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Bounds for a Čebyšev type functional in terms of Riemann-Stieltjes integral

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Abstract. Upper and lower bounds for a Čebyšev type functional in terms of Riemann-Stieltjes integral are given. Applications for functions of selfadjoint operators in Hilbert spaces are also provided.

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§1. Introduction

In [16], the authors have considered the following functional:

$$D(f;u) := \int_a^b f(x)\,du(x) - [u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt, \tag{1.1}$$

provided that the Riemann-Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

It has been shown in [16], that, if $f,u:[a,b]\to\mathbb{R}$ are such that u is *Lipschitzian* on $[a,b]$, i.e.,

$$|u(x)-u(y)| \le L|x-y| \quad \text{for any } x,y\in[a,b] \quad (L>0) \tag{1.2}$$

and f is *Riemann integrable* on $[a,b]$ with

$$m \le f(x) \le M \quad \text{for any } x\in[a,b], \tag{1.3}$$

for some $m,M\in\mathbb{R}$, then we have the inequality

$$|D(f;u)| \le \frac{1}{2}L(M-m)(b-a). \tag{1.4}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

We recall that a function $u : [a, b] \to \mathbb{R}$ is of *bounded variation* on $[a, b]$ if for any division $d \in Div[a, b]$ with $d : a = x_0 < x_1 < ... < x_n = b$ we have $\sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)| < \infty$. For a function of bounded variation $u : [a, b] \to \mathbb{R}$ we define the *total variation* of u on $[a, b]$ by

$$\bigvee_a^b (u) = \sup_{d \in Div[a,b]} \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)| < \infty.$$

In [15], the following result complementing the above has been obtained as well:

$$|D(f; u)| \le \frac{1}{2} L (b - a) \bigvee_a^b (u), \tag{1.5}$$

where $f, u : [a, b] \to \mathbb{R}$ are such that u is of bounded variation on $[a, b]$ and f is Lipschitzian with the constant $L > 0$. The constant $\frac{1}{2}$ in (1.5) is sharp in the above sense.

In the case of convex integrators $u : [a, b] \to \mathbb{R}$, we have [11]:

$$0 \le D(f; u) \le 2 \cdot \frac{u'_-(b) - u'_+(a)}{b - a} \int_a^b \left(t - \frac{a + b}{2}\right) f(t)\, dt, \tag{1.6}$$

where $f : [a, b] \to \mathbb{R}$ is a monotonic nondecreasing function on $[a, b]$. Here 2 is also best possible.

For other related results for the functional $D(\cdot; \cdot)$, see [1]-[5], [7]-[14] and [18].

In this paper some new lower and upper bounds for $D(\cdot; \cdot)$ are provided. Applications for functions of selfadjoint operators on complex Hilbert spaces are also given.

§2. Some New Bounds

The following lemma may be stated:

Lemma 2.1. *Let $g : [a, b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) The function $g - \frac{l+L}{2} \cdot \ell$, where $\ell(t) = t$, $t \in [a, b]$ is $\frac{1}{2}(L - l)$-Lipschitzian;

(ii) We have the inequalities

$$l \leq \frac{g(t) - g(s)}{t - s} \leq L \quad \text{for each } t, s \in [a, b] \quad \text{with } t \neq s; \tag{2.1}$$

(iii) We have the inequalities

$$l(t - s) \leq g(t) - g(s) \leq L(t - s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s. \tag{2.2}$$

Following [18], we can introduce the definition of (l, L)-Lipschitzian functions:

Definition 1. The function $g : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 2.1 is said to be (l, L)-Lipschitzian on $[a, b]$.

If $L > 0$ and $l = -L$, then $(-L, L)$-Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l, L)-Lipschitzian functions.

Proposition 2.2. *Let $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $-\infty < l = \inf_{t \in (a,b)} g'(t)$ and $\sup_{t \in (a,b)} g'(t) = L < \infty$, then g is (l, L)-Lipschitzian on $[a, b]$.*

We have the following result:

Theorem 2.3. *Let $u : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a (l, L)-Lipschitzian function on $[a, b]$. Then*

$$\begin{aligned} l\left[\frac{u(a) + u(b)}{2}(b - a) - \int_a^b u(t)\,dt\right] &\leq D(f; u) \\ &\leq L\left[\frac{u(a) + u(b)}{2}(b - a) - \int_a^b u(t)\,dt\right]. \end{aligned} \tag{2.3}$$

The inequalities in (2.3) are sharp.

Proof. Consider the auxiliary function $f_L : [a, b] \to \mathbb{R}$, $f_L = L\ell - f$, where ℓ is the identity function $\ell(t) = t$, $t \in [a, b]$. Since $f : [a, b] \to \mathbb{R}$ a (l, L)-Lipschitzian function on $[a, b]$ then $f(t) - f(s) \leq L(t - s)$ for each $t, s \in [a, b]$ with $t > s$ which shows that f_L is monotonic nondecreasing on $[a, b]$.

Utilizing the first inequality in (1.6) we have

$$0 \leq D(L\ell - f, u) = LD(\ell, u) - D(f, u)$$

showing that

$$D(f,u) \le LD(\ell,u). \tag{2.4}$$

A similar argument applied for the auxiliary function $f_l : [a,b] \to \mathbb{R}$, $f_L = f - l\ell$ produces the reverse inequality

$$lD(\ell,u) \le D(f,u). \tag{2.5}$$

On the other hand, integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} D(\ell,u) &= \int_a^b t du(t) - \frac{1}{b-a}[u(b)-u(a)]\int_a^b tdt \\ &= bu(b) - au(a) - \int_a^b u(t)\,dt - \frac{a+b}{2}[u(b)-u(a)] \\ &= \frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt, \end{aligned}$$

which together with (2.4) and (2.5) produce the desired result (2.3).

If we take $f_0(t) = t$, and $\varepsilon \in (0,1)$ then for each $t,s \in [a,b]$ with $t > s$ we have

$$(1-\varepsilon)(t-s) \le f_0(t) - f_0(s) = t - s \le (1+\varepsilon)(t-s),$$

which shows that f is a $(1-\varepsilon, 1+\varepsilon)$-Lipschitzian function on $[a,b]$.

Assume that there exists $A, B > 0$ such that

$$lAD(\ell,u) \le D(f,u) \le LBD(\ell,u) \tag{2.6}$$

for $u : [a,b] \to \mathbb{R}$ a convex function on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a (l,L)-Lipschitzian function on $[a,b]$.

If we write the inequality (2.6) for f_0 and u strictly convex, we get

$$(1-\varepsilon)AD(\ell,u) \le D(\ell,u) \le (1+\varepsilon)BD(\ell,u)$$

and dividing by $D(\ell,u) > 0$ we get

$$(1-\varepsilon)A \le 1 \le (1+\varepsilon)B. \tag{2.7}$$

Letting $\varepsilon \to 0+$ in (2.7) we get $A \le 1 \le B$, which proves the sharpness of the inequality (2.3). ∎

Remark 1. The double inequality in (2.3) is equivalent to

$$\begin{aligned} &\left| D(f;u) - \frac{l+L}{2}\left(\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right)\right| \\ &\le \frac{1}{2}(L-l)\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right]. \end{aligned} \tag{2.8}$$

The constant $\frac{1}{2}$ is best possible.

Corollary 2.4. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $-\infty < l = \inf_{t\in(a,b)} f'(t)$ and $\sup_{t\in(a,b)} f'(t) = L < \infty$. If $u : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then the inequality (2.8) holds true.*

If $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$, then

$$|D(f;u)| \le \|f'\|_\infty \left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right]. \tag{2.9}$$

The inequality is sharp.

The proof follows from (2.8) by taking $L = \|f'\|_\infty$ and $l = -\|f'\|_\infty$.

For two Lebesgue integrable functions f and g we can define the Čebyšev functional:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt.$$

Corollary 2.5. *Let $w : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a (l,L)-Lipschitzian function on $[a,b]$. Then*

$$\frac{l}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) w(t)\,dt \le C(f,w) \le \frac{L}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) w(t)\,dt. \tag{2.10}$$

The inequalities in (2.10) are sharp.

Proof. Choose $u(t) := \int_a^t w(s)\,ds$, $t \in [a,b]$. Since $w : [a,b] \to \mathbb{R}$ is a monotonic nondecreasing function on $[a,b]$, then u is convex on $[a,b]$.

We also have

$$\begin{aligned} &\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt \\ &= \frac{1}{2}(b-a)\int_a^b w(s)\,ds - \left[t\int_a^t w(s)\,ds\Big|_a^b - \int_a^b s w(s)\,ds \right] \\ &= \int_a^b \left(s - \frac{a+b}{2}\right) w(s)\,ds. \end{aligned} \tag{2.11}$$

Writing the inequalities (2.3) for these functions we deduce the desired result (2.10). ∎

Remark 2. The inequalities (2.10) are equivalent to

$$\begin{aligned} &\left| C(f,w) - \frac{l+L}{2}\frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) w(t)\,dt \right| \\ &\le \frac{1}{2}(L-l)\frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right) w(t)\,dt. \end{aligned} \tag{2.12}$$

The constant $\frac{1}{2}$ is best possible.

If $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$, then

$$|C(f,w)| \le \|f'\|_\infty \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) w(t)\,dt. \tag{2.13}$$

The inequality is sharp.

Definition 2. For two constants δ, Δ with $\delta < \Delta$, we say that the function $g : [a,b] \to \mathbb{R}$ is (δ,Δ)-convex (see also [6] for more general concepts) if $g-\frac{1}{2}\delta\ell^2$ and $\frac{1}{2}\Delta\ell^2 - g$ are convex functions on $[a,b]$.

It is easy to see that, if g is twice differentiable on (a,b) and the second derivative satisfies the condition

$$\delta \le g''(t) \le \Delta \text{ for any } t \in (a,b),$$

then g is (δ,Δ)-convex.

The following result also holds:

Theorem 2.6. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and for δ,Δ with $\delta < \Delta$, a (δ,Δ)-convex function $u : [a,b] \to \mathbb{R}$. Then we have the double inequality*

$$\delta\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt \le D(f;u) \le \Delta\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt. \tag{2.14}$$

The inequalities are sharp.

Proof. Since the function f is monotonic nondecreasing and $u-\frac{1}{2}\delta\ell^2$ is convex, then from the first inequality in (1.6) we have

$$D\left(f; u-\frac{1}{2}\delta\ell^2\right) \ge 0,$$

which is equivalent with

$$\frac{1}{2}\delta D\left(f;\ell^2\right) \le D(f;u).$$

From the convexity of $\frac{1}{2}\Delta\ell^2 - g$ we also have

$$D(f;u) \le \frac{1}{2}\Delta D\left(f;\ell^2\right).$$

However

$$\begin{aligned} D\left(f;\ell^2\right) &= \int_a^b f(t)\, d\ell^2(t) - \frac{\ell^2(b)-\ell^2(a)}{b-a}\int_a^b f(t)\,dt \\ &= 2\int_a^b f(t)\,dt - (b+a)\int_a^b f(t)\,dt \\ &= 2\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt. \end{aligned}$$

If we take $u_0(t) := \frac{1}{2}t^2$, and $\varepsilon \in (0,1)$, then for $\delta = 1-\varepsilon$ and $\Delta = 1+\varepsilon$ we have that u_0 is $(1-\varepsilon, 1+\varepsilon)$-convex on $[a,b]$.

Assume that there exists the constants $P, Q > 0$ such that

$$\delta P\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt \le D(f;u) \le \Delta Q\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt, \tag{2.15}$$

for $f : [a,b] \to \mathbb{R}$ a monotonic nondecreasing function on $[a,b]$ and (δ,Δ)-convex function $u : [a,b] \to \mathbb{R}$.

Since

$$D(f;u_0) = \int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt$$

then by replacing $u_0, \delta = 1-\varepsilon$ and $\Delta = 1+\varepsilon$ in (2.15) we get

$$\begin{aligned} (1-\varepsilon)P\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt &\le \int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,d \\ &\le (1+\varepsilon)Q\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt, \end{aligned} \tag{2.16}$$

and by division with $\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt$ that is positive for many functions f (for instance $f(t) = t - \frac{a+b}{2}$), we obtain

$$(1-\varepsilon)P \le 1 \le (1+\varepsilon)Q.$$

Letting $\varepsilon \to 0+$ we deduce $P \le 1 \le Q$, and the sharpness of the inequalities are proved. ∎

Remark 3. Integrating by parts in the Riemann-Stieltjes integral we have

$$
\begin{aligned}
&D(f;u) \\
&= f(b)u(b) - f(a)u(a) - \int_a^b u(t)\,df(t) \\
&- \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt \\
&= u(b)\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) + u(a)\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right) \\
&- \int_a^b u(t)\,df(t).
\end{aligned} \tag{2.17}
$$

The inequality (2.3) is then equivalent with

$$
\begin{aligned}
&l\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right] \\
&\leq u(b)\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) + u(a)\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right) \\
&- \int_a^b u(t)\,df(t) \\
&\leq L\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right]
\end{aligned} \tag{2.18}
$$

while (2.14) is equivalent with

$$
\begin{aligned}
&\delta\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \\
&\leq u(b)\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) + u(a)\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right) \\
&- \int_a^b u(t)\,df(t) \\
&\leq \Delta\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt.
\end{aligned} \tag{2.19}
$$

§3. Applications for Selfadjoint Operators

Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \leq \lambda, \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are summed up in the following fundamental result concerning the spectral decomposition of bounded selfadjoint operators in Hilbert spaces, see for instance [17, p. 256]

Theorem 3.1 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m := \min\{\lambda \,|\lambda \in Sp(A)\} = \min Sp(A)$ and $M := \max\{\lambda \,|\lambda \in Sp(A)\} = \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

b) $E_{m-0} = 0, E_M = 1_H$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$;

c) We have the representation

$$A = \int_{m-0}^{M} \lambda dE_\lambda. \tag{3.2}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon \tag{3.3}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases} \tag{3.4}$$

this means that

$$\varphi(A)=\int_{m-0}^{M}\varphi(\lambda)\,dE_{\lambda}, \tag{3.5}$$

where the integral is of Riemann-Stieltjes type.

Corollary 3.2. *With the assumptions of Theorem 3.1 for A, E_λ and φ we have the representations*

$$\varphi(A)\,x=\int_{m-0}^{M}\varphi(\lambda)\,dE_{\lambda}x \quad \text{for all } x\in H \tag{3.6}$$

and

$$\langle\varphi(A)\,x,y\rangle=\int_{m-0}^{M}\varphi(\lambda)\,d\langle E_{\lambda}x,y\rangle \quad \text{for all } x,y\in H. \tag{3.7}$$

In particular,

$$\langle\varphi(A)\,x,x\rangle=\int_{m-0}^{M}\varphi(\lambda)\,d\langle E_{\lambda}x,x\rangle \quad \text{for all } x\in H. \tag{3.8}$$

Moreover, we have the equality

$$\|\varphi(A)\,x\|^{2}=\int_{m-0}^{M}|\varphi(\lambda)|^{2}\,d\|E_{\lambda}x\|^{2} \quad \text{for all } x\in H. \tag{3.9}$$

Utilising the Spectral Representation Theorem we can prove the following inequalities for functions of selfadjoint operators:

Theorem 3.3. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m=:\min\{\lambda\,|\lambda\in Sp(A)\}=\min Sp(A)$ and $M:=\max\{\lambda\,|\lambda\in Sp(A)\}$ $=\max Sp(A)$. Assume that the function $f:I\to\mathbb{R}$ is differentiable on the interior of I denoted $\mathring{I}$ and $[m,M]\subset\mathring{I}$. If the derivative f' is (δ,Δ)-Lipschitzian with $\delta<\Delta$, then*

$$\begin{aligned}\frac{1}{2}\delta\,(M1_H-A)\,(A-m1_H) &\\ \le\frac{1}{M-m}\,[f(M)\,(A-m1_H)+f(m)\,(M1_H-A]-f(A) &\\ \le\frac{1}{2}\Delta\,(M1_H-A)\,(A-m1_H) &\end{aligned} \tag{3.10}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ the spectral family of A and $x \in H$. Utilising the inequality (2.10) for the (δ,Δ)-Lipschitzian function f' and the monotonic nondecreasing function $w(t) = \langle E_t x, x\rangle$, $t \in [m-\varepsilon, M]$ for a small positive ε, we have

$$
\begin{aligned}
(3.11)\qquad & \frac{\delta}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\left(t-\frac{m-\varepsilon+M}{2}\right)\langle E_t x, x\rangle\, dt \\
& \le \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M} f'(t)\langle E_t x, x\rangle\, dt \\
& - \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M} f'(t)\, dt \cdot \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\langle E_t x, x\rangle\, dt \\
& \le \frac{\Delta}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\left(t-\frac{a+b}{2}\right) w(t)\, dt.
\end{aligned}
$$

Letting $\varepsilon \to 0+$ in (3.11) we get

$$
\begin{aligned}
(3.12)\qquad & \delta\int_{m-0}^{M}\left(t-\frac{m+M}{2}\right)\langle E_t x, x\rangle\, dt \\
& \le \int_{m-0}^{M} f'(t)\langle E_t x, x\rangle\, dt - \frac{1}{M-m}\int_{m-0}^{M} f'(t)\, dt \cdot \int_{m-0}^{M}\langle E_t x, x\rangle\, dt \\
& \le \Delta\int_{m-0}^{M}\left(t-\frac{a+b}{2}\right) w(t)\, dt
\end{aligned}
$$

for any $x \in H$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_{m-0}^{M}\left(t-\frac{m+M}{2}\right)\langle E_t x,x\rangle\,dt \\
&=\frac{1}{2}\int_{m-0}^{M}\langle E_t x,x\rangle\,d\left(\left(t-\frac{m+M}{2}\right)^2\right) \\
&=\frac{1}{2}\left[\langle E_t x,x\rangle\left(t-\frac{m+M}{2}\right)^2\Big|_{m-0}^{M}-\int_{m-0}^{M}\left(t-\frac{m+M}{2}\right)^2 d\left(\langle E_t x,x\rangle\right)\right] \\
&=\frac{1}{2}\left[\|x\|^2\left(\frac{M-m}{2}\right)^2-\int_{m-0}^{M}\left(t-\frac{m+M}{2}\right)^2 d\left(\langle E_t x,x\rangle\right)\right] \\
&=\frac{1}{2}\left[\int_{m-0}^{M}\left[\left(\frac{M-m}{2}\right)^2-\left(t-\frac{m+M}{2}\right)^2\right]d\left(\langle E_t x,x\rangle\right)\right] \\
&=\frac{1}{2}\int_{m-0}^{M}(M-t)(t-m)\,d\left(\langle E_t x,x\rangle\right)=\frac{1}{2}\left\langle\left(M1_H-A\right)\left(A-m1_H\right)x,x\right\rangle
\end{aligned}\tag{3.13}$$

for any $x\in H$.

We also have

$$\begin{aligned}
\int_{m-0}^{M}f'(t)\langle E_t x,x\rangle\,dt&=f(t)\langle E_t x,x\rangle|_{m-0}^{M}-\int_{m-0}^{M}f(t)\,d\left(\langle E_t x,x\rangle\right) \\
&=f(M)\|x\|^2-\int_{m-0}^{M}f(t)\,d\left(\langle E_t x,x\rangle\right) \\
&=\int_{m-0}^{M}\left[f(M)-f(t)\right]d\left(\langle E_t x,x\rangle\right) \\
&=\left\langle\left[f(M)1_H-f(A)\right]x,x\right\rangle
\end{aligned}\tag{3.14}$$

and, similarly

$$\int_{m-0}^{M}\langle E_t x,x\rangle\,dt=\left\langle\left(M1_H-A\right)x,x\right\rangle\tag{3.15}$$

for any $x\in H$.

Utilising (3.14) and (3.15) we have

(3.16)
$$\begin{aligned}
&\int_{m-0}^{M} f'(t)\langle E_t x, x\rangle\, dt-\frac{1}{M-m}\int_{m-0}^{M} f'(t)\, dt\cdot\int_{m-0}^{M}\langle E_t x, x\rangle\, dt\\
&=\langle [f(M)1_H-f(A)]x,x\rangle-\frac{f(M)-f(m)}{M-m}\langle (M1_H-A)x,x\rangle\\
&=\left\langle\left[\frac{(M-m)f(M)1_H-[f(M)-f(m)](M1_H-A)}{M-m}-f(A)\right]x,x\right\rangle\\
&=\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\right]x,x\right\rangle
\end{aligned}$$
for any $x\in H$.

From (3.12) we deduce the desired result (3.10). ∎

From (1.6), we have for $h:[a,b]\to\mathbb{R}$ a convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$,
$$\begin{aligned}
0 &\le D(g;h)\\
&\le 2\cdot\frac{h'_-(b)-h'_+(a)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)g(t)\,dt.
\end{aligned}\tag{3.17}$$
Since, by (2.17) we have

(3.18)
$$\begin{aligned}
&0\le D(g;h)\\
&=h(b)\left(g(b)-\frac{1}{b-a}\int_a^b g(t)\,dt\right)+h(a)\left(\frac{1}{b-a}\int_a^b g(t)\,dt-g(a)\right)\\
&-\int_a^b h(t)\,df(t)
\end{aligned}$$
and, as in (3.13), we also have
$$\int_a^b\left(t-\frac{a+b}{2}\right)g(t)\,dt=\frac{1}{2}\int_a^b(b-t)(t-a)\,dg(t),\tag{3.19}$$
then by (3.17) we have

(3.20)
$$\begin{aligned}
&0\le h(b)\left(g(b)-\frac{1}{b-a}\int_a^b g(t)\,dt\right)+h(a)\left(\frac{1}{b-a}\int_a^b g(t)\,dt-g(a)\right)\\
&-\int_a^b h(t)\,df(t)\\
&\le\frac{h'_-(b)-h'_+(a)}{b-a}\int_a^b(b-t)(t-a)\,dg(t).
\end{aligned}$$

We can state the following result as well:

Theorem 3.4. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m := \min\{\lambda \,|\lambda \in Sp(A)\} = \min Sp(A)$ and $M := \max\{\lambda \,|\lambda \in Sp(A)\} = \max Sp(A)$. Assume that the function $f : I \to \mathbb{R}$ is convex on the interior of I denoted $\mathring{I}$ and $[m, M] \subset \mathring{I}$. Then*

$$0 \leq \frac{1}{M-m}\left[f(M)(A - m1_H) + f(m)(M1_H - A\right] - f(A) \leq \frac{f'_-(M) - f'_+(m)}{M-m}(M1_H - A)(A - m1_H). \tag{3.21}$$

The proof follows by (3.20) by choosing $h = f$ and $g = \langle E_t x, x\rangle$, $t \in \mathbb{R}$, where $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of A.

Consider the exponential function $f : \mathbb{R} \to \mathbb{R}$, then by (3.10) we have:

Theorem 3.5. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m := \min\{\lambda \,|\lambda \in Sp(A)\}$ and $M := \max\{\lambda \,|\lambda \in Sp(A)\}$. Then we have*

$$\begin{aligned} &\frac{1}{2}\exp(m)(M1_H - A)(A - m1_H) \\ &\leq \frac{1}{M-m}\left[\exp(M)(A - m1_H) + \exp(m)(M1_H - A\right] - \exp(A) \\ &\leq \frac{1}{2}\exp(M)(M1_H - A)(A - m1_H). \end{aligned} \tag{3.22}$$

Consider the function $f : [m, M] \to \mathbb{R}$, $f(t) = -\ln t$ and $[m, M] \subset (0, \infty)$. Then by (3.10) we have:

Theorem 3.6. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m := \min\{\lambda \,|\lambda \in Sp(A)\}$ and $M := \max\{\lambda \,|\lambda \in Sp(A)\}$ with $[m, M] \subset (0, \infty)$, then*

$$\begin{aligned} &\frac{1}{2M^2}(M1_H - A)(A - m1_H) \\ &\leq \ln(A) - \frac{1}{M-m}\left[\ln(M)(A - m1_H) + \ln(m)(M1_H - A\right] \\ &\leq \frac{1}{2m^2}(M1_H - A)(A - m1_H). \end{aligned} \tag{3.23}$$

If we take the power function $f : [m, M] \to \mathbb{R}$, $f(t) = t^p, p \geq 2$ and $[m, M] \subset [0, \infty)$ then by (3.10) we also have:

Theorem 3.7. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m := \min\{\lambda\,|\lambda \in Sp(A)\}$ and $M := \max\{\lambda\,|\lambda \in Sp(A)\}$ with $[m, M] \subset [0, \infty)$, then*

$$\begin{aligned}\tag{3.24} &\frac{1}{2}p(p-1)\,m^{p-2}(M1_H - A)(A - m1_H)\\ &\le \frac{1}{M-m}\left[M^p(A - m1_H) + m^p(M1_H - A\right] - A^p\\ &\le \frac{1}{2}p(p-1)\,M^{p-2}(M1_H - A)(A - m1_H).\end{aligned}$$

Finally, consider the convex function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \left|t - \frac{m+M}{2}\right|$. Utilizing the inequality (3.21) we have:

Theorem 3.8. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m := \min\{\lambda\,|\lambda \in Sp(A)\}$ and $M := \max\{\lambda\,|\lambda \in Sp(A)\}$, then*

$$\tag{3.25} 0 \le \frac{M-m}{2} - \left|A - \frac{m+M}{2}\right| \le \frac{2}{M-m}(M1_H - A)(A - m1_H).$$

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