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A measure of departure from second-order marginal symmetry for multi-way tables with nominal categories

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Abstract. For multi-way tables, Bhapkar and Darroch (1990) gave the second-order marginal symmetry model. The present paper proposes a measure to represent the degree of departure from the second-order marginal symmetry model. The measure is expressed as the weighted sum of the Shannon entropy. The paper also gives the approximate confidence interval of the measure, and shows relationship between the measure and the trivariate normal distribution. Examples are given.

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§1. Introduction

Consider an r^T contingency table ($T \geq 3$) with nominal categories. Let X_k denote the kth variable ($k = 1, \ldots, T$).

The first-order marginal symmetry (MS(1)) model is defined by

$$p_i^{(1)} = p_i^{(2)} = \cdots = p_i^{(T)} \qquad (i = 1, \ldots, r),$$

where $p_i^{(k)} = \Pr(X_k = i)$ (Agresti, 2013, p.439; Tahata and Tomizawa, 2014).

The second-order marginal symmetry (MS(2)) model is defined by

$$\begin{cases} p_{ij}^{(s,t)} = p_{ij}^{(1,2)} \\ p_{ij}^{(s,t)} = p_{ji}^{(s,t)} \end{cases} \qquad (i, j = 1, \ldots, r; 1 \leq s < t \leq T),$$

where $p_{ij}^{(s,t)} = \Pr(X_s = i, X_t = j)$ (Bhapkar and Darroch, 1990). If the MS(2) model holds then the MS(1) model holds, but the converse is not always true. Thus the MS(2) model has stronger constraint than the MS(1) model.

The data in Tables 1(a) and 1(b) taken from the 2014 General Social Survey (Smith et al., 2014) are conducted by the National Opinion Research Center at the University of Chicago. Tables 1(a) and 1(b) describe the cross-classifications of subjects' opinions about government spending on the environment, health, assistance to big city and law enforcement in 2004 and 2014, respectively. The response categories are (1) 'too little', (2) 'about right' and (3) 'too much'. So Tables 1(a) and 1(b) are the 3^4 contingency tables.

For these data, some statisticians may be interested in determining degrees of various unbalances of the opinions among items. There is one considerable analysis as comparing (first-order) marginal distributions of the opinions, e.g., analyzing which item tends to be regarded 'too little' by relatively more subjects than the other items. Moreover there is one of the other analyses as comparing higher-order marginal distributions than first-order. For example, determining whether there is MS(2) may be motivated by joint distributions of pairs of opinions being able to unbalanced even though the first-order marginal distributions are similar. Indeed, the analyses for second-order marginal structure are developed by several statisticians. Becker and Agresti (1992) discussed the log-linear models that describe second-order marginal structure for determining degree of agreement among multiple observers. Fleiss, Levin and Paik (2003, Chap.18), and Agresti (2013, Sec.11.5) reviewed the measurement for pairwise agreement or multiple agreement. Balagtas, Becker and Lang (1995) analyzed the crossover experiment data with three-treatments, three-periods and binary responses using the models for log-odds-ratio of second-order marginal probability. Lang and Agresti (1994) discussed simultaneously modeling for joint and any-order marginal distributions.

For the data in Tables 1(a) and 1(b), the MS(1) model indicates that first-order marginal distributions of the opinions are identical among items. The MS(2) model indicates (i) the probabilities that a subject has opinion k about both items are identical among all pairs of items, and (ii) the probability that a subject has opinion i about sth item and has opinion j about tth item, is equal to the probability that the subject has opinion j about sth item and has opinion i about tth item for $k, i, j = 1, 2, 3; i \neq j; 1 \leq s < t \leq 4$. If the goodness-of-fit of the MS(2) model applied to the data is poorly, we are interested in measuring the degree of departure from MS(2). Such measure may be interpreted as the degree of unbalance of opinions for second-order marginal distributions.

Tomizawa and Makii (2001) gave the measure which represents degree of departure from MS(1). However Tomizawa and Makii's measure cannot determine the degree of unbalance of more detailed structure of opinions than first-order marginal distributions. Therefore, the present paper gives the measure to represent the degree of departure from MS(2). The proposed measure enables us to compare the degrees of departure from MS(2) between two different tables (see Section 4).

§2. Measure

The MS(2) model can also be expressed as

$$\begin{cases} p_{kk}^{(s,t)} = p_{kk}^{(1,2)} \\ p_{ij}^{(s,t)} = p_{ji}^{(s,t)} = p_{ij}^{(1,2)} \end{cases} \qquad (k, i, j = 1, \ldots, r; i \neq j; 1 \leq s < t \leq T).$$

Let

$$C_{ij} = \begin{cases} \sum_{l=1}^{T-1} \sum_{m=l+1}^{T} p_{ij}^{(l,m)} & (i = j), \\ \sum_{l=1}^{T-1} \sum_{m=l+1}^{T} \left(p_{ij}^{(l,m)} + p_{ji}^{(l,m)} \right) & (i \neq j), \end{cases}$$

$$\pi_{ij} = \frac{C_{ij}}{\binom{T}{2}} \qquad (i, j = 1, \ldots, r).$$

Assume that $\{C_{ij} > 0\}$. Let

$$p_{ij}^{*(s,t)} = \frac{p_{ij}^{(s,t)}}{C_{ij}} \qquad (i, j = 1, \ldots, r; 1 \leq s < t \leq T).$$

Consider the measure to represent degree of departure from MS(2) as follows:

$$\Phi = \sum_{k=1}^{r} \pi_{kk} \left[1 - \frac{1}{\log \binom{T}{2}} H_{kk} \right] + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \pi_{ij} \left[1 - \frac{1}{\log \left(2 \binom{T}{2} \right)} H_{ij} \right],$$

where

$$H_{ij} = \begin{cases} - \sum_{s=1}^{T-1} \sum_{t=s+1}^{T} p_{ij}^{*(s,t)} \log p_{ij}^{*(s,t)} & (i = j), \\ - \sum_{s=1}^{T-1} \sum_{t=s+1}^{T} \left(p_{ij}^{*(s,t)} \log p_{ij}^{*(s,t)} + p_{ji}^{*(s,t)} \log p_{ji}^{*(s,t)} \right) & (i < j), \end{cases}$$

and 0log0 = 0. Thus Φ is the weighted sum of the Shannon entropy.

We obtain the following theorem.

Theorem 1.
(i) $0 \le \Phi < 1$,
(ii) $\Phi = 0$ *if and only if there is a structure of MS(2) in the* r^T *table.*

Proof. We see

$$H_{kk} \le \log\binom{T}{2} \qquad (k = 1, \dots, r),$$

and

$$H_{ij} \le \log\left(2\binom{T}{2}\right) \qquad (1 \le i < j \le r).$$

These lead to

$$0 \le \sum_{k=1}^{r} \pi_{kk}\left[1 - \frac{1}{\log\binom{T}{2}} H_{kk}\right],$$

and

$$0 \le \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} \pi_{ij}\left[1 - \frac{1}{\log\left(2\binom{T}{2}\right)} H_{ij}\right].$$

Therefore $0 \le \Phi$. Next, we shall show $\Phi < 1$. Let $p_{i_1 \dots i_T}$ denote the probability that an observation will fall in $(i_1, \dots, i_T)$ cell of an r^T table ($i_k = 1, \dots, r; k = 1, \dots, T$). From the assumption $\{C_{ij} > 0\}$, $p_{kk}^{(s,t)} > 0$ for at least one $s < t$ ($k = 1, \dots, r$). Assume that $p_{kk}^{(s_0,t_0)} > 0$ for fixed s_0 and t_0. If $p_{kk\dots k} > 0$ then $H_{kk} > 0$. And if $p_{kk\dots k} = 0$ (i.e., $p_{i_1,\dots,i_{s_0-1},k,i_{s_0+1},\dots,i_{t_0-1},k,i_{t_0+1},\dots,i_T} > 0$ for at least one $(i_1, \dots, i_{s_0-1}, i_{s_0+1}, \dots, i_{t_0-1}, i_{t_0+1}, \dots, i_T)$), then (1) $H_{i_s k} > 0$ for $i_s < k$ or (2) $H_{k i_s} > 0$ for $i_s > k$ ($s \ne s_0, t_0$). Therefore

$$\sum_{k=1}^{r} \pi_{kk}\left[1 - \frac{1}{\log\binom{T}{2}} H_{kk}\right] + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} \pi_{ij}\left[1 - \frac{1}{\log\left(2\binom{T}{2}\right)} H_{ij}\right] < 1.$$

Thus we obtain (i). If the MS(2) model holds, $\Phi = 0$. Assuming that $\Phi = 0$, then $H_{kk} = \log\binom{T}{2}$ for $k = 1, \dots, r$, and $H_{ij} = \log\left(2\binom{T}{2}\right)$ for $1 \le i < j \le r$, namely the MS(2) model holds. Thus (ii) holds. The proof is completed.

§3. Approximate confidence interval of measure

Let $n_{i_1 \dots i_T}$ denote the observed frequency in the $(i_1, \dots, i_T)$ cell, and let $\hat{p}_{i_1 \dots i_T} = n_{i_1 \dots i_T}/n$, where $n = \sum \cdots \sum n_{i_1 \dots i_T}$ ($i_k = 1, \dots, r; k = 1, \dots, T$).

Assuming that a multinomial distribution applies to the r^T table, we consider an approximate standard error and large-sample confidence interval of Φ. The sample version of Φ, denoted by $\hat{\Phi}$, is given by Φ with $(p_{i_1 \ldots i_T})$ replaced by $(\hat{p}_{i_1 \ldots i_T})$. We obtain the following theorem.

Theorem 2. *$\sqrt{n}(\hat{\Phi} - \Phi)$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance $\sigma^2[\Phi]$. The asymptotic variance $\sigma^2[\Phi]$ is obtained as follows:*

$$\sigma^2[\Phi] = \sum_{i_1=1}^{r} \cdots \sum_{i_T=1}^{r} p_{i_1 \ldots i_T} \gamma^2_{i_1 \ldots i_T} - \Phi^2,$$

where

$$\gamma_{i_1 \ldots i_T} = 1 + \frac{1}{\binom{T}{2}} \sum_{s=1}^{T-1} \sum_{t=s+1}^{T} \left\{ I(i_s = i_t) \frac{\log p^{*(s,t)}_{i_s i_t}}{\log \binom{T}{2}} + I(i_s \neq i_t) \frac{\log p^{*(s,t)}_{i_s i_t}}{\log \left(2\binom{T}{2} \right)} \right\},$$

and $I(\cdot)$ is an indicator function.

Proof. Let $\boldsymbol{p} = (p_{1 \ldots 11}, \ldots, p_{1 \ldots 1r}, p_{1 \ldots 21}, \ldots, p_{1 \ldots 2r}, \ldots, p_{r \ldots rr})'$ where '$'$' means the transpose, and let $\hat{\boldsymbol{p}}$ denote $\boldsymbol{p}$ with $(p_{i_1 \ldots i_T})$ replaced by $(\hat{p}_{i_1 \ldots i_T})$. Note that $\sqrt{n}(\hat{\boldsymbol{p}} - \boldsymbol{p})$ has a normal distribution with mean zero vector and covariance matrix $\boldsymbol{D} - \boldsymbol{p}\boldsymbol{p}'$ where $\boldsymbol{D}$ means a diagonal matrix with ith component of $\boldsymbol{p}$ as ith diagonal component. From Taylor expansion of the estimated measure $\hat{\Phi}$ about $\hat{\boldsymbol{p}} = \boldsymbol{p}$,

$$\hat{\Phi} = \Phi + \frac{\partial \Phi}{\partial \boldsymbol{p}'} (\hat{\boldsymbol{p}} - \boldsymbol{p}) + o\left(\|\hat{\boldsymbol{p}} - \boldsymbol{p}\|\right).$$

Using the delta method (Agresti, 2013, p.587), $\sqrt{n}(\hat{\Phi} - \Phi)$ has asymptotically a normal distribution with mean zero and variance

$$\sigma^2[\Phi] = \left(\frac{\partial \Phi}{\partial \boldsymbol{p}'}\right) (\boldsymbol{D} - \boldsymbol{p}\boldsymbol{p}') \left(\frac{\partial \Phi}{\partial \boldsymbol{p}'}\right)'.$$

The proof is completed.

Let $\hat{\sigma}^2[\Phi]$ denote $\sigma^2[\Phi]$ with $\{p_{i_1 \ldots i_T}\}$ replaced by $\{\hat{p}_{i_1 \ldots i_T}\}$. Then an estimated standard error of $\hat{\Phi}$ is $\hat{\sigma}[\Phi]/\sqrt{n}$. Therefore, we obtain an approximate $100(1-\alpha)\%$ confidence interval of Φ as $\hat{\Phi} \pm z_{\alpha/2} \hat{\sigma}[\Phi]/\sqrt{n}$, where $z_{\alpha/2}$ is the percentage point from standard normal distribution corresponding to a two-tail probability equal to α.

§4. Examples

The estimated measures $\hat{\Phi}$ applied to the data in Tables 1(a) and 1(b) are 0.104 and 0.064, respectively. The approximate 95% confidence interval of Φ

for Table 1(a) is (0.094, 0.115) with the estimated approximate standard error 0.005, and that for Table 1(b) is (0.055, 0.073) with the estimated approximate standard error 0.005.

Thus it is inferred that the degree of departure from MS(2) for Table 1(a) is larger than that for Table 1(b), since lower limit of the 95% confidence interval of Φ for Table 1(a) is greater than upper limit of the 95% confidence interval of Φ for Table 1(b). Namely, subjects' opinions about government spending on the environment, health, assistance to big city and law enforcement in 2004 may be more unbalanced than in 2014, in the sense structure of the pairwise opinions in 2004 are more distant from MS(2) in terms of Φ than in 2014.

§5. Relationship between measure and normal distribution

Assume that $\sum_{l=1}^{T} p_i^{(l)} > 0$ for $i = 1, \dots, r$. Let

$$\pi_i = \frac{\sum_{l=1}^{T} p_i^{(l)}}{T}, \qquad p_i^{*(s)} = \frac{p_i^{(s)}}{\sum_{l=1}^{T} p_i^{(l)}} \qquad (i = 1, \dots, r; s = 1, \dots, T).$$

Tomizawa and Makii (2001) gave the measure to represent degree of departure from MS(1), defined by

$$\Phi_{TM} = \sum_{i=1}^{r} \pi_i \left(1 - \frac{1}{\log T} H_i \right),$$

where

$$H_i = -\sum_{s=1}^{T} p_i^{*(s)} \log p_i^{*(s)} \qquad (i = 1, \dots, r),$$

and $0\log 0 = 0$. H_i is the Shannon entropy. Note that Tomizawa and Makii (2001) also gave more general measure to represent the degree of departure from MS(1).

We suppose that there is an underlying trivariate normal distribution for the variables of the r^3 contingency table. Consider random variables U_1, U_2 and U_3 having a joint trivariate normal distribution with means $\mathrm{E}[U_k] = \mu_k$, variances $\mathrm{Var}[U_k] = \sigma^2$ $(k = 1, 2, 3)$, and correlations $\mathrm{Corr}[U_s, U_t] = \rho_{st}$ $(1 \le s < t \le 3)$. Denote the probability density function of (U_1, U_2, U_3) by $f(u_1, u_2, u_3)$. Let Y_k denote the kth variable of the r^3 table $(k = 1, 2, 3)$, and let $D_{ij}^{(s,t)}$ denote the integral interval corresponding to U_1, U_2 and U_3 for obtaining the second-order marginal probability that Y_s takes i and Y_t takes j $(i, j = 1, \dots, r; 1 \le s < t \le 3)$. We shall consider the second-order marginal probability obtained by multiple integral of $f(u_1, u_2, u_3)$ as follows:

$$q_{ij}^{(s,t)} = \iiint_{D_{ij}^{(s,t)}} f(u_1, u_2, u_3) du_1 du_2 du_3.$$

Tables 2, 3, 4 and 5 give the second-order marginal probability tables based on the $\left\{q_{ij}^{(s,t)}\right\}$, formed by using cutpoints for each variable at μ_1, $\mu_1 \pm 0.6\sigma$, for the underlying trivariate normal distribution with the conditions given in the tables themselves. For examples,

$$D_{11}^{(2,3)} = \{(u_1, u_2, u_3) | -\infty < u_1 < +\infty, -\infty < u_2 \leq \mu_1 - 0.6\sigma, \\ -\infty < u_3 \leq \mu_1 - 0.6\sigma\},$$

and

$$D_{23}^{(1,3)} = \{(u_1, u_2, u_3) | \mu_1 - 0.6\sigma < u_1 \leq \mu_1, -\infty < u_2 < +\infty, \\ \mu_1 < u_3 \leq \mu_1 + 0.6\sigma\}.$$

Note that values of $\left\{q_{ij}^{(s,t)}\right\}$ are calculated using cubature package in the statistical software R version 3.2.3. Tables 6(a) and 6(b) give the values of Φ_{TM} and Φ for each of Tables 2, 3, 4 and 5.

Let $f_{U_s U_t}$ denote the second-order marginal probability density function of U_s and U_t $(s < t)$. We see

$$\frac{f_{U_s U_t}(u_s, u_t)}{f_{U_s U_t}(u_t, u_s)} = \exp\left[\frac{(u_s - u_t)(\mu_s - \mu_t)}{(1 - \rho_{st})\sigma^2}\right] \quad \text{for} \quad u_s < u_t.$$

From this equation and Tables 2, 3 and 4, it follows that if $\mu_s < \mu_t$, then it tends to be $q_{ij}^{(s,t)}/q_{ji}^{(s,t)} > 1$ for $i < j$, and $q_{ij}^{(s,t)}/q_{ji}^{(s,t)}$ tends to increase (i) as the difference of means $\mu_s - \mu_t$ decreases for fixed σ^2 and ρ_{st}, or (ii) as the correlation ρ_{st} increases for fixed μ_s, μ_t and σ^2. Also, if $\mu_s > \mu_t$, then $q_{ij}^{(s,t)}/q_{ji}^{(s,t)} < 1$ for $i < j$, and $q_{ij}^{(s,t)}/q_{ji}^{(s,t)}$ tends to decrease (i) as the difference of means $\mu_s - \mu_t$ increases for fixed σ^2 and ρ_{st}, or (ii) as the correlation ρ_{st} increases for fixed μ_s, μ_t and σ^2. Moreover, if $\mu_s = \mu_t$, then $q_{ij}^{(s,t)}/q_{ji}^{(s,t)} = 1$ for $i < j$. We see from Tables 2, 3, 4 and 6(a), as all the differences of means of latent variables, i.e. $\mu_1 - \mu_2$, $\mu_1 - \mu_3$ and $\mu_2 - \mu_3$, decrease for fixed σ^2, ρ_{12}, ρ_{13} and ρ_{23}, each of Φ_{TM} and Φ tends to increase. Also, as all the correlations of latent variables, i.e. ρ, where $\rho = \rho_{st}$ $(1 \leq s < t \leq 3)$, increases for fixed μ_1, μ_2, μ_3 and σ^2, Φ tends to increase, while Φ_{TM} is constant. It seems natural to assume that the degree of departure from symmetry of the second-order marginal probabilities becomes larger (i) as all the differences of means increase, or (ii) as all the correlations increase, because $q_{ij}^{(s,t)}/q_{ji}^{(s,t)}$ (> 1) tends to increase for $i < j$ (see Tables 2, 3 and 4). Thus Φ may be appropriate for measuring the degree of departure from symmetry of second-order marginal probabilities.

We see from Tables 5 and 6(b), as the correlation ρ_{12} increases for fixed μ_1, μ_2, μ_3, σ^2, ρ_{13} and ρ_{23}, Φ tends to increase, while Φ_{TM} is constant. It seems natural to assume that the degree of departure from homogeneity of the second-order marginal probabilities becomes larger as the correlation ρ_{12} increases, because $q_{kk}^{(1,2)}/q_{kk}^{(s,t)}$ (≥ 1) increases for $k = 1, \ldots, 4$, and $q_{ij}^{(1,2)}/q_{ij}^{(s,t)}$ (≤ 1) decreases for $|i - j| \geq 2$ (see Table 5). Thus Φ may be appropriate for measuring the degree of departure from homogeneity of second-order marginal probabilities.

Therefore Φ may be appropriate for measuring the degree of departure from MS(2), because Φ may simultaneously measure the degrees of departure from symmetry and homogeneity of second-order marginal probabilities. Also Φ_{TM} may not be appropriate for measuring the degree of departure from MS(2).

§6. Concluding remarks

For an r^T contingency table $(T \geq 3)$, we have proposed the measure to represent the degree of departure from the MS(2) model. Note that, the proposed measure Φ is invariant under arbitrary same permutations of categories of variables. Thus the measure Φ is appropriate for the nominal contingency table because this measure does not use information about the order of the categories.

We have shown that the measure Φ is useful for comparing the degrees of departure from MS(2) between two different tables in Section 4. Also we have shown how the measure Φ takes the values when there is an underlying trivariate normal distribution with various conditions on three-way tables, and have discussed the appropriation of the measure Φ in Section 5.

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Table 1. Opinions about government spending (a) in 2004 with sample size $n = 1172$ and (b) in 2014 with sample size $n = 1061$

(a)

Big city		1			2			3		
Law enforcement		1	2	3	1	2	3	1	2	3
Environment	Health									
1	1	83	48	14	187	115	23	109	54	21
	2	4	5	3	12	23	3	9	11	5
	3	4	3	0	7	7	0	4	2	2
2	1	21	15	3	59	34	7	42	34	4
	2	2	2	2	18	24	3	12	10	3
	3	3	0	0	4	4	2	6	9	1
3	1	4	1	2	10	5	2	13	6	6
	2	0	0	2	2	6	2	2	4	2
	3	2	0	1	1	0	1	8	3	5

(b)

Big city		1			2			3		
Law enforcement		1	2	3	1	2	3	1	2	3
Environment	Health									
1	1	59	34	22	99	74	16	79	35	18
	2	10	6	9	24	35	7	12	15	8
	3	6	6	9	15	14	4	21	7	4
2	1	8	5	1	30	27	5	29	18	4
	2	10	6	1	13	23	4	6	18	2
	3	3	0	0	11	17	1	18	15	7
3	1	5	4	0	4	4	0	13	10	1
	2	1	1	0	3	5	2	6	8	3
	3	1	2	4	4	5	1	15	16	13

Note: These data are from the 2014 General Social Survey, with categories 1 is 'too little', 2 is 'about right' and 3 is 'too much'.

Table 2. The second-order marginal probability tables, formed by using cutpoints for each variable at μ_1, $\mu_1 \pm 0.6$, for an underlying trivariate normal distribution with the conditions $\mu_2 = \mu_1 + 0.4$, $\mu_3 = \mu_1 + 0.8$, $\sigma^2 = 1$ and $\rho_{st} = \rho$ $(1 \leq s < t \leq 3)$ with $\rho = 0, 0.3, 0.6$ and 0.9. T_{st} is the marginal table of Y_s and Y_t.

(a) $\rho = 0$

T_{12}				T_{13}				T_{23}			
0.04	0.05	0.06	0.12	0.02	0.04	0.06	0.16	0.01	0.02	0.03	0.09
0.04	0.04	0.05	0.09	0.02	0.03	0.05	0.13	0.02	0.02	0.04	0.11
0.04	0.04	0.05	0.09	0.02	0.03	0.05	0.13	0.02	0.03	0.05	0.14
0.04	0.05	0.06	0.12	0.02	0.04	0.06	0.16	0.03	0.06	0.09	0.24

(b) $\rho = 0.3$

T_{12}				T_{13}				T_{23}			
0.07	0.06	0.06	0.08	0.04	0.05	0.07	0.12	0.03	0.03	0.04	0.06
0.04	0.05	0.06	0.09	0.02	0.03	0.05	0.12	0.02	0.03	0.04	0.09
0.03	0.04	0.05	0.10	0.01	0.03	0.05	0.14	0.02	0.03	0.05	0.13
0.02	0.04	0.06	0.16	0.01	0.02	0.05	0.20	0.02	0.04	0.07	0.29

Table 2. (continued)

(c) $\rho = 0.6$

T_{12}				T_{13}				T_{23}			
0.10	0.08	0.06	0.04	0.06	0.07	0.07	0.07	0.04	0.04	0.04	0.03
0.03	0.05	0.07	0.07	0.01	0.03	0.06	0.12	0.02	0.04	0.05	0.07
0.02	0.04	0.06	0.11	0.01	0.02	0.05	0.15	0.01	0.03	0.06	0.13
0.01	0.02	0.05	0.20	1.56E-03	0.01	0.03	0.24	4.60E-03	0.02	0.06	0.34

(d) $\rho = 0.9$

T_{12}				T_{13}				T_{23}			
0.15	0.09	0.03	2.62E-03	0.08	0.10	0.08	0.02	0.07	0.06	0.02	2.21E-03
0.01	0.07	0.10	0.04	1.55E-03	0.03	0.10	0.10	0.01	0.06	0.09	0.03
6.97E-04	0.02	0.08	0.12	3.36E-05	2.38E-03	0.03	0.19	5.20E-04	0.01	0.08	0.14
7.65E-06	7.91E-04	0.02	0.26	1.34E-07	4.33E-05	2.57E-03	0.27	6.47E-06	7.60E-04	0.02	0.40

Table 3. The second-order marginal probability tables, formed by using cutpoints for each variable at μ_1, $\mu_1 \pm 0.6$, for an underlying trivariate normal distribution with the conditions $\mu_2 = \mu_1 + 0.6$, $\mu_3 = \mu_1 + 1.2$, $\sigma^2 = 1$ and $\rho_{st} = \rho$ $(1 \leq s < t \leq 3)$ with $\rho = 0, 0.3, 0.6$ and 0.9. T_{st} is the marginal table of Y_s and Y_t.

(a) $\rho = 0$

T_{12}				T_{13}				T_{23}			
0.03	0.04	0.06	0.14	0.01	0.02	0.04	0.20	4.13E-03	0.01	0.02	0.08
0.03	0.04	0.05	0.11	0.01	0.02	0.04	0.16	0.01	0.01	0.03	0.12
0.03	0.04	0.05	0.11	0.01	0.02	0.04	0.16	0.01	0.02	0.04	0.16
0.03	0.04	0.06	0.14	0.01	0.02	0.04	0.20	0.02	0.04	0.08	0.36

(b) $\rho = 0.3$

T_{12}				T_{13}				T_{23}			
0.05	0.06	0.07	0.10	0.02	0.03	0.06	0.16	0.01	0.02	0.03	0.06
0.03	0.04	0.05	0.10	0.01	0.02	0.04	0.16	0.01	0.02	0.03	0.10
0.02	0.03	0.05	0.12	0.01	0.01	0.03	0.17	0.01	0.02	0.04	0.16
0.01	0.03	0.05	0.18	3.40E-03	0.01	0.03	0.23	0.01	0.03	0.06	0.40

Table 3. (continued)

(c) $\rho = 0.6$

T_{12}				T_{13}				T_{23}			
0.08	0.07	0.07	0.05	0.03	0.05	0.07	0.12	0.02	0.03	0.03	0.04
0.02	0.04	0.06	0.09	4.96E-03	0.02	0.04	0.16	0.01	0.02	0.04	0.09
0.01	0.03	0.06	0.13	1.62E-03	0.01	0.03	0.19	4.96E-03	0.02	0.04	0.16
3.03E-03	0.01	0.04	0.22	3.53E-04	2.67E-03	0.01	0.26	1.97E-03	0.01	0.04	0.45

(d) $\rho = 0.9$

T_{12}				T_{13}				T_{23}			
0.11	0.10	0.05	0.01	0.04	0.07	0.10	0.06	0.03	0.05	0.03	4.97E-03
4.80E-03	0.05	0.11	0.06	9.55E-05	4.71E-03	0.05	0.17	2.26E-03	0.03	0.07	0.05
1.68E-04	0.01	0.06	0.16	7.63E-07	1.67E-04	0.01	0.22	9.55E-05	4.71E-03	0.05	0.17
1.11E-06	2.03E-04	0.01	0.27	1.06E-09	1.11E-06	2.03E-04	0.27	7.64E-07	1.68E-04	0.01	0.49

Table 4. The second-order marginal probability tables, formed by using cutpoints for each variable at μ_1, $\mu_1 \pm 0.6$, for an underlying trivariate normal distribution with the conditions $\mu_2 = \mu_1 + 0.8$, $\mu_3 = \mu_1 + 1.6$, $\sigma^2 = 1$ and $\rho_{st} = \rho$ $(1 \le s < t \le 3)$ with $\rho = 0, 0.3, 0.6$ and 0.9. T_{st} is the marginal table of Y_s and Y_t.

(a) $\rho = 0$

T_{12}				T_{13}				T_{23}			
0.02	0.04	0.06	0.16	3.81E-03	0.01	0.03	0.23	1.12E-03	3.30E-03	0.01	0.07
0.02	0.03	0.05	0.13	3.14E-03	0.01	0.02	0.19	1.82E-03	0.01	0.01	0.11
0.02	0.03	0.05	0.13	3.14E-03	0.01	0.02	0.19	2.90E-03	0.01	0.02	0.18
0.02	0.04	0.06	0.16	3.81E-03	0.01	0.03	0.23	0.01	0.02	0.06	0.49

(b) $\rho = 0.3$

T_{12}				T_{13}				T_{23}			
0.04	0.05	0.07	0.12	0.01	0.02	0.04	0.20	3.53E-03	0.01	0.01	0.05
0.02	0.03	0.05	0.12	3.05E-03	0.01	0.03	0.19	3.22E-03	0.01	0.02	0.10
0.01	0.03	0.05	0.14	1.88E-03	0.01	0.02	0.20	3.29E-03	0.01	0.03	0.17
0.01	0.02	0.05	0.20	1.07E-03	4.65E-03	0.02	0.25	3.86E-03	0.01	0.04	0.52

Table 4. (continued)

(c) $\rho = 0.6$

T_{12}				T_{13}				T_{23}			
0.06	0.07	0.07	0.07	0.01	0.03	0.06	0.17	0.01	0.01	0.02	0.04
0.01	0.03	0.06	0.12	1.36E-03	0.01	0.03	0.19	3.49E-03	0.01	0.03	0.09
0.01	0.02	0.05	0.15	3.65E-04	2.77E-03	0.01	0.21	1.88E-03	0.01	0.03	0.17
1.56E-03	0.01	0.03	0.24	6.38E-05	7.00E-04	4.80E-03	0.27	7.27E-04	0.01	0.03	0.55

(d) $\rho = 0.9$

T_{12}				T_{13}				T_{23}			
0.08	0.10	0.08	0.02	0.01	0.04	0.09	0.13	0.01	0.03	0.03	0.01
1.55E-03	0.03	0.10	0.10	2.79E-06	4.19E-04	0.01	0.21	4.82E-04	0.01	0.05	0.07
3.36E-05	2.38E-03	0.03	0.19	7.95E-09	5.57E-06	6.92E-04	0.23	1.34E-05	1.22E-03	0.02	0.19
1.34E-07	4.33E-05	2.57E-03	0.27	3.81E-12	1.32E-08	7.64E-06	0.27	6.83E-08	2.85E-05	2.19E-03	0.58

Table 5. The second-order marginal probability tables, formed by using cutpoints for each variable at μ_1, $\mu_1 \pm 0.6$, for an underlying joint trivariate normal distribution with the conditions $\mu_1 = \mu_2 = \mu_3$, $\sigma^2 = 1$, $\rho_{12} = 0, 0.3, 0.6, 0.9$ and $\rho_{13} = \rho_{23} = 0$. T_{st} is the marginal table of Y_s and Y_t.

(a) $\rho_{12} = 0$, $\rho_{13} = \rho_{23} = 0$

T_{12}				T_{13}				T_{23}			
0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08
0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08

(b) $\rho_{12} = 0.3$, $\rho_{13} = \rho_{23} = 0$

T_{12}				T_{13}				T_{23}			
0.11	0.07	0.05	0.04	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08
0.07	0.05	0.05	0.05	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
0.05	0.05	0.05	0.07	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
0.04	0.05	0.07	0.11	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08

Table 5. (continued)

(c) $\rho_{12} = 0.6,\ \rho_{13} = \rho_{23} = 0$

T_{12}				T_{13}				T_{23}			
0.15	0.07	0.04	0.02	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08
0.07	0.06	0.06	0.04	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
0.04	0.06	0.06	0.07	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
0.02	0.04	0.07	0.15	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08

(d) $\rho_{12} = 0.9,\ \rho_{13} = \rho_{23} = 0$

T_{12}				T_{13}				T_{23}			
0.21	0.05	0.01	2.04E-04	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08
0.05	0.11	0.06	0.01	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
0.01	0.06	0.11	0.05	0.06	0.05	0.05	0.06	0.06	0.05	0.05	0.06
2.04E-04	0.01	0.05	0.21	0.08	0.06	0.06	0.08	0.08	0.06	0.06	0.08

Table 6. Values of Φ_{TM} and Φ (a) for each of Tables 2, 3 and 4 and (b) for Table 5

(a)

	Table 2		Table 3		Table 4	
ρ	Φ_{TM}	Φ	Φ_{TM}	Φ	Φ_{TM}	Φ
0	0.038	0.051	0.078	0.103	0.123	0.161
0.3	0.038	0.059	0.078	0.116	0.123	0.176
0.6	0.038	0.078	0.078	0.147	0.123	0.211
0.9	0.038	0.156	0.078	0.241	0.123	0.292

(b)

ρ_{12}	Φ_{TM}	Φ
0	0	0
0.3	0	0.005
0.6	0	0.025
0.9	0	0.084

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