

LP-Sasakian manifolds with quasi-conformal curvature tensor

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Abstract. The object of the present paper is to study *LP*-Sasakian manifolds with quasi-conformal curvature tensor.

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§1. Introduction

In 1989 Matsumoto [7] introduced the notion of Lorentzian para-Sasakian manifolds. Then Mihai and Rosca [10] defined the same notion independently and they obtained several results in this manifold. *LP*-Sasakian manifolds have also been studied by Matsumoto and Mihai [8], Matsumoto, Mihai and Rosca [9], De and Shaikh [3], Ozgur [12] and many others.

The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [16]. According to them a quasi-conformal curvature tensor is defined by

$$(1.1) \quad \begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z \\ &+ b[S(Y, Z)X - S(X, Z)Y \\ &+ g(Y, Z)QX - g(X, Z)QY] \\ &- \frac{r}{n} \left(\frac{a}{n-1} + 2b \right) [g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where a and b are non-zero constants, R is the curvature tensor, S is the Ricci tensor, Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$ and r is the

scalar curvature of the Riemannian manifold (M^n, g) ($n \geq 3$). If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the form

$$\begin{aligned}
 (1.2) \quad \tilde{C}(X, Y)Z &= R(X, Y)Z \\
 &\quad - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y \\
 &\quad + g(Y, Z)QX - g(X, Z)QY] \\
 &\quad + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] \\
 &= C(X, Y)Z,
 \end{aligned}$$

where C is the conformal curvature tensor [15]. Thus the conformal curvature tensor C is a particular case of the tensor \tilde{C} . For this reason \tilde{C} is called the quasi-conformal curvature tensor. A Riemannian manifold (M^n, g) ($n > 3$) shall be called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C} = 0$. It is known [1] that the quasi-conformally flat Riemannian manifold is either conformally flat if $a \neq 0$ or, Einstein if $a = 0$ and $b \neq 0$. Since they give no restrictions if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$.

In [5], De and Matsuyama studied quasi-conformally flat Riemannian manifolds satisfying a certain condition on the Ricci tensor. From Theorem 5 of [5], it can be proved that a 4-dimensional quasiconformally flat semi-Riemannian manifold is the Robertson-Walker space time. Robertson-Walker spacetime is the warped product $I \times_f M^*$, where M^* is a space of constant curvature and I is an open interval [11]. From (1.1), we obtain

$$\begin{aligned}
 (1.3) \quad (\nabla_W \tilde{C})(X, Y)Z &= a(\nabla_W R)(X, Y)Z + b[(\nabla_W S)(Y, Z)X \\
 &\quad - (\nabla_W S)(X, Z)Y \\
 &\quad + g(Y, Z)(\nabla_W Q)(X) - g(X, Z)(\nabla_W Q)(Y)] \\
 &\quad - \frac{dr(W)}{n} \left[\frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y],
 \end{aligned}$$

where ∇ denotes the Levi-Civita connection. If the condition

$$\nabla R = 0$$

holds on M^n , then M^n is called locally symmetric. An LP -Sasakian manifold (M^n, g) is said to be locally ϕ -symmetric if

$$(1.4) \quad \phi^2((\nabla_X R)(Y, Z)W) = 0$$

for all vector fields X, Y, Z, W orthogonal to ξ . This notion was introduced for Sasakian manifolds by Takahashi [14]. Later in [2], Blair, Koufogiorgos and Sharma studied locally ϕ -symmetric contact metric manifolds.

In (1.4), if X, Y, Z and W are not horizontal vectors then we call the manifold globally ϕ -symmetric.

In this paper, we study locally ϕ -quasiconformally symmetric and globally ϕ -quasiconformally symmetric LP -Sasakian manifolds. An LP -Sasakian manifold is called locally ϕ -quasiconformally symmetric if the condition

$$(1.5) \quad \phi^2 \left((\nabla_X \tilde{C})(Y, Z)W \right) = 0$$

holds on M^n , where X, Y, Z and W are horizontal vectors. If X, Y, Z and W are arbitrary vectors then the manifold is called globally ϕ -quasiconformally symmetric.

A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([13], [6]) if $R(X, Y) \cdot R = 0$, where R is the Riemannian curvature tensor and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y . If a Riemannian manifold satisfies $R(X, Y) \cdot \tilde{C} = 0$, then the manifold is said to be quasi-conformally semi-symmetric manifold.

The paper is organized as follows.

After introduction in Section 2, we give a brief account of LP -Sasakian manifolds. In the next two sections, we prove that in a complete simply connected LP -Sasakian manifold if M is quasi-conformally flat, then M is isometric to the Lorentz sphere $\mathbb{S}_1^n(1)$, and if M is a quasi-conformally semi-symmetric and $a + (n - 2)b \neq 0$, then M is isometric to the Lorentz sphere $\mathbb{S}_1^n(1)$. In Section 5, we study globally ϕ -quasiconformally symmetric LP -Sasakian manifolds. We prove that a globally ϕ -quasiconformally symmetric LP -Sasakian manifold is globally ϕ -symmetric if $a \neq 0$. In the next Section, we study 3-dimensional locally ϕ -quasiconformally symmetric LP -Sasakian manifolds. We prove that a 3-dimensional LP -Sasakian manifold is locally ϕ -quasiconformally symmetric if and only if the scalar curvature r is constant if $a + b \neq 0$ and $r \neq 6$. Finally, we construct an example of a 3-dimensional quasi-conformally flat LP -Sasakian manifold.

§2. Preliminaries

Let M^n be an n -dimensional differentiable manifold endowed with a $(1, 1)$ tensor field ϕ , a contravariant vector field ξ , a covariant vector field η and a Lorentzian metric g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p: T_pM \times T_pM \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, \dots, +)$, where T_pM denotes the tangent space of M at p and \mathbb{R} is the real number space which satisfies

$$(2.1) \quad \phi^2(X) = X + \eta(X)\xi, \quad \eta(\xi) = -1,$$

$$(2.2) \quad g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all vector fields X, Y . Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [7]. In the Lorentzian almost paracontact manifold M^n , the following relations hold [7] :

$$(2.3) \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$(2.4) \quad \Omega(X, Y) = \Omega(Y, X),$$

where $\Omega(X, Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor S and the scalar curvature r are defined by

$$S(X, Y) = \sum_{i=1}^n \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^n \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1$, $\epsilon_2 = \dots = \epsilon_n = 1$.

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called Lorentzian paracontact manifold if

$$\Omega(X, Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}.$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called an LP -Sasakian manifold [7] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In an LP -Sasakian manifold the 1-form η is closed. Also in [7], it is proved that if an n -dimensional Lorentzian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then M^n admits an LP -Sasakian structure. Also since the 1-form η is closed in an LP -Sasakian manifold, we have ([7], [8])

$$(2.5) \quad (\nabla_X \eta)Y = \Omega(X, Y),$$

$$(2.6) \quad \Omega(X, \xi) = 0,$$

$$(2.7) \quad \nabla_X \xi = \phi X$$

for any vector field X and Y .

Further, on such an LP -Sasakian manifold $M^n(\phi, \xi, \eta, g)$, the following relations hold [7]:

$$(2.8) \quad \eta(R(X, Y)Z) = [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$

$$(2.9) \quad S(X, \xi) = (n - 1)\eta(X),$$

$$(2.10) \quad R(X, Y)\xi = [\eta(Y)X - \eta(X)Y],$$

$$(2.11) \quad R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(2.12) \quad (\nabla_X \phi)(Y) = [g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]$$

for all vector fields X, Y, Z , where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

§3. Quasi-conformally flat LP -Sasakian manifold

When the quasi-conformal curvature tensor vanishes identically on the Lorentzian manifold, then we find from (1.1)

$$(3.1) \quad \begin{aligned} a\tilde{R}(X, Y, Z, W) = & b\{S(X, Z)g(Y, W) - S(Y, Z)g(X, W) \\ & + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)\} \\ & + \frac{r}{n} \left(\frac{a}{n-1} + 2b \right) \\ & \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}, \end{aligned}$$

which implies that

$$(3.2) \quad \{a + (n - 2)b\} \left\{ S(Y, Z) - \frac{r}{n}g(Y, Z) \right\} = 0.$$

Thus we obtain $a + (n - 2)b = 0$ or $S(Y, Z) = \frac{r}{n}g(Y, Z)$. If $a + (n - 2)b = 0$, then the conformal curvature tensor vanishes identically. It is known that a conformally flat LP -Sasakian manifold is of constant curvature [4]. When M is an Einstein LP -Sasakian manifold, we get $r = n(n - 1)$. It is easy to see

form (3.1) that M is of constant curvature 1. Conversely, if M is of constant curvature, then the quasi-conformal curvature tensor vanishes. Hence we have

Theorem 3.1. *Let M^n ($n > 3$) be an LP-Sasakian manifold. Then M is quasi-conformally flat if and only if it is of constant curvature.*

From [11], we have

Theorem 3.2. *Let M^n ($n > 3$) be a complete simply connected LP-Sasakian manifold. If M is quasi-conformally flat, then M is isometric to the Lorentz sphere $\mathbb{S}_1^n(1)$.*

§4. LP-Sasakian manifolds satisfying $R(\xi, Y) \cdot \tilde{C} = 0$

In this section we consider an LP-Sasakian manifold M^n ($n > 3$) satisfying the condition

$$(4.1) \quad (R(\xi, Y) \cdot \tilde{C})(U, V)W = 0,$$

which yields from (2.11) that

$$\begin{aligned} &g(\tilde{C}(U, V)W, Y)\xi - \eta(\tilde{C}(U, V)W)Y - g(Y, U)\tilde{C}(\xi, V)W \\ &+ \eta(U)\tilde{C}(Y, V)W - g(Y, V)\tilde{C}(U, \xi)W + \eta(V)\tilde{C}(U, Y)W \\ &- g(Y, W)\tilde{C}(U, V)\xi + \eta(W)\tilde{C}(U, V)Y = 0. \end{aligned}$$

Operating η to the above equation and using of (1.1), (2.8) \sim (2.11) we obtain

$$(4.2) \quad \begin{aligned} &g(\tilde{C}(U, V)W, Y) + bg(U, Y)\{S(V, W) + (n-1)\eta(V)\eta(W)\} \\ &- bg(Y, V)\{S(U, W) + (n-1)\eta(U)\eta(W)\} \\ &+ b\{S(V, Y)\eta(U) - S(Y, U)\eta(V)\}\eta(W) \\ &- \left\{ a + (n-1)b - \frac{r}{n} \left(\frac{a}{n-1} + 2b \right) \right\} \\ &\{g(V, W)g(Y, U) - g(U, W)g(V, Y)\} = 0. \end{aligned}$$

Putting $Y = U = e_i$ in the above equation and taking summation over i , we get

$$(4.3) \quad \begin{aligned} &(a-b)S(V, W) - \{(n-1)a + (n-1)^2b - br\}g(V, W) \\ &+ b\{r - n(n-1)\}\eta(V)\eta(W) = 0, \end{aligned}$$

moreover, we find $\{a + (n-2)b\}\{r - n(n-1)\} = 0$.

We can consider the two cases. At first, in the case of $r = n(n-1)$, we have form (4.3)

$$(a-b)\{S(V, W) - (n-1)g(V, W)\} = 0.$$

If $a \neq b$, then $S(V, W) = (n - 1)g(V, W)$. Therefore it is clear from (4.2) that the quasi-conformal curvature tensor vanishes, namely, M is of constant curvature 1 from Theorem 3.1. Also, if $a = b(\neq 0)$, then we get from (1.1) and (4.2)

$$\begin{aligned} &g(R(U, V)W, Y) + \{2S(V, W) - ng(V, W) + (n - 1)\eta(V)\eta(W)\}g(U, Y) \\ &- \{2S(U, W) - ng(U, W) + (n - 1)\eta(U)\eta(W)\}g(V, Y) \\ &+ \{g(V, W) - \eta(V)\eta(W)\}S(U, Y) - \{g(U, W) - \eta(U)\eta(W)\}S(V, Y) = 0. \end{aligned}$$

If we put $W = \xi$, then we have $\eta(V)S(U, Y) - \eta(U)S(V, Y) = 0$. Furthermore, putting $U = \xi$, we get $S(V, Y) = -(n - 1)\eta(V)\eta(Y)$, that is $r = n - 1$. This is the contradiction. Thus $a \neq b$ holds.

Secondly, in the case of $a + (n - 2)b = 0$, equation (4.2) is rewritten as follows:

$$\begin{aligned} (4.4) \quad &(n - 2)g(R(U, V)W, Y) \\ &- \{2S(V, W) - g(V, W) + (n - 1)\eta(V)\eta(W)\}g(U, Y) \\ &+ \{2S(U, W) - g(U, W) + (n - 1)\eta(U)\eta(W)\}g(V, Y) \\ &- \{g(V, W) - \eta(V)\eta(W)\}S(U, Y) \\ &+ \{g(U, W) - \eta(U)\eta(W)\}S(V, Y) = 0. \end{aligned}$$

We put $U = W = \xi$. Then we find $S(V, Y) = -(n - 1)\eta(V)\eta(Y)$, which yields from (4.2) that

$$\begin{aligned} (4.5) \quad R(U, V)W &= \frac{1}{4}(c + 3)\{g(V, W)U - g(U, W)V\} \\ &+ \frac{1}{4}(c - 1)\{\eta(V)\eta(W)U - \eta(U)\eta(W)V\} \\ &+ g(V, W)\eta(U)\xi - g(U, W)\eta(V)\xi, \end{aligned}$$

where $c = -\frac{3n-2}{n-2}$. Hence we have

Theorem 4.1. *Let $M^n(n > 3)$ be an LP-Sasakian manifold satisfying $R(\xi, Y) \cdot \tilde{C} = 0$ for any Y .*

- (1) *If $a + (n - 2)b \neq 0$, then M is of constant curvature 1.*
- (2) *If $a + (n - 2)b = 0$, then M is a space satisfying (4.5).*

Corollary 4.1. *Let $M^n(n > 3)$ be an LP-Sasakian manifold. If M is a quasi-conformally semi-symmetric, then*

- (1) *when $a + (n - 2)b \neq 0$, then M is of constant curvature 1.*
- (2) *when $a + (n - 2)b = 0$, then M is a space satisfying (4.5).*

From [11], we have

Theorem 4.2. *Let $M^n(n > 3)$ be a complete simply connected LP-Sasakian*

manifold satisfying $R(\xi, Y) \cdot \tilde{C} = 0$ for any Y . If $a + (n - 2)b \neq 0$, then M is isometric to the Lorentz sphere $\mathbb{S}_1^n(1)$.

Corollary 4.2. *Let $M^n(n > 3)$ be a complete simply connected LP-Sasakian manifold. If M is a quasi-conformally semi-symmetric and $a + (n - 2)b \neq 0$, then M is isometric to the Lorentz sphere $\mathbb{S}_1^n(1)$.*

§5. Globally ϕ -quasiconformally symmetric LP-Sasakian manifolds

Let us suppose that M is a globally ϕ -quasiconformally symmetric LP-Sasakian manifold. Then by definition

$$\phi^2 \left((\nabla_W \tilde{C})(X, Y)Z \right) = 0.$$

Using (2.1) we have

$$(\nabla_W \tilde{C})(X, Y)Z + \eta \left((\nabla_W \tilde{C})(X, Y)Z \right) \xi = 0.$$

From (1.2) it follows from (2.7) and (2.9) that

$$(5.1) \quad \begin{aligned} & a\{g((\nabla_W R)(X, Y)Z, U) + \eta(U)\eta((\nabla_W R)(X, Y)Z)\} \\ & + b\{g(X, U) + \eta(X)\eta(U)\}(\nabla_W S)(Y, Z) \\ & - \{g(Y, U) + \eta(Y)\eta(U)\}(\nabla_W S)(X, Z) \\ & + g(Y, Z)\{(\nabla_W S)(X, U) - \eta(U)S(X, \phi W) + (n - 1)\eta(U)g(X, \phi W)\} \\ & - g(X, Z)\{(\nabla_W S)(Y, U) - \eta(U)S(Y, \phi W) + (n - 1)\eta(U)g(Y, \phi W)\} \\ & - \frac{dr(W)}{n} \left(\frac{a}{n - 1} + 2b \right) [g(Y, Z)\{g(X, U) + \eta(X)\eta(U)\} \\ & \quad - g(X, Z)\{g(Y, U) + \eta(Y)\eta(U)\}] = 0. \end{aligned}$$

Putting $Z = \xi$, in (5.1) and using of (2.7), (2.9) and (2.10), we obtain

$$(5.2) \quad \begin{aligned} & a\{g(Y, \phi W)g(X, U) - g(X, \phi W)g(Y, U) - g(R(X, Y)\phi W, U)\} \\ & + b[\eta(Y)(\nabla_W S)(X, U) - \eta(X)(\nabla_W S)(Y, U) \\ & - g(X, U)\{S(Y, \phi W) - (n - 1)g(Y, \phi W)\} \\ & + g(Y, U)\{S(X, \phi W) - (n - 1)g(X, \phi W)\}] \\ & - \frac{dr(W)}{n} \left(\frac{a}{n - 1} + 2b \right) \{ \eta(Y)g(X, U) \\ & \quad - \eta(X)g(Y, U) \} = 0. \end{aligned}$$

Moreover, putting $X = U = e_i$ in (5.2) and taking summation over i , we obtain

$$\{a + (n - 2)b\} \left\{ S(Y, \phi W) - (n - 1)g(Y, \phi W) + \frac{dr(W)}{n}\eta(Y) \right\} = 0.$$

Thus if $a + (n - 2)b \neq 0$, then we find $S(Y, \phi W) = (n - 1)g(Y, \phi W) - \frac{dr(W)}{n}\eta(Y)$. Setting $Y = \xi$, we have $dr(W) = 0$, that is, the scalar curvature is constant. It is easy to see from (2.9) that $S(Y, W) = (n - 1)g(Y, W)$, that is, M is an Einstein. Since (5.2), we find that M is of constant curvature 1. Next, when $a + (n - 2)b = 0$, it is clear from (5.2) that

$$\begin{aligned} & (n - 2)\{g(R(X, Y)\phi W, U) - g(Y, \phi W)g(X, U) + g(X, \phi W)g(Y, U)\} \\ & + \eta(Y)(\nabla_W S)(X, U) - g(X, U)\{S(Y, \phi W) - (n - 1)g(Y, \phi W)\} \\ & - \eta(X)(\nabla_W S)(Y, U) + g(Y, U)\{S(X, \phi W) - (n - 1)g(X, \phi W)\} \\ & - \frac{dr(W)}{n - 1}\{\eta(Y)g(X, U) - \eta(X)g(Y, U)\} = 0. \end{aligned}$$

Setting $Y = \xi$ in the above equation, we get

$$\begin{aligned} (\nabla_W S)(X, U) &= \eta(X)\{S(U, \phi W) - (n - 1)g(U, \phi W)\} \\ &+ \eta(U)\{S(X, \phi W) - (n - 1)g(X, \phi W)\} \\ &+ \frac{dr(W)}{n - 1}\{g(X, U) + \eta(X)\eta(U)\}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} (5.3) \quad R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y \\ &+ \frac{1}{n - 2}\{S(Y, Z) - (n - 1)g(Y, Z)\}\phi^2 X \\ &- \frac{1}{n - 2}\{S(X, Z) - (n - 1)g(X, Z)\}\phi^2 Y. \end{aligned}$$

Hence we have

Theorem 5.1. *Let M^n ($n > 3$) be a globally ϕ -quasiconformally symmetric LP-Sasakian manifold.*

- (1) *If $a + (n - 2)b \neq 0$, then M is of constant curvature 1.*
- (2) *If $a + (n - 2)b = 0$, then M is a space satisfying (5.3).*

From [11], we have

Theorem 5.2. *Let M^n ($n > 3$) be a complete simply connected LP-Sasakian manifold. If M is globally ϕ -quasiconformally symmetric and $a + (n - 2)b \neq 0$,*

then M is isometric to the Lorentz sphere $\mathbb{S}_1^n(1)$.

Moreover, by virtue of (5.1) and Theorem 5.1, we find

$$a[(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi] = 0,$$

which implies that $\phi^2((\nabla_W R)(X, Y)Z) = 0$ if $a \neq 0$. Hence we can state:

Theorem 5.3. *A globally ϕ -quasiconformally symmetric LP-Sasakian manifold is globally ϕ -symmetric if $a \neq 0$.*

§6. 3-dimensional locally ϕ -quasiconformally symmetric LP-Sasakian manifolds

Let us consider a 3-dimensional LP-Sasakian manifold. It is known that the conformal curvature tensor vanishes identically in the 3-dimensional Riemannian manifold. Thus we find

$$(6.1) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y],$$

where Q is the Ricci operator, that is, $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold.

Putting $Z = \xi$ in (6.1) and using (2.10) we have

$$(6.2) \quad \eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} - 1\right) [\eta(Y)X - \eta(X)Y].$$

Putting $Y = \xi$ in (6.2) and using (2.1) and (2.9), we get

$$(6.3) \quad QX = \frac{1}{2}[(r - 2)X + (r - 6)\eta(X)\xi],$$

that is,

$$(6.4) \quad S(X, Y) = \frac{1}{2}[(r - 2)g(X, Y) + (r - 6)\eta(X)\eta(Y)].$$

Using (6.3) in (6.1), we get

$$(6.5) \quad R(X, Y)Z = \left(\frac{r - 4}{2}\right) [g(Y, Z)X - g(X, Z)Y] + \left(\frac{r - 6}{2}\right) [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].$$

Putting (6.3), (6.4) and (6.5) into (1.1) we have

$$(6.6) \quad \begin{aligned} \tilde{C}(X, Y)Z &= (a+b)(r-6) \left[\frac{1}{3} \{g(Y, Z)X - g(X, Z)Y\} \right. \\ &\quad + \frac{1}{2} \{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &\quad \left. + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \right]. \end{aligned}$$

Thus we have

Lemma 6.1. *Let M be a 3-dimensional LP-Sasakian manifold. If $a+b=0$ or $r=6$, then the quasi-conformal curvature tensor vanishes identically.*

Next, we assume that $a+b \neq 0$ or $r \neq 6$. Taking the covariant differentiation of (6.6), we get

$$\begin{aligned} (\nabla_W \tilde{C})(X, Y)Z &= \frac{dr(W)}{3} (a+b) \{g(Y, Z)X - g(X, Z)Y\} \\ &\quad + \frac{dr(W)}{2} (a+b) \{g(Y, Z)\eta(X)\xi \\ &\quad - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ &\quad + \frac{1}{2} (r-6)(a+b) [\{g(Y, Z)g(X, \phi W) \\ &\quad - g(X, Z)g(Y, \phi W)\}\xi \\ &\quad + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\phi W \\ &\quad + \{\eta(Z)g(Y, \phi W) + \eta(Y)g(Z, \phi W)\}X \\ &\quad - \{\eta(Z)g(X, \phi W) + \eta(X)g(Z, \phi W)\}Y]. \end{aligned}$$

Operating ϕ^2 to the above equation, then we find

$$\begin{aligned} \phi^2((\nabla_W \tilde{C})(X, Y)Z) &= \frac{dr(W)}{3} (a+b) \{g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y\} \\ &\quad + \frac{dr(W)}{2} (a+b)\eta(Z) \{\eta(Y)\phi^2 X - \eta(X)\phi^2 Y\} \\ &\quad + \frac{1}{2} (r-6)(a+b) [\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\phi W \\ &\quad + \{\eta(Z)g(Y, \phi W) + \eta(Y)g(Z, \phi W)\}\phi^2 X \\ &\quad - \{\eta(Z)g(X, \phi W) + \eta(X)g(Z, \phi W)\}\phi^2 Y]. \end{aligned}$$

If the vector fields X , Y and Z are horizontal, then the above equation is rewritten as follows:

$$\phi^2((\nabla_W \tilde{C})(X, Y)Z) = \frac{dr(W)}{3} (a+b) \{g(Y, Z)X - g(X, Z)Y\}.$$

Hence we conclude the following theorem:

Theorem 6.1. *A 3-dimensional LP-Sasakian manifold is locally ϕ -quasiconformally symmetric if and only if the scalar curvature r is constant if $a+b \neq 0$ and $r \neq 6$.*

§7. Example

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}_1^3\}$, where (x, y, z) are standard coordinates of \mathbb{R}_1^3 .

The vector fields

$$e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M .

Let g be the Lorentzian metric defined by

$$\begin{aligned} g(e_1, e_1) &= g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \\ g(e_1, e_2) &= g(e_1, e_3) = g(e_2, e_3) = 0. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any vector field $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g we have

$$\begin{aligned} \eta(e_3) &= -1, \\ \phi^2 Z &= Z + \eta(Z)e_3, \\ g(\phi Z, \phi W) &= g(Z, W) + \eta(Z)\eta(W) \end{aligned}$$

for any vector fields $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M .

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g . Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1$$

and

$$[e_2, e_3] = -e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for the Lorentzian metric g , we can easily calculate

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_3, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= -e_1, \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= -e_3, & \nabla_{e_2} e_3 &= -e_2, \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an *LP*-Sasakian manifold. With the help of the above results it can be easily verified that

$$\begin{aligned} R(e_1, e_2)e_3 &= 0, & R(e_2, e_3)e_3 &= -e_2, & R(e_1, e_3)e_3 &= -e_1, \\ R(e_1, e_2)e_2 &= e_1, & R(e_2, e_3)e_2 &= -e_3, & R(e_1, e_3)e_2 &= 0, \\ R(e_1, e_2)e_1 &= -e_2, & R(e_2, e_3)e_1 &= 0, & R(e_1, e_3)e_1 &= -e_3. \end{aligned}$$

From the above expressions of the curvature tensor we obtain

$$\begin{aligned} S(e_1, e_1) &= g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) \\ &= 2. \end{aligned}$$

Similarly we have

$$S(e_2, e_2) = 2, S(e_3, e_3) = -2$$

and

$$S(e_i, e_j) = 0(i \neq j).$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.$$

Thus the 3-dimensional *LP*-Sasakian manifold is quasi-conformally flat. Therefore Lemma 6.1. is verified.

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