{i+1}) = 4n+5+8i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$,
$f^*(v_{n+1-i} v_{n-i}) = 4n - 5 + 8i$, for $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $f^*(v_n v_{n-1}) = 4n + 5$,

$f^*(v_iv'_i) = 4n + 1 + 8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$,
$f^*(v_{n+1-i}v'_{n+1-i}) = 4n-10+8i$, for $2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $f^*(v_nv'_n) = 4n+1$.

Case (ii) $n \equiv 1(mod\ 4)$.

We define $f : V(G \odot S_1) \to \{1, 2, 3, \ldots, 8n - 1\}$ as follows:

$f(u_i) = 4i - 1$, for $1 \le i \le n$, $f(u'_i) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \le i \le n, \end{cases}$

$f(v_i) = \begin{cases} 4n - 4 + 8i & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is odd} \\ 4n - 2 + 8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is even,} \end{cases}$

$f(v_{n+1-i}) = 4n - 5 + 8i$ for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$,

$f(v'_i) = \begin{cases} 4n - 2 + 8i & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is odd} \\ 4n - 4 + 8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is even and} \end{cases}$

$f(v'_{n+1-i}) = 4n - 8 + 8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$.

The induced edge labeling is as follows:

$f^*(u_iu_{i+1}) = 4i + 1$, for $1 \le i \le n - 1$, $f^*(u_iu'_i) = 4i - 2$, for $1 \le i \le n$,
$f^*(u_iv_i) = 4n+1$, for $i = \left\lfloor \frac{n}{2} \right\rfloor + 1$, $f^*(v_iv_{i+1}) = 4n+1+8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_{n+1-i}v_{n-i}) = 4n - 1 + 8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_iv'_i) = 4n - 3 + 8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$ and
$f^*(v_{n+1-i}v'_{n+1-i}) = 4n - 6 + 8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Case (iii) $n \equiv 2(mod\ 4)$.

We define $f : V(G \odot S_1) \to \{1, 2, 3, \ldots, 8n - 1\}$ as follows:

$f(u_i) = 4i - 1$, for $1 \le i \le n$, $f(u'_i) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \le i \le n, \end{cases}$

$f(v_i) = \begin{cases} 4n + 8i & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 2 \text{ and } i \text{ is odd} \\ 4n + 2 + 8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even,} \end{cases}$

$f(v_{n+1-i}) = \begin{cases} 4n + 2 & i = 1 \\ 4n - 9 + 8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1, \end{cases}$

$f(v'_i) = \begin{cases} 4n + 2 + 8i & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 2 \text{ and } i \text{ is odd} \\ 4n + 8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even and} \end{cases}$

$f(v'_{n+1-i}) = \begin{cases} 4n & i = 1 \\ 4n - 12 + 8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$

The induced edge labeling is as follows:

$f^*(u_iu_{i+1}) = 4i + 1$, for $1 \le i \le n - 1$, $f^*(u_iu'_i) = 4i - 2$, for $1 \le i \le n$,
$f^*(u_{i+1}v_i) = 4n+3$, for $i = \left\lfloor \frac{n}{2} \right\rfloor$, $f^*(v_iv_{i+1}) = 4n+5+8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$,
$f^*(v_{n+1-i}v_{n-i}) = 4n - 5 + 8i$, for $2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, $f^*(v_nv_{n-1}) = 4n + 5$,
$f^*(v_iv'_i) = 4n + 1 + 8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1$,
$f^*(v_{n+1-i}v'_{n+1-i}) = 4n-10+8i$, for $2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $f^*(v_nv'_n) = 4n+1$.

Case (iv) $n \equiv 3(mod\ 4)$.

We define $f : V(G \odot S_1) \to \{1, 2, 3, \ldots, 8n - 1\}$ as follows:

$f(u_i) = 4i - 1$, for $1 \le i \le n$, $f(u'_i) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \le i \le n, \end{cases}$

$$f(v_i) = \begin{cases} 4n-2+8i & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is odd} \\ 4n-4+8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even,} \end{cases}$$

$f(v_{n+1-i}) = 4n-5+8i$ for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$,

$$f(v_i') = \begin{cases} 4n-4+8i & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \text{ is odd} \\ 4n-2+8i & 2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ is even and} \end{cases}$$

$f(v'_{n+1-i}) = 4n-8+8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$.

The induced edge labeling is as follows:

$f^*(u_iu_{i+1}) = 4i+1$, for $1 \le i \le n-1$, $f^*(u_iu_i') = 4i-2$, for $1 \le i \le n$,
$f^*(u_iv_i) = 4n+1$, for $i = \left\lfloor \frac{n}{2} \right\rfloor + 1$, $f^*(v_iv_{i+1}) = 4n+1+8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_{n+1-i}v_{n-i}) = 4n-1+8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_iv_i') = 4n-3+8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$ and
$f^*(v_{n+1-i}v'_{n+1-i}) = 4n-6+8i$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Hence, f is a super geometric mean labeling of $G \odot S_1$. Thus the graph the graph $G \odot S_1$ is a super geometric mean graph. □

The Super geometric mean labeling of $G_1 \odot S_1$ and $G_2 \odot S_1$ are shown in Figure 9.

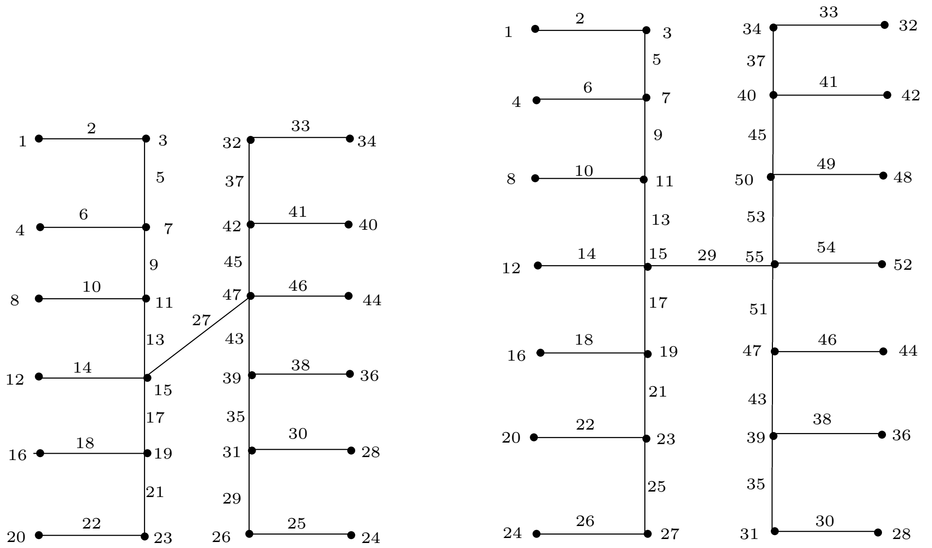

Figure 9.

$G_1 \odot S_1$ $G_2 \odot S_1$

Theorem 2.10. *For a $H-$graph G, $G \odot S_2$ is a super geometric mean graph.*

Proof. Let $u_1, u_2, \dots u_n$ and $v_1, v_2, \dots, v_n$ be the vertices of G. Let $V(G)$ together with $u_1', u_2', \dots, u_n', u_1'', u_2'', \dots, u_n'', v_1', v_2' \dots, v_n', v_1'', v_2'', \dots, v_n''$ form the vertex set of $G \odot S_2$ and $E(G)$ together with $\{u_iu_i', u_iu_i'', v_iv_i', v_iv_i''; 1 \le i \le n\}$ form the edge set of $G \odot S_2$.

Case (i) n is odd.

We define $f : V(G \odot S_2) \to \{1, 2, 3, \dots, 12n-1\}$ as follows:

$f(u_i) = 6i - 3$, for $1 \leq i \leq n$, $f(u'_i) = 6i - 5$, for $1 \leq i \leq n$,
$f(u''_i) = 6i - 1$, for $1 \leq i \leq n$, $f(v_i) = 6n - 6 + 12i$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,
$f(v_{n+1-i}) = 6n - 7 + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$,
$f(v'_i) = 6n - 10 + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$,
$f(v'_{n+1-i}) = 6n - 11 + 12i$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,
$f(v''_i) = \begin{cases} 6n - 2 + 12i & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 12n - 5 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and} \end{cases}$
$f(v''_{n+1-i}) = 6n - 3 + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

The induced edge labeling is as follows:
$f^*(u_iu_{i+1}) = 6i$, for $1 \leq i \leq n - 1$, $f^*(u_iu'_i) = 6i - 4$, for $1 \leq i \leq n$,
$f^*(u_iu''_i) = 6i - 2$, for $1 \leq i \leq n$, $f^*(u_iv_i) = 6n$, for $i = \left\lfloor \frac{n}{2} \right\rfloor + 1$,
$f^*(v_iv_{i+1}) = 6n + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_{n+1-i}v_{n-i}) = 6n - 1 + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_iv'_i) = f(v'_i) + 2$, for $1 \leq i \leq n$ and
$f^*(v_iv''_i) = \begin{cases} f(v''_i) - 2, & \text{for } 1 \leq i \leq n \text{ and } i \neq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ f(v''_i) + 2, & i = \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$

Case (ii) n is even.

We define $f : V(G \odot S_2) \rightarrow \{1, 2, 3, \ldots, 12n - 1\}$ as follows:
$f(u_i) = 6i - 3$, for $1 \leq i \leq n$, $f(u'_i) = 6i - 5$, for $1 \leq i \leq n$,
$f(u''_i) = 6i - 1$, for $1 \leq i \leq n$, $f(v_i) = 6n + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$,
$f(v_{n+1-i}) = \begin{cases} 6n + 2 & i = 1 \\ 6n - 13 + 12i & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \end{cases}$
$f(v'_i) = 6n - 4 + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$,
$f(v'_{n+1-i}) = \begin{cases} 6n - 6 + 6i & 1 \leq i \leq 2 \\ 6n - 17 + 12i & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \end{cases}$
$f(v''_i) = 6n + 4 + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and
$f(v''_{n+1-i}) = \begin{cases} 6n + 5 & i = 1 \\ 6n - 9 + 12i & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 12n - 4 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$

The induced edge labeling is as follows:
$f^*(u_iu_{i+1}) = 6i$, for $1 \leq i \leq n - 1$, $f^*(u_iu'_i) = 6i - 4$, for $1 \leq i \leq n$,
$f^*(u_iu''_i) = 6i - 2$, for $1 \leq i \leq n$, $f^*(u_{i+1}v_i) = 6n + 3$, for $i = \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_iv_{i+1}) = 6n + 6 + 12i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$,
$f^*(v_{n+1-i}v_{n-i}) = 6n - 7 + 12i$, for $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$,
$f^*(v_nv_{n-1}) = 6n + 7$, $f^*(v_iv'_i) = \begin{cases} f(v_i) - 2, & 1 \leq i \leq n - 1 \\ f(v_i) - 1 & i = n \text{ and} \end{cases}$
$f^*(v_iv''_i) = \begin{cases} f(v_i) + 2, & i \neq \left\lfloor \frac{n}{2} \right\rfloor \\ f(v_i) - 1 & i = \left\lfloor \frac{n}{2} \right\rfloor. \end{cases}$

Hence, f is a super geometric mean labeling of $G \odot S_2$. Thus the graph the graph $G \odot S_2$ is a super geometric mean graph. □

The super geometric mean labeling of $G_1 \odot S_2$ and $G_2 \odot S_2$ are shown in Figure 10.

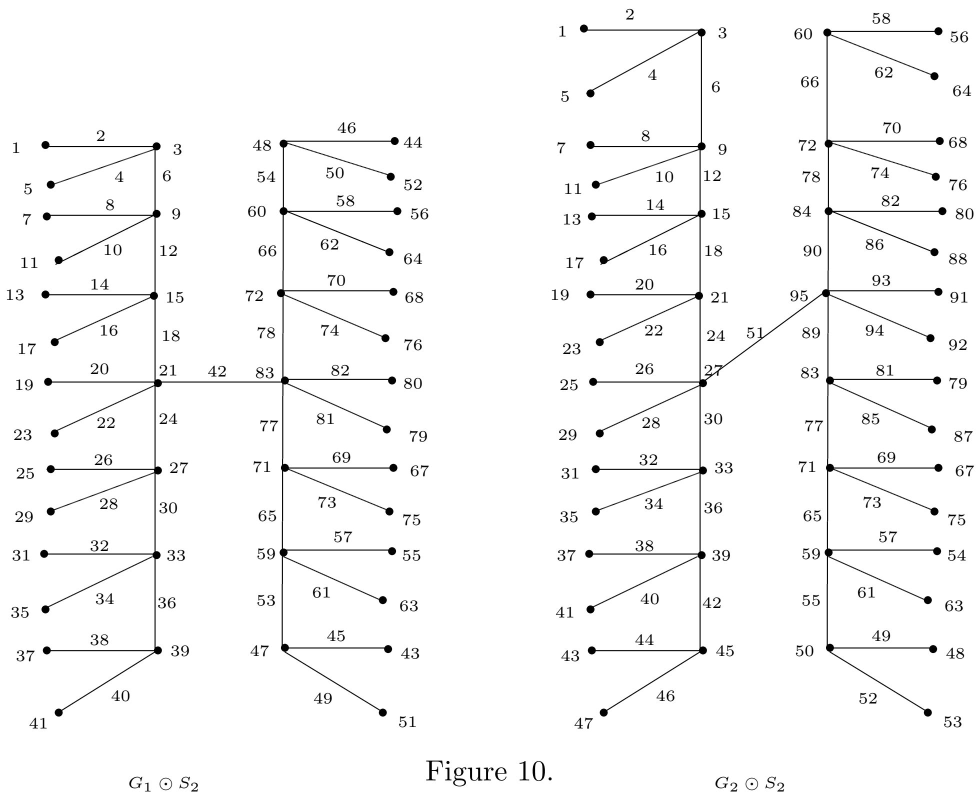

Figure 10.

Theorem 2.11. *$S(K_{1,3})$ is a super geometric mean graph.*

Proof. Let v_0, v_1, v_2 and v_3 be the vertices of $S(K_{1,3})$ in which v_0 is the central vertex and v_1, v_2 and v_3 are the pendant vertices of $K_{1,3}$.

Let the edges v_0v_1, v_0v_2 and v_0v_3 of $K_{1,3}$ be subdivided by p_1, p_2 and p_3 number of vertices respectively.

Let $v_0, v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots, v_{p_1+1}^{(1)}(= v_1), v_0, v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots, v_{p_2+1}^{(2)}(= v_2)$ and $v_0, v_1^{(3)}, v_2^{(3)}, v_3^{(3)}, \ldots, v_{p_3+1}^{(3)}(= v_3)$ be the vertices of $S(K_{1,3})$ and $v_0 = v_0^{(i)}$, for $1 \leq i \leq 3$. Let $e_j^{(i)} = v_{j-1}^{(i)}v_j^{(i)}, 1 \leq j \leq p_i + 1$ and $1 \leq i \leq 3$ be the edges of $S(K_{1,3})$ and it has $p_1 + p_2 + p_3 + 4$ vertices and $p_1 + p_2 + p_3 + 3$ edges with $p_1 \leq p_2 \leq p_3$.

Case (i) $p_1 = p_2$.

We define $f : V(S(K_{1,3})) \to \{1, 2, 3, \ldots, 2(p_1 + p_2 + p_3) + 7\}$ as follows:

$f(v_0) = 2(p_1 + p_2) + 5$, $f(v_j^{(1)}) = 2(p_1 + p_2) + 5 - 4j$, for $1 \leq j \leq p_1 + 1$,

$f(v_j^{(2)}) = 2(p_1 + p_2) + 6 - 4j$, for $1 \leq j \leq p_2 + 1$ and

$f(v_j^{(3)}) = 2(p_1 + p_2) + 5 + 2j$, for $1 \leq j \leq p_3 + 1$.

The induced edge labeling is as follows:

$f^*\left(v_j^{(1)}v_{j+1}^{(1)}\right) = 2(p_1+p_2)+3-4j$, for $1 \le j \le p_1$,

$f^*\left(v_j^{(2)}v_{j+1}^{(2)}\right) = 2(p_1+p_2)+4-4j$, for $1 \le j \le p_2$,

$f^*\left(v_j^{(3)}v_{j+1}^{(3)}\right) = 2(p_1+p_2)+6+2j$, for $1 \le j \le p_3$,

$f^*\left(v_0v_1^{(1)}\right) = 2(p_1+p_2)+3$, $f^*\left(v_0v_1^{(2)}\right) = 2(p_1+p_2)+4$ and

$f^*\left(v_0v_1^{(3)}\right) = 2(p_1+p_2)+6$.

Case (ii) $p_1 < p_2 < p_3$.

We define $f : V(S(K_{1,3})) \to \{1, 2, 3, \ldots, 2(p_1+p_2+p_3)+7\}$ as follows:

$f(v_0) = 2(p_1+p_2)+5$, $f(v_j^{(1)}) = 2(p_1+p_2)+6-4j$, for $1 \le j \le p_1+1$,

$$f(v_j^{(2)}) = \begin{cases} 2(p_1+p_2)+5-4j & 1 \le j \le p_1+1 \\ 2p_2+3-2j & p_1+2 \le j \le p_2+1 \text{ and} \end{cases}$$

$f(v_j^{(3)}) = 2(p_1+p_2)+5+2j$, for $1 \le j \le p_3+1$.

The induced edge labeling is as follows:

$f^*\left(v_j^{(1)}v_{j+1}^{(1)}\right) = 2(p_1+p_2)+4-4j$, for $1 \le j \le p_1$,

$$f^*\left(v_j^{(2)}v_{j+1}^{(2)}\right) = \begin{cases} 2(p_1+p_2)+3-4j & 1 \le j \le p_1 \\ 2p_2+2-2j & p_1+1 \le j \le p_2, \end{cases}$$

$f^*\left(v_j^{(3)}v_{j+1}^{(3)}\right) = 2(p_1+p_2)+6+2j$, for $1 \le j \le p_3$,

$f^*\left(v_0v_1^{(1)}\right) = 2(p_1+p_2)+4$,

$f^*\left(v_0v_1^{(2)}\right) = 2(p_1+p_2)+3$ and $f^*\left(v_0v_1^{(3)}\right) = 2(p_1+p_2)+6$.

Hence, f is a super geometric mean labeling of $S(K_{1,3})$. Thus the graph the graph $S(K_{1,3})$ is a super geometric mean graph. □

A Super geometric mean labeling of $S(K_{1,3})$ is shown in Figure 11.

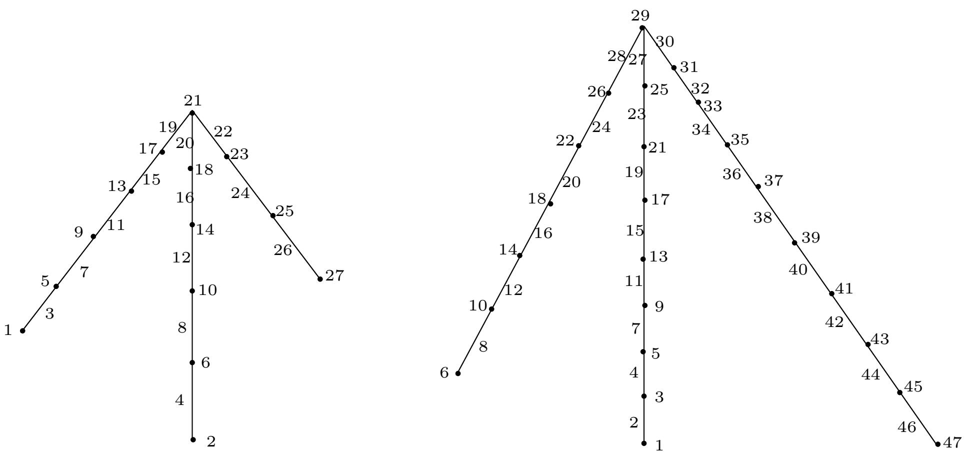

Figure 11.

Theorem 2.12. *$\widehat{G}(p_1, k_1, p_2, k_2, \ldots, k_{n-1}, p_n)$ is a super geometric mean graph with $p_i \neq 4$ for $2 \leq i \leq n$ and for any k_i.*

Proof. Let $\{v_j^{(i)}; 1 \leq i \leq n$ and $1 \leq j \leq p_i\}$ be the vertices of the n number of cycles in $\widehat{G}$ with $p_i \neq 4$ for $2 \leq i \leq n$.

Let $\{u_j^{(i)}; 1 \leq i \leq n-1$ and $1 \leq j \leq k_i\}$ be the vertices of the $(n-1)$ number of paths in $\widehat{G}$. For $1 \leq i \leq n-1$, the i^{th} cycle and i^{th} path are identified by a vertex $v^{(i)}_{\left(\frac{p_i+3}{2}\right)}$ and $u_1^{(i)}$ while p_i is odd and $v^{(i)}_{\left(\frac{p_i+2}{2}\right)}$ and $u_1^{(i)}$ while p_i is even and the i^{th} path and the $(i+1)^{th}$ cycle are identified by a vertex $u_{k_i}^{(i)}$ and $v_1^{(i+1)}$ in $\widehat{G}$.

We define $f : V\left(\widehat{G}\right) \to \{1, 2, 3, \ldots, \sum\limits_{i=1}^{n-1}(2p_i + 2k_i) + 2p_n - 3n + 3\}$ as follows:

When p_1 is odd,

$$f\left(v_j^{(1)}\right) = \begin{cases} 1 & j = 1 \\ 4j - 4 & 2 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4j - 5 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4j - 6 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \\ 4p_1 + 5 - 4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \leq j \leq p_1 \text{ and} \end{cases}$$

$$f\left(u_j^{(1)}\right) = f\left(v^{(1)}_{\left\lfloor \frac{p_1}{2} \right\rfloor + 2}\right) + 2j - 2, \quad \text{for } 2 \leq j \leq k_1.$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \begin{cases} 4j - 2 & 1 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4j - 3 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4j - 8 & j = \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \\ 4p_1 + 3 - 4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \leq j \leq p_1 - 1, \end{cases}$$

$$f^*\left(v_1^{(1)} v_{p_1}^{(1)}\right) = 3 \text{ and}$$

$$f^*\left(u_j^{(1)} u_{j+1}^{(1)}\right) = f\left(v^{(1)}_{\left\lfloor \frac{p_1}{2} \right\rfloor + 2}\right) + 2j - 1, \quad \text{for } 1 \leq j \leq k_1 - 1.$$

When p_1 is even,

$$f\left(v_j^{(1)}\right) = \begin{cases} 1 & j = 1 \\ 4j - 4 & 2 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4p_1 + 5 - 4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \leq j \leq p_1 \text{ and} \end{cases}$$

$$f\left(u_j^{(1)}\right) = f\left(v^{(1)}_{\left\lfloor \frac{p_1}{2} \right\rfloor + 1}\right) + 2j - 2, \quad \text{for } 2 \leq j \leq k_1.$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \begin{cases} 4j - 2 & 1 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4p_1 + 3 - 4j & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq j \leq p_1 - 1, \end{cases}$$

$$f^*\left(v_1^{(1)} v_{p_1}^{(1)}\right) = 3 \text{ and}$$

$$f^*\left(u_j^{(1)} u_{j+1}^{(1)}\right) = f\left(v^{(1)}_{\left\lfloor \frac{p_1}{2} \right\rfloor + 1}\right) + 2j - 1, \quad \text{for } 1 \leq j \leq k_1 - 1.$$

For $2 \leq i \leq n-1$,

$$f\left(u_j^{(i)}\right) = \begin{cases} f\left(v^{(i)}_{\left\lfloor \frac{p_i}{2} \right\rfloor + 2}\right) + 2j - 2 & 2 \leq j \leq k_i \text{ and } p_i \text{ is odd} \\ f\left(v^{(i)}_{\left\lfloor \frac{p_i}{2} \right\rfloor + 1}\right) + 2j - 2 & 2 \leq j \leq k_i \text{ and } p_i \text{ is even.} \end{cases}$$

For $2 \leq i \leq n$,

$$f\left(v_j^{(i)}\right) = \begin{cases} \begin{cases} f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 6 & 2 \leq j \leq \left\lfloor \frac{p_i}{2} \right\rfloor + 1 \text{ and } p_i \text{ is odd} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4p_i + 5 - 4j & \left\lfloor \frac{p_i}{2} \right\rfloor + 2 \leq j \leq p_i \text{ and } p_i \text{ is odd} \end{cases} \\ \begin{cases} f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 6 & 2 \leq j \leq \left\lfloor \frac{p_i}{2} \right\rfloor \text{ and } p_i \text{ is even} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 5 & j = \left\lfloor \frac{p_i}{2} \right\rfloor + 1 \text{ and } p_i \text{ is even} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 12 & j = \left\lfloor \frac{p_i}{2} \right\rfloor + 2 \text{ and } p_i \text{ is even} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4p_i + 5 - 4j & \left\lfloor \frac{p_i}{2} \right\rfloor + 3 \leq j \leq p_i \\ & \text{and } p_i \text{ is even.} \end{cases} \end{cases}$$

The induced edge labeling is as follows:
For $2 \leq i \leq n-1$,

$$f^*\left(u_j^{(i)} u_{j+1}^{(i)}\right) = \begin{cases} f\left(v^{(i)}_{\left\lfloor \frac{p_i}{2} \right\rfloor + 2}\right) + 2j - 1 & 1 \leq j \leq k_i - 1 \text{ and } p_i \text{ is odd} \\ f\left(v^{(i)}_{\left\lfloor \frac{p_i}{2} \right\rfloor + 1}\right) + 2j - 1 & 1 \leq j \leq k_i - 1 \text{ and } p_i \text{ is even.} \end{cases}$$

For $2 \leq i \leq n$,

$$f^*\left(v_j^{(i)} v_{j+1}^{(i)}\right) = \begin{cases} \begin{cases} f\left(u^{(i-1)}_{k_{i-1}}\right) + 1 & j = 1 \text{ and } p_i \text{ is odd} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 4 & 2 \leq j \leq \left\lfloor \frac{p_i}{2} \right\rfloor + 1 \\ & \text{and } p_i \text{ is odd} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4p_i + 3 - 4j & \left\lfloor \frac{p_i}{2} \right\rfloor + 2 \leq j \leq p_i - 1 \\ & \text{and } p_i \text{ is odd} \end{cases} \\ \begin{cases} f\left(u^{(i-1)}_{k_{i-1}}\right) + 1 & j = 1 \text{ and } p_i \text{ is even} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 4 & 2 \leq j \leq \left\lfloor \frac{p_i}{2} \right\rfloor - 1 \\ & \text{and } p_i \text{ is even} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 3 & j = \left\lfloor \frac{p_i}{2} \right\rfloor \\ & \text{and } p_i \text{ is even} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4j - 6 & j = \left\lfloor \frac{p_i}{2} \right\rfloor + 1 \\ & \text{and } p_i \text{ is even} \\ f\left(u^{(i-1)}_{k_{i-1}}\right) + 4p_i + 3 - 4j & \left\lfloor \frac{p_i}{2} \right\rfloor + 2 \leq j \leq p_i - 1 \\ & \text{and } p_i \text{ is even and} \end{cases} \end{cases}$$

$f^*\left(v_1^{(i)}v_{p_i}^{(i)}\right) = f\left(u_{k_{i-1}}^{(i-1)}\right) + 3.$

Hence, f is a super geometric mean labeling of $\widehat{G}(p_1, k_1, p_2, k_2, \dots, k_{n-1}, p_n)$.

Thus the graph $\widehat{G}(p_1, k_1, p_2, k_2, \dots, k_{n-1}, p_n)$ is a super geometric mean graph with $p_i \neq 4$ for $2 \leq i \leq n$ and for any k_i. □

A Super geometric mean labeling of $\widehat{G}(9, 5, 12, 3, 6)$ is shown in Figure 12.

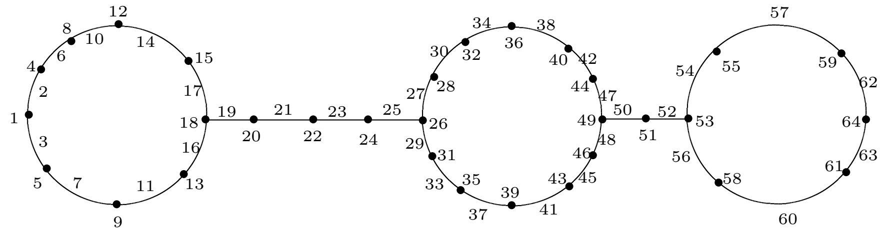

Figure 12.

Corollary 2.13. *$G^*(p_1, p_2, \dots, p_n)$ is a super geometric mean graph with $p_i \neq 4$, for all $2 \leq i \leq n$.*

Corollary 2.14. *Every triangular snake is a super geometric mean graph.*

Proof. By Corollary 2.13, if $p_1 = p_2 = p_3 = \cdots = p_n = 3$, then the triangular snake $T_n \cong G^*(3, 3, \dots, 3)$ is a super geometric mean graph. □

Corollary 2.15. *Tadpoles $T(n, k)$ is a super geometric mean graph, for $n \geq 3$ and $k \geq 2$.*

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