

An application of wave packet transform to scattering theory for Schrödinger equations with variable coefficients

Taisuke Yoneyama

(Received July 10, 2016; Revised October 4, 2016)

Abstract. In this paper, we show the existence of the wave operators for the Schrödinger equation with time-dependent variable coefficients by using the method introduced by the author and K. Kato [14] and give characterizations of their ranges by wave packet transform similar to those in [14].

AMS 2010 Mathematics Subject Classification. 35P25, 81U05.

Key words and phrases. Scattering theory, wave packet transform, wave operator.

§1. Introduction

In this paper, we prove the existence of the wave operators for the Schrödinger equation with time-dependent variable coefficients

$$H(t) = A(t) + V(t)$$

with unperturbed system $H_0 \equiv -1/2\Delta$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ and characterize their ranges. Here $A(t)$ is the differential operator defined by

$$A(t) = -\frac{1}{2} \sum_{j,k}^n \partial_{x_k} a_{jk}(t, x) \partial_{x_j}$$

and $V(t)$ is the multiplication operator of a function $V(t, x)$ and the domain $D(A(t)) = H^2(\mathbb{R}^n)$ is the Sobolev space of order two.

In the case that $a_{jk}(t, x) \equiv \delta_{jk}$, H. Kitada and K. Yajima [11] have characterized the ranges of the wave operators. In the previous paper [14], the author and K. Kato have proved the existence of the wave operators and have characterized their ranges by using the wave packet transform.

We assume that $a_{jk}(t, x)$ and $V(t, x)$ satisfy the following conditions:

Assumption (A). (i) (a_{jk}) is symmetric, that is, $a_{jk} = a_{kj}$ and in $C^\infty(\mathbb{R}_t \times \mathbb{R}_x^n; \mathbb{R})$ for $j, k = 1, \dots, n$.

(ii) There exists a positive constant ρ for any multi-index α such that

$$(1.1) \quad |\partial_x^\alpha (a_{jk}(t, x) - \delta_{jk})| \leq C_\alpha (1 + |x|)^{-1-\rho},$$

for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, where δ_{jk} is the Kronecker delta.

(iii) $V(t, x)$ is a real-valued Lebesgue measurable function on $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

(iv) There exists a positive constant $\tilde{\rho}$ such that

$$(1.2) \quad |V(t, x)| \leq C(1 + |x|)^{-1-\tilde{\rho}}$$

for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

We assume the existence of the propagator generated by $H(t)$.

Assumption (B). There exists a family of unitary operators $(U(t, \tau))_{(t, \tau) \in \mathbb{R}^2}$ in \mathcal{H} satisfying the following conditions.

(i) For $f \in \mathcal{H}$, $U(t, \tau)f$ is strongly continuous function with respect to t and satisfies

$$U(t, \tau')U(\tau', \tau) = U(t, \tau), \quad U(t, t) = I \quad \text{for all } t, \tau', \tau \in \mathbb{R},$$

where I is the identity operator on \mathcal{H} .

(ii) For $f \in H^2(\mathbb{R}^n)$, $U(t, \tau)f$ is strongly continuously differentiable in \mathcal{H} with respect to t and satisfies

$$\frac{\partial}{\partial t} U(t, \tau)f = -iH(t)U(t, \tau)f \quad \text{for all } t, \tau \in \mathbb{R}.$$

Remark 1. $H = A + V$ is self-adjoint operator on \mathcal{H} if $A = A(t)$ and $V = V(t)$ do not depend on t and $\det(a_{jk}) \neq 0$. Then Assumption (B) is satisfied by the Stone theorem.

Theorem 1. Suppose that (A) and (B) be satisfied. Then the wave operators

$$W_\pm^A(\tau) = \text{s-lim}_{t \rightarrow \pm\infty} U(t, \tau)^* e^{-i(t-\tau)H_0}$$

exist for any $\tau \in \mathbb{R}$, where $*$ denotes the adjoint of the operator.

Let \mathcal{S} be the Schwartz space of all rapidly decreasing functions on \mathbb{R}^n and \mathcal{S}' be the space of tempered distributions on \mathbb{R}^n . For positive constants a and R , we put $\Gamma_{a,R} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x| \geq R \text{ or } |\xi| \leq a \text{ or } |\xi| \geq R\}$.

Definition 1 (Wave packet transform). Let $\varphi \in \mathcal{S} \setminus \{0\}$ and $f \in \mathcal{S}'$. We define the wave packet transform $W_\varphi f(x, \xi)$ of f with the wave packet generated by a function φ as follows:

$$W_\varphi f(x, \xi) = \int_{\mathbb{R}^n} \overline{\varphi(y-x)} f(y) e^{-iy\xi} dy \quad \text{for } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Its inverse is the operator W_φ^{-1} which is defined by

$$W_\varphi^{-1} F(x) = \frac{1}{(2\pi)^n \|\varphi\|_{\mathcal{H}}^2} \int \int_{\mathbb{R}^{2n}} \varphi(x-y) F(y, \xi) e^{ix\xi} dy d\xi$$

for $x \in \mathbb{R}^n$ and a function $F(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$. This transform is introduced by Córdoba and C. Fefferman ([2]).

Definition 2. Let $\tau \in \mathbb{R}$ and $\Phi \in \mathcal{S}_0 \equiv \left\{ \Phi \in \mathcal{S} \mid \|\Phi\|_{\mathcal{H}} = 1 \text{ and } \hat{\Phi}(0) \neq 0 \right\}$ and we put $\Phi(t) = e^{-itH_0} \Phi$. We define $\tilde{D}_{A_{scat}}^{\pm, \Phi}(\tau)$ by the set of all functions in \mathcal{H} such that

$$\lim_{t \rightarrow \pm\infty} \left\| \chi_{\Gamma_{a,R}}(x - (t - \tau)\xi, \xi) W_{\Phi(t-\tau)} [U(t, \tau)f](x, \xi) \right\|_{L^2(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)} = 0$$

for some positive constants a and R , where $\chi_A(x)$ is the characterization function of a measurable set A , which is defined by $\chi_A(x) = 1$ on A and $\chi_A(x) = 0$ otherwise. For $\tau \in \mathbb{R}$, $D_{A_{scat}}^{\pm, \Phi}(\tau)$ is defined by the closure of $\tilde{D}_{A_{scat}}^{\pm, \Phi}(\tau)$ in the topology of \mathcal{H} .

Theorem 2. Suppose that (A) and (B) be satisfied. Then the ranges of the wave operators $\mathcal{R}(W_\pm^A(\tau))$ coincide with $D_{A_{scat}}^{\pm, \Phi}(\tau)$ for any $\Phi \in \mathcal{S}_0$. In particular, $D_{A_{scat}}^{\pm, \Phi}(\tau)$ is independent of Φ .

We use the following notations throughout the paper. $i = \sqrt{-1}$, $n \in \mathbb{N}$. For a subset Ω in \mathbb{R}^n or in \mathbb{R}^{2n} , the standard inner product and the standard norm on $L^2(\Omega)$ are denoted by $(f, g)_{L^2(\Omega)} = \int_\Omega f \bar{g} dx$ and $\|f\|_{L^2(\Omega)} = (f, f)_{L^2(\Omega)}^{1/2}$ for $f, g \in L^2(\Omega)$, respectively. We write $\partial_{x_j} = \partial/\partial x_j$, $\partial_t = \partial/\partial t$, $L_{x, \xi}^2 = L^2(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, $(\cdot, \cdot) = (\cdot, \cdot)_{L_{x, \xi}^2}$, $\|\cdot\| = \|\cdot\|_{L_{x, \xi}^2}$, $\langle t \rangle = 1 + |t|$, $\|f\|_{\Sigma(l)} = \sum_{|\alpha+\beta|=l} \|x^\beta \partial_x^\alpha f\|_{\mathcal{H}}$ and $W_\varphi u(t, x, \xi) = W_\varphi[u(t)](x, \xi)$. $\|\cdot\|_{\mathcal{B}(X)}$ denotes the operator norm on the Hilbert space X . \mathcal{F} and \mathcal{F}^{-1} are the Fourier transform and the inverse Fourier transform defined by $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$ and $\mathcal{F}^{-1}f(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi$, respectively. We often write $\{\xi = 0\}$ as $\{(x, \xi) \in \mathbb{R}^{2n} \mid \xi = 0\}$. For sets A and B , $A \setminus B$ denotes the set $\{a \in A \mid a \notin B\}$. $F(\cdots)$ denotes the multiplication operator of a function $\chi_{\{x \in \mathbb{R}^n \mid \cdots\}}(x)$.

The idea of the proofs of the main theorems is as follows. Splitting the principal part $H(t)$ into $(A(t) - H_0) + (H_0 + V(t))$, applying the wave packet

transform to the equation and estimating the terms including $A(t) - H_0$ and $H_0 + V(t)$ by using the wave packet transform, we prove the existence of the wave operators and characterize their ranges. In order to estimate the term including $A(t) - H_0$, we use the duality argument and Lemma 3. That is, taking $f \in W_{\Phi}^{-1}[C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\})]$, we have

$$(1.3) \quad \|(A(t) - H_0)e^{-itH_0}f\|_{\mathcal{H}} \leq \|(A(t) - H_0)e^{-itH_0}W_{\Phi}^{-1}\chi_{\Gamma_{a,R}^c}\|_{\mathcal{B}(\mathcal{H})}\|W_{\Phi}f\|_{\mathcal{H}}$$

for some positive numbers a and R . Lemma 3 shows that

$$(1.4) \quad \|(A(t) - H_0)e^{-itH_0}W_{\Phi}^{-1}\chi_{\Gamma_{a,R}^c}\|_{\mathcal{B}(\mathcal{H})} \leq C\langle t \rangle^{-1-\rho},$$

which is the key of the proofs of the main theorems. The term including $H_0 + V(t)$ can be estimated by the same argument as in [14]. The existence of the wave operators and the characterizations of their ranges are obtained by (1.3), (1.4) and the Cook-Kuroda method ([1], [12]).

In the case that the coefficients depend only on x , R. Melrose [13], J. Wunsch and A. Hassell [5] and K. Ito and S. Nakamura [8], study the microlocal singularity with the solution of the equation. In [13], they characterize the scattering operator, which is defined by the wave operator and its adjoint operator. In [5], they use the same modifier as the modified wave operator in [3]. In [8], they represent the wave operators by the Fourier integral operator which is introduced by L. Hörmander [6]. All the above works do not treat the characterization of the ranges of the wave operators.

The plan of the paper is as follows. In section 2, we recall properties of the wave packet transform and prove a propagation estimate using the wave packet transform. In section 3, we give a proof of the existence of the wave operators (Theorem 1) and characterize the ranges of them (Theorem 2).

§2. Preliminaries

In this section, we recall the representation via the wave packet transform which is used in the proofs of the main theorems and give a propagation estimate via the wave packet transform.

Proposition 1. Let $\varphi, \psi \in \mathcal{S} \setminus \{0\}$ and $f \in \mathcal{S}'$. Then the wave packet transform $W_{\varphi}f(x, \xi)$ has the following properties:

- (i) $W_{\varphi}f(x, \xi) \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$.
- (ii) If $f, g \in \mathcal{H}$, we have

$$(2.1) \quad (W_{\varphi}f, W_{\psi}g) = \overline{(\varphi, \psi)_{\mathcal{H}}(f, g)_{\mathcal{H}}} = (\psi, \varphi)_{\mathcal{H}}(f, g)_{\mathcal{H}}.$$

(iii) If $(\psi, \varphi)_{\mathcal{H}} \neq 0$, the inversion formula

$$(\psi, \varphi)_{\mathcal{H}}^{-1} W_{\psi}^{-1} [W_{\varphi} f] = f$$

holds for $f \in \mathcal{S}'$.

Proof. See [4]. □

Let $\varphi_0 \in \mathcal{S} \setminus \{0\}$, $\tilde{\varphi}(t, t_0, x) = e^{-i(t-t_0)H_0} \varphi_0(x)$ and $\psi \in \mathcal{H}$ for $t, t_0 \in \mathbb{R}$. Since

$$\begin{aligned} & W_{\tilde{\varphi}(t, t_0)} [H_0 u](t, x, \xi) \\ &= W_{H_0 \tilde{\varphi}(t, t_0)} u(t, x, \xi) - i\xi \cdot \nabla_x W_{\tilde{\varphi}(t, t_0)} u(t, x, \xi) + \frac{1}{2} |\xi|^2 W_{\tilde{\varphi}(t, t_0)} u(t, x, \xi) \end{aligned}$$

and

$$W_{\tilde{\varphi}(t, t_0)} [i\partial_t u](t, x, \xi) = i\partial_t W_{\tilde{\varphi}(t, t_0)} u(t, x, \xi) + W_{i\partial_t \tilde{\varphi}(t, t_0)} u(t, x, \xi)$$

for $u \in C(\mathbb{R}; \mathcal{S})$, $i\partial_t U(t, t_0)\psi = H(t)U(t, t_0)\psi$ is transformed to

$$\left(i\partial_t + i\xi \cdot \nabla_x - \frac{1}{2} |\xi|^2 \right) W_{\tilde{\varphi}(t, t_0)} [U(t, t_0)\psi](t, x, \xi) = \tilde{G}(t, t_0, x, \xi, U(t, t_0)\psi)$$

for $t, t_0 \in \mathbb{R}$, where $\tilde{G}(t, t_0, x, \xi, \psi) = W_{\tilde{\varphi}(t, t_0)} [H_0 - (A(t) - V(t))\psi](x, \xi)$. We have by the method of characteristic curve that

$$\begin{aligned} (2.2) \quad & W_{\tilde{\varphi}(t, t_0)} [U(t, t_0)\psi](x, \xi) = e^{-i\frac{1}{2}(t-t_0)|\xi|^2} W_{\varphi_0} \psi(x - (t-t_0)\xi, \xi) \\ & - i \int_{t_0}^t e^{-i\frac{1}{2}(t-s)|\xi|^2} \tilde{G}(s, t_0, x - (t-s)\xi, \xi, U(s, t_0)\psi) ds. \end{aligned}$$

In particular, we have

$$(2.3) \quad W_{\tilde{\varphi}(t, 0)} [e^{-itH_0} \psi](x + t\xi, \xi) = e^{-i\frac{1}{2}t|\xi|^2} W_{\varphi_0} \psi(x, \xi).$$

Taking $V \equiv 0$, $t = 0$, $t_0 = t$ and φ_0 as $e^{-itH_0} \varphi_0$, we obtain the following representation of e^{itH_0} :

$$(2.4) \quad W_{\varphi_0} [e^{itH_0} \psi](x, \xi) = e^{i\frac{1}{2}t|\xi|^2} W_{\tilde{\varphi}(t, 0)} \psi(x + t\xi, \xi).$$

Substituting ψ in (2.2) for $e^{-it'H_0} \psi$ and φ_0 as $e^{-it'H_0} \varphi_0$, we have for $t, t' \in \mathbb{R}$

(2.5)

$$\begin{aligned} & W_{\varphi(t)} [U(t, t') e^{-it'H_0} \psi](x + t\xi, \xi) \\ &= e^{-i\frac{1}{2}(t-t')|\xi|^2} W_{\varphi(t')} [e^{-it'H_0} \psi](x + t'\xi, \xi) \\ & \quad + i \int_t^{t'} e^{-i\frac{1}{2}(t-s)|\xi|^2} G(s, x + s\xi, \xi, U(s, t') e^{-it'H_0} \psi) ds. \\ &= e^{-i\frac{1}{2}t|\xi|^2} W_{\varphi_0} \psi(x, \xi) + i \int_t^{t'} e^{-i\frac{1}{2}(t-s)|\xi|^2} G(s, x + s\xi, \xi, U(s, t') e^{-it'H_0} \psi) ds, \end{aligned}$$

where $\varphi(t, x) = \tilde{\varphi}(t, 0, x)$, $G(s, x, \xi, \psi) = \tilde{G}(s, 0, x, \xi, \psi)$. Integration by parts and the fact that $\nabla e^{-itH_0} = e^{-itH_0} \nabla$ yield that

$$\begin{aligned}
 (2.6) \quad & -2G(t, x, \xi, \psi) \\
 &= \int \overline{\varphi(t, y-x)} (\Delta - 2A(t) - 2V(t)) \psi(y) e^{-i\xi y} dy \\
 &= \sum_{j,k=1}^n \int \overline{\varphi(t, y-x)} \{ \partial_{y_k} (\delta_{jk} - a_{jk}(t, y)) \partial_{y_j} - 2V(t, y) \} \psi(y) e^{-i\xi y} dy \\
 &= \sum_{\substack{|\alpha|=2, |\alpha_2| \leq 1 \\ \alpha = \alpha_1 + \alpha_2 + \alpha_3}} \int \overline{(\partial_y^{\alpha_1} \varphi)(t, y-x)} \partial_y^{\alpha_2} (\delta_{jk} - a_{jk}(t, y)) \\
 &\quad \times \psi(y) e^{-i\xi y} (-i\xi)^{\alpha_3} e^{-i\xi y} dy - 2 \int \overline{\varphi(t, y-x)} V(t, y) \psi(y) e^{-i\xi y} dy.
 \end{aligned}$$

The following well-known lemma is used in the proof of Lemma 3.

Lemma 2. Let L_1, L_2 be positive constants and $f \in \mathcal{S}$. Suppose that $\text{supp } \hat{f} \subset \{ \xi \in \mathbb{R}^n \mid L_1 < |\xi| < L_2 \}$. Then for any non-negative integer l , there exists a positive constant C_l such that

$$\left| \left(\left| F \left(|x| < \frac{L_1}{2} t \right) \right| + \left| F \left(|x| > 2L_2 t \right) \right| \right) e^{-itH_0} f(x) \right| \leq C_l (1 + |x| + |t|)^{-l} \|f\|_{\Sigma(l)}$$

for any $t > 0$ and $x \in \mathbb{R}^n$.

Proof. See [7]. □

The following propagation estimate plays an important role in the proof of the main theorems.

Lemma 3. Suppose that (A) be satisfied. Let a and R be positive constants. Then for any multi-index α, β , a constant $L \in (0, a/6]$ and $\varphi_0 \in \mathcal{S} \setminus \{0\}$ with $\text{supp } \hat{\varphi}_0 \subset \{ \xi \in \mathbb{R}^n \mid L/2 < |\xi| < L \}$, there exists a positive constant C satisfying

$$(2.7) \quad \left\| \xi^\beta W_{\varphi(s)} [\partial^\alpha (a_{jk}(s) - \delta_{jk}) \psi] (x + s\xi, \xi) \right\|_{L^2(\mathbb{R}^{2n} \setminus \Gamma_{a,R})} \leq C \langle s \rangle^{-1-\rho} \|\psi\|_{\mathcal{H}}$$

for any $s \geq 0$ and any $\psi \in \mathcal{H}$.

Proof. Let $\sigma = a/6$ and let l be an integer satisfying $l \geq \rho + 1 + (n + 1)/2$. We put $\zeta_\sigma(t, x) = \chi_0(\frac{1}{\sigma(t)} x) \{ \partial_x^\alpha (a_{jk}(t, x) - \delta_{jk}) \}$ where $\chi_0 \in C^\infty(\mathbb{R}^n)$ satisfies

$\chi_0(x) = 1$ for $|x| \geq 1$ and $\chi_0(x) = 0$ for $|x| \leq 1/2$. Thus by (1.1) for any multi-index α , there exists a positive constant C_α such that $|\zeta_\sigma(t, x)| \leq C_\alpha \langle t \rangle^{-1-\rho}$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

For $(x, \xi) \in \mathbb{R}^{2n} \setminus \Gamma_{a,R}$, we put $\mathcal{O}_1 \equiv \{y \in \mathbb{R}^n \mid |y - (x + s\xi)| \leq as/3\}$ and $\mathcal{O}_2 \equiv \{y \in \mathbb{R}^n \mid |y - (x + s\xi)| > as/3\}$. If $(x, \xi) \in \mathbb{R}^{2n} \setminus \Gamma_{a,R}$, $s \geq \max\{3R/a, 3\}$ and $y \in \mathcal{O}_1$, we have

$$(2.8) \quad |y| \geq (as - R) - \frac{as}{3} \geq \sigma \langle s \rangle.$$

Since $\zeta_\sigma(s, y) = \partial_y^\alpha (a_{jk}(s, y) - \delta_{jk})$ for $|y| \geq \sigma \langle s \rangle$, we have by Plancherel's theorem and (2.8)

$$\begin{aligned} & \left\| \xi^\beta \int_{y \in \mathcal{O}_1} \overline{e^{-isH_0} \varphi_0(y - (x + s\xi))} \partial_y^\alpha (a_{jk}(s, y) - \delta_{jk}) \psi(y) e^{-i\xi y} dy \right\|_{L^2(\mathbb{R}^{2n} \setminus \Gamma_{a,R})} \\ & \leq R^{|\beta|} \left\| \int_{y \in \mathcal{O}_1} \overline{e^{-isH_0} \varphi_0(y - (x + s\xi))} \zeta_\sigma(s, y) \psi(y) e^{-i\xi y} dy \right\|_{L^2(\mathbb{R}^{2n} \setminus \Gamma_{a,R})} \\ & \leq R^{|\beta|} \left\| \overline{e^{-isH_0} \varphi_0(y - x)} \zeta_\sigma(s, y) \psi(y) \right\|_{L^2(\mathbb{R}_x^n \times \mathbb{R}_y^n)} \\ & \leq C \langle s \rangle^{-1-\rho} \|\psi\|_{\mathcal{H}}. \end{aligned}$$

On the other hand, Lemma 2 shows that

$$\begin{aligned} & \left\| \xi^\beta \int_{y \in \mathcal{O}_2} \overline{e^{-isH_0} \varphi_0(y - (x + s\xi))} \partial_y^\alpha (a_{jk}(s, y) - \delta_{jk}) \psi(y) e^{-i\xi y} dy \right\| \\ & \leq R^{|\beta|} \left\| \left(F \left(|y - x| > \frac{as}{3} \right) \overline{e^{-isH_0} \varphi_0(y - x)} \right) \partial_y^\alpha (a_{jk}(s, y) - \delta_{jk}) \psi(y) \right\|_{L^2(\mathbb{R}_{x,y}^{2n})} \\ & \leq C \langle s \rangle^{-l+(n+1)/2} \|\varphi_0\|_{\Sigma(l)} \left\| \langle y - x \rangle^{-(n+1)/2} \partial_y^\alpha (a_{jk}(s, y) - \delta_{jk}) \psi(y) \right\|_{L^2(\mathbb{R}_{x,y}^{2n})} \\ & \leq C \langle s \rangle^{-1-\rho} \|\psi\|_{\mathcal{H}}, \end{aligned}$$

since $\text{supp } \hat{\varphi}_0 \subset \{\xi \in \mathbb{R}^n \mid 0 < |\xi| < a/6\}$. □

The following lemma is obtained in [14].

Lemma 4. Suppose that (A) be satisfied. Let a and R be positive constants. Then for any $L \in (0, a/6]$ and $\varphi_0 \in \mathcal{S} \setminus \{0\}$ with $\text{supp } \hat{\varphi}_0 \subset \{\xi \in \mathbb{R}^n \mid L/2 < |\xi| < L\}$, there exists a positive constant C satisfying

$$(2.9) \quad \left\| W_{\varphi(s)} [V(s)\psi](x + s\xi, \xi) \right\|_{L^2(\mathbb{R}^{2n} \setminus \Gamma_{a,R})} \leq C \langle s \rangle^{-1-\tilde{\rho}} \|\psi\|_{\mathcal{H}}$$

for any $s \geq 0$ and any $\psi \in \mathcal{H}$.

Proof. See [14]. □

§3. Proofs of main theorems

In this section, we prove Theorem 1 and 2 by using Lemma 3 and the representation via the wave packet transform which are proved in the previous section. We give a proof only for the case that $V \equiv 0$ since the last term of (2.6) can be estimated by Lemma 4.

Proof of Theorem 1. Substituting $A(t)$ for $A(t-\tau)$, it suffices to show the case $\tau = 0$. We prove the existence in the case $t \rightarrow +\infty$ only.

Let $\Phi \in \mathcal{S}_0$ and $u_0 \in \mathcal{H}$. In order to apply the Cook-Kuroda method, we shall prove

$$(3.1) \quad \|(A(t) - H_0)e^{-itH_0}u_0\|_{\mathcal{H}} \leq C\langle t \rangle^{-1-\rho}$$

for $t \geq 0$ and $u_0 \in W_{\Phi}^{-1}[C_0^{\infty}(\mathbb{R}^{2n} \setminus \{\xi = 0\})]$.

Let a and R be positive constants satisfying

$$(3.2) \quad \text{supp } W_{\Phi}u_0 \subset \mathbb{R}^{2n} \setminus \Gamma_{a,R}$$

and $\varphi_0 \in \mathcal{S} \setminus \{0\}$ satisfying

$$(3.3) \quad \text{supp } \hat{\varphi}_0 \subset \left\{ \xi \in \mathbb{R}^n \mid \frac{L}{2} < |\xi| < L \right\} \text{ with } 0 < L \leq \frac{a}{6}, C_{\Phi, \varphi_0} \equiv \overline{(\Phi, \varphi_0)}_{\mathcal{H}} \neq 0.$$

By (2.1) and (2.4), we have for $t \geq 0$

$$(3.4) \quad \begin{aligned} ((A(t) - H_0)e^{-itH_0}u_0, \psi)_{\mathcal{H}} &= (u_0, e^{itH_0}(A(t) - H_0)\psi)_{\mathcal{H}} \\ &= C_{\Phi, \varphi_0}^{-1} (W_{\Phi}u_0, W_{\varphi_0}[e^{itH_0}(A(t) - H_0)\psi]) \\ &= C_{\Phi, \varphi_0}^{-1} (W_{\Phi}u_0, e^{it|\xi|^2/2}G(t, x + t\xi, \xi, \psi)). \end{aligned}$$

Lemma 3, (2.6) and (3.3) show that for $t' \geq t > 0$

$$\begin{aligned} |(W_{\Phi}u_0, e^{it|\xi|^2/2}G(t, x + t\xi, \xi, \psi))| \\ \leq \|W_{\Phi}u_0\| \|G(t, x + t\xi, \xi, \psi)\|_{L^2(\mathbb{R}^{2n} \setminus \Gamma_{a,R})} \\ \leq C \|u_0\|_{\mathcal{H}} \langle t \rangle^{-1-\rho} \|\psi\|_{\mathcal{H}} \\ \leq C \langle t \rangle^{-1-\rho} \|u_0\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}}. \end{aligned}$$

The above inequality and (3.4) imply (3.1).

Hence we obtain the existence of $W_{+}^A(0)u_0$ since $W_{\Phi}^{-1}(C_0^{\infty}(\mathbb{R}^{2n} \setminus \{\xi = 0\}))$ is dense in \mathcal{H} . \square

Proof of Theorem 2. From the same reason in the proof of Theorem 1, it suffices to show the case $\tau = 0$ and $t \rightarrow +\infty$ only. Let $\Phi \in \mathcal{S}_0$. We abbreviate $W_+ = W_+^A(0)$, $D_{scat}^+ = D_{A_{scat}}^{+, \Phi}(0)$ and $\tilde{D}_{scat}^+ = \tilde{D}_{A_{scat}}^{+, \Phi}(0)$ until the end of the proof.

The part that $\mathcal{R}(W_+) \subset D_{scat}^+$ can be proved by the same argument in Proposition 6 of [14]. That is, for $g \in W_{\Phi}^{-1}(C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\}))$, we can prove $W_+g \in \tilde{D}_{scat}^+$.

Thus for completing the proof of Theorem 2, we shall prove the other part that

$$(3.5) \quad \mathcal{R}(W_+) \supset D_{scat}^+.$$

Since D_{scat}^\pm is the closure of \tilde{D}_{scat}^\pm and U is a bounded operator on \mathcal{H} , it suffices to prove the existence of the inverse wave operator $W_+^{-1}u_0 = \lim_{t \rightarrow +\infty} e^{itH_0}U(t, 0)u_0$ for $u_0 \in \tilde{D}_{scat}^+$.

Let $u_0 \in \tilde{D}_{scat}^+$ and let a and R be positive constants satisfying

$$(3.6) \quad \lim_{t \rightarrow \infty} \|\chi_{\Gamma_{a,R}}(x - t\xi, \xi)W_{\Phi(t)}[U(t, 0)u_0]\| = 0.$$

We abbreviate $\Gamma = \Gamma_{a,R}$ and $\Gamma^c = \mathbb{R}^{2n} \setminus \Gamma$ until the end of the proof. Taking $\varphi_0 \in \mathcal{S} \setminus \{0\}$ satisfying (3.3), we have for $t' \geq t > 0$

$$(3.7) \quad \begin{aligned} & \left(e^{itH_0}U(t, 0)u_0 - e^{it'H_0}U(t', 0)u_0, \psi \right)_{\mathcal{H}} \\ &= C_{\Phi, \varphi_0}^{-1} \left(\chi_{\Gamma}(x - t\xi, \xi)W_{\Phi(t)}[U(t, 0)u_0], W_{\varphi(t)}[e^{-itH_0}\psi - U(t, t')e^{-it'H_0}\psi] \right) \\ &+ C_{\Phi, \varphi_0}^{-1} \left(W_{\Phi(t)}[U(t, 0)u_0], \chi_{\Gamma^c}(x - t\xi, \xi) \left(W_{\varphi(t)}[e^{-itH_0}\psi - U(t, t')e^{-it'H_0}\psi] \right) \right). \end{aligned}$$

Using (3.6), we obtain

$$(3.8) \quad \begin{aligned} & \sup_{\|\psi\|_{\mathcal{H}}=1} |(\text{the first term of the right hand side in (3.7)})| \\ & \leq \sup_{\|\psi\|_{\mathcal{H}}=1} \|\chi_{\Gamma}(x - t\xi, \xi)W_{\Phi(t)}[U(t, 0)u_0]\| \|W_{\varphi(t)}[e^{-itH_0}\psi - U(t, t')e^{-it'H_0}\psi]\| \\ & \leq 2\|\varphi_0\|_{\mathcal{H}} \|\chi_{\Gamma}(x - t\xi, \xi)W_{\Phi(t)}[U(t, 0)u_0]\| \rightarrow 0 \text{ as } t \rightarrow \infty. \end{aligned}$$

By Lemma 3, (2.3), (2.5), (2.6) and (3.3), we have for $t' \geq t > 0$

(3.9)

$$\begin{aligned}
 & |(\text{the second term of the right hand side in (3.7)})| \\
 = & \left| \left(W_{\Phi(t)}[U(t, 0)u_0](x + t\xi, \xi), \right. \right. \\
 & \left. \left. \chi_{\Gamma^c} \int_t^{t'} e^{-i\frac{1}{2}(t-s)|\xi|^2} G(s, x + s\xi, \xi, U(s, t')e^{-it'H_0}\psi) ds \right) \right| \\
 \leq & C \|u_0\|_{\mathcal{H}} \int_t^{t'} \left\| G(s, x + s\xi, \xi, U(s, t')e^{-it'H_0}\psi) \right\|_{L^2(\Gamma^c)} ds \\
 \leq & C \|u_0\|_{\mathcal{H}} \int_t^{t'} \langle s \rangle^{-1-\rho} ds \|\psi\|_{\mathcal{H}}.
 \end{aligned}$$

(3.5) follows from (3.8) and (3.9). \square

Acknowledgments

The author is grateful to Professor Keiichi Kato for useful discussions. I also thank the referee for several valuable comments suggestions.

References

- [1] J. Cook, *Convergence to the Møller wave matrix*, J. Math. Phys. **36** (1957), 82–87.
- [2] A. Córdoba and C. Fefferman, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations **3** (1978), 979–1005.
- [3] J. D. Dollard, *Quantum-mechanical scattering theory for short-range and Coulomb interactions*, Rocky Mountain J. Math. **1** (1971), 5–88.
- [4] K. Gröchenig, *Foundations of time-frequency analysis* (Appl. Numer. Harmon. Anal., Birkhäuser Boston, Inc., Boston, MA, 2001).
- [5] A. Hassell and J. Wunsch, *On the structure of the Schrödinger propagator*, in Partial differential equations and inverse problems, Contemp. Math., Amer. Math. Soc., Providence, RI **362** (2004), 199–209.
- [6] L. Hörmander, *Fourier integral operators. I*, Acta Math. **127** (1971), No. 1–2, 79–183.
- [7] L. Hörmander, *The existence of wave operators in scattering theory*, Math. Z. **146** (1976), No. 1, 69–91.

- [8] K. Ito and S. Nakamura, *Remarks on the fundamental solution to Schrödinger equation with variable coefficients*, Ann. Inst. Fourier (Grenoble) **62**, No. 3 (2012), 1091–1121.
- [9] K. Kato, M. Kobayashi and S. Ito, *Remark on wave front sets of solutions to Schrödinger equation of a free particle and a harmonic oscillator*, SUT J. of Math. **47**, No. 2 (2011), 175–183.
- [10] K. Kato, M. Kobayashi and S. Ito, *Representation of Schrödinger operator of a free particle via short time Fourier transform and its applications*, Tohoku Math. J. **64**, No. 2 (2012), 223–231.
- [11] H. Kitada and K. Yajima, *A scattering theory for time-dependent long-range potentials*, Duke Math. J. **49** (1982), 341–376.
- [12] S. T. Kuroda, *On the existence and the unitary property of the scattering operator*, Nuovo Cimento, X. **12** (1959), 431–454.
- [13] R. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, Spectral and scattering theory (Sanda, 1992), Lecture Notes in Pure and Appl. Math. **161** (1994), 85–130.
- [14] T. Yoneyama and K. Kato, *Characterization of the ranges of wave operators for Schrödinger equations with time-dependent short-range potentials via wave packet transform*, *arXiv*: 1510.03255v2.

Taisuke Yoneyama
Department of Mathematics, Tokyo University of Science
Kagurazaka 1-3, Shinjuku, Tokyo 162-0825, Japan
E-mail: 1114705@ed.tus.ac.jp