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Generalized marginal cumulative logistic model for multi-way contingency tables

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Abstract. For multi-way contingency tables of same classifications with ordered categories, this paper proposes two generalized marginal cumulative logistic models. Those indicate that the difference between two marginal cumulative logits, or two conditional marginal cumulative logits on condition that all values of the variables are not identical, is a polynomial function of the category value. This paper also gives a theorem that the marginal homogeneity model holds if and only if both the proposed model and moment equality model hold. An example is given.

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§1. Introduction

Consider a multi-way r^T contingency table of same classifications having ordered categories. Let X_t denote the t-th random variable ($t = 1, \dots, T$), and let $\Pr(X_1 = i_1, \dots, X_T = i_T) = p_{i_1 \cdots i_T}$ ($i_t = 1, \dots, r$). The marginal homogeneity (MH) model is defined by

$$p_i^{(1)} = \cdots = p_i^{(T)} \quad (i = 1, \dots, r),$$

where

$$p_i^{(t)} = \Pr(X_t = i);$$

see e.g., Stuart (1955), Bishop, Fienberg and Holland (1975, p.303) and Agresti (2002, p.440). The MH model also may be expressed as

$$p_i^{c(1)} = \cdots = p_i^{c(T)} \quad (i = 1, \dots, r),$$

where

$$p_i^{c(t)} = \Pr(X_t = i | (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, r).$$

Let $L_i^{(t)}$ denote the marginal cumulative logit of X_t ($i = 1, \ldots, r-1; t = 1, \ldots, T$). Thus

$$L_i^{(t)} = \operatorname{logit}\left(F_i^{(t)}\right) = \log\left(\frac{F_i^{(t)}}{1 - F_i^{(t)}}\right),$$

where

$$F_i^{(t)} = \sum_{k=1}^{i} p_k^{(t)}\ [= \Pr(X_t \leq i)].$$

Agresti (2002, p.442) considered the marginal cumulative logistic (L) model which indicates that the difference between two marginal cumulative logits is constant. Define the conditional marginal cumulative logits by

$$L_i^{c(t)} = \operatorname{logit}\left(F_i^{c(t)}\right) \quad (i = 1, \ldots, r-1; t = 1, \ldots, T),$$

where

$$F_i^{c(t)} = \sum_{k=1}^{i} p_k^{c(t)}\ [= \Pr(X_t \leq i | (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, r)].$$

On condition that all values of the variables are not identical, Tahata, Katakura and Tomizawa (2007) proposed the conditional marginal cumulative logistic (CL) model by using $\{L_i^{c(t)}\}$.

Kurakami, Tahata and Tomizawa (2010) proposed the extended marginal cumulative logistic (EL) model defined by

$$L_i^{(t)} = L_i^{(1)} - \left(\delta_0^{(t)} + i\delta_1^{(t)}\right) \quad (i = 1, \ldots, r-1; t = 2, \ldots, T).$$

Kurakami et al. (2010) also proposed the extended conditional marginal cumulative logistic (ECL) model defined by

$$L_i^{c(t)} = L_i^{c(1)} - \left(\delta_0^{(t)*} + i\delta_1^{(t)*}\right) \quad (i = 1, \ldots, r-1; t = 2, \ldots, T).$$

These models indicate that the difference between two marginal cumulative logits (or two conditional marginal cumulative logits) is a linear function of the category value. If these models do not hold, we are interested in applying more extended L or CL models which indicate that the difference between such two logits (or such two conditional logits) is a polynomial function of the category value.

Tahata et al. (2007) gave the decomposition of the MH model using the L (or CL) model, and Kurakami et al. (2010) gave the decomposition of it using the EL (or ECL) model. Note that when $T = 2$, some other decompositions of the MH model are considered by, e.g., Tomizawa (1993, 1998), Tahata and Tomizawa (2008). We are also interested in giving decompositions of the MH model using the proposed models.

For the multi-way tables, the present paper (i) proposes a model which is a generalization of the L model, (ii) proposes a similar model on condition that all values of the variables are not identical, and (iii) gives the decompositions of the MH model using the proposed models.

§2. Models

For a given m ($m = 1, \ldots, r-1$), consider a new model defined by

$$L_i^{(t)} = L_i^{(1)} - \Delta_i^{(m,t)} \quad (i = 1, \ldots, r-1; t = 2, \ldots, T), \tag{2.1}$$

where

$$\Delta_i^{(m,t)} = \sum_{k=0}^{m-1} i^k \delta_k^{(t)},$$

and $\{\delta_k^{(t)}\}$ are unspecified. A special case of this model obtained by putting $\delta_0^{(t)} = \cdots = \delta_{m-1}^{(t)} = 0$ for every t is the MH model. Note that when $m = 1$, this model indicates the L model, and when $m = 2$, this model indicates the EL model. We shall refer to model (2.1) as the m-th generalized marginal cumulative logistic (L(m)) model. Note that when $m = r-1$, the L($r-1$) model is saturated model, i.e., no restriction is imposed.

This model indicates that the odds that the value of X_1 is i or below, instead of $i+1$ or above, is $\exp(\Delta_i^{(m,t)})$ times higher than the odds that the value of X_t is i or below, instead of $i+1$ or above ($i = 1, \ldots, r-1; t = 2, \ldots, T$). Under the L($m$) model, $\Delta_i^{(m,t)} > 0$ is equivalent to $F_i^{(1)} > F_i^{(t)}$ ($i = 1, \ldots, r-1; t = 2, \ldots, T$). Therefore, the parameters $\{\Delta_i^{(m,t)}\}$ would be useful for making inference the relationship between the $\{F_i^{(1)}\}$ and $\{F_i^{(t)}\}$ ($t = 2, \ldots, T$).

Let $L_i^{(1)} = \theta_i$ ($i = 1, \ldots, r-1$). Using the logistic functions, the L(m) model may be expressed as

$$F_i^{(1)} = \frac{\exp(\theta_i)}{1 + \exp(\theta_i)},$$

and

$$F_i^{(t)} = \frac{\exp\left(\theta_i - \Delta_i^{(m,t)}\right)}{1 + \exp\left(\theta_i - \Delta_i^{(m,t)}\right)} \quad (i = 1, \ldots, r-1; t = 2, \ldots, T).$$

Next, using conditional marginal cumulative logits, for a given m ($m = 1, \ldots, r-1$), consider a model defined by

$$L_i^{c(t)} = L_i^{c(1)} - \Delta_i^{(m,t)*} \quad (i = 1, \ldots, r-1; t = 2, \ldots, T), \tag{2.2}$$

where

$$\Delta_i^{(m,t)*} = \sum_{k=0}^{m-1} i^k \delta_k^{(t)*},$$

and $\{\delta_k^{(t)*}\}$ are unspecified. A special case of this model obtained by putting $\delta_0^{(t)*} = \cdots = \delta_{m-1}^{(t)*} = 0$ for every t is the MH model. Note that when $m = 1$, this model indicates the CL model, and when $m = 2$, this model indicates the ECL model. We shall refer to model (2.2) as the m-th generalized conditional marginal cumulative logistic (CL(m)) model. Note that when $m = r-1$, the CL($r-1$) model is saturated model. The CL(m) model has similar properties to the L(m) model in the sense that the parameters in the two models have similar interpretation, on condition that $(X_1, \ldots, X_T) \neq (s, \ldots, s)$, $s = 1, \ldots, r$.

§3. Decompositions of marginal homogeneity model

For given k-th moments ($k = 1, \ldots, r-1$), consider a model defined by

$$E(X_1^k) = \cdots = E(X_T^k), \tag{3.1}$$

where

$$E(X_t^k) = \sum_{i=1}^{r} i^k p_i^{(t)} \quad (t = 1, \ldots, T).$$

When $k = 1$, this model indicates the marginal mean equality (ME) model. We shall refer to (3.1) as the k-th marginal moment equality I (M-I(k)) model. Note that {M-I(k)}, $k = 1, \ldots, r-1$, models hold if and only if the MH model holds. Also note that the M-I(k) model does not depend on the main diagonal probabilities $\{p_{ii\cdots i}\}$.

We obtain the following theorem:

Theorem 1. *For a given m ($m = 1, \ldots, r-1$), the MH model holds if and only if the L(m) and {M-I(k)}, $k = 1, \ldots, m$, models hold.*

Proof. For a given m, if the MH model holds, the L(m) and {M-I(k)}, $k = 1, \ldots, m$, models hold. Assuming that the L(m) and {M-I(k)}, $k = 1, \ldots, m$, models hold, then we shall show that the MH model holds.

We shall consider the relationship between X_1 and X_2. The L(m) model may be expressed as

$$G_i^{(1)} = \Theta_i^{(m)} G_i^{(2)} \quad (i = 1, \ldots, r-1),$$

where

$$\begin{aligned} G_i^{(1)} &= F_i^{(1)} \left(1 - F_i^{(2)}\right), \\ G_i^{(2)} &= \left(1 - F_i^{(1)}\right) F_i^{(2)}, \\ \Theta_i^{(m)} &= \prod_{k=0}^{m-1} \alpha_k^{i^k}, \\ \alpha_k &= \exp\left(\delta_k^{(2)}\right). \end{aligned}$$

Let

$$H_{s(i)} = \frac{G_i^{(s)}}{\sum_{l=1}^{r-1} \left(G_l^{(1)} + G_l^{(2)}\right)} \quad (s = 1, 2; i = 1, \ldots, r-1).$$

Note that $\sum_{i=1}^{r-1} (H_{1(i)} + H_{2(i)}) = 1$. The L($m$) model may be further expressed as

$$H_{1(i)} = \Theta_i^{(m)} H_{2(i)} \quad (i = 1, \ldots, r-1);$$

namely,

$$H_{1(i)} = \Theta_i^{(m)} \gamma_{1(i)} \text{ and } H_{2(i)} = \gamma_{2(i)} \quad (i = 1, \ldots, r-1),$$

with $\gamma_{1(i)} = \gamma_{2(i)}$. Let $\{p^*_{i_1 \cdots i_T}\}$ denote the cell probabilities which satisfy the L(m) and {M-I(k)}, $k = 1, \ldots, m$, models, and let $H^*_{s(i)}$ be $H_{s(i)}$ with $\{p_{i_1 \cdots i_T}\}$ replaced by $\{p^*_{i_1 \cdots i_T}\}$ $(s = 1, 2)$. Since the L(m) model holds, we see

$$\log H^*_{s(i)} = \left(\sum_{k=0}^{m-1} i^k \log \alpha_k \right) \lambda_s + \log \gamma_{s(i)} \quad (s = 1, 2; i = 1, \ldots, r-1),$$

where $\lambda_1 = 1$ and $\lambda_2 = 0$. Let $\pi_{s(i)} = c^{-1} \gamma_{s(i)}$ $(s = 1, 2; i = 1, \ldots, r-1)$ with

$$c = \sum_{l=1}^{r-1} (\gamma_{1(l)} + \gamma_{2(l)}).$$

Note that $\sum_{i=1}^{r-1}(\pi_{1(i)} + \pi_{2(i)}) = 1$ with $0 < \pi_{s(i)} < 1$ $(s = 1, 2)$. Then the L(m) model is expressed as

$$\text{(3.2)}\quad \log\left(\frac{H^*_{s(i)}}{\pi_{s(i)}}\right) = \left(\sum_{k=0}^{m-1} i^k \log\alpha_k\right)\lambda_s + \log c \quad (s = 1, 2; i = 1, \ldots, r-1).$$

For $k = 1, \ldots, m$,

$$\begin{aligned} E\left(X_2^k\right) - E\left(X_1^k\right) &= \sum_{i=1}^{r-1}\left\{(i+1)^k - i^k\right\}\left(F_i^{(1)} - F_i^{(2)}\right) \\ &= \sum_{i=1}^{r-1}\left\{(i+1)^k - i^k\right\}\left(G_i^{(1)} - G_i^{(2)}\right). \end{aligned}$$

Since $\{p^*_{i_1\cdots i_T}\}$ satisfy the $\{$M-I(k)$\}$ models, we have

$$\text{(3.3)}\qquad \nu_1^{(u)*} = \nu_2^{(u)*} \quad (u = 0, 1, \ldots, m-1),$$

where

$$\nu_s^{(u)*} = \sum_{i=1}^{r-1} i^u H^*_{s(i)} \quad (s = 1, 2).$$

We denote $\nu_1^{(u)*} (= \nu_2^{(u)*})$ by $\nu^{(u)}$.

Consider arbitrary cell probabilities $\{p_{i_1\cdots i_T}\}$ that satisfy

$$\text{(3.4)}\qquad \nu_1^{(u)} = \nu_2^{(u)} = \nu^{(u)} \quad (u = 0, 1, \ldots, m-1),$$

where

$$\nu_s^{(u)} = \sum_{i=1}^{r-1} i^u H_{s(i)} \quad (s = 1, 2).$$

From (3.2), (3.3) and (3.4), we have

$$\text{(3.5)}\qquad \sum_{s=1}^{2}\sum_{i=1}^{r-1}\left(H_{s(i)} - H^*_{s(i)}\right)\log\left(\frac{H^*_{s(i)}}{\pi_{s(i)}}\right) = 0.$$

Let

$$K\left(H, \pi\right) = \sum_{s=1}^{2}\sum_{i=1}^{r-1} H_{s(i)}\log\left(\frac{H_{s(i)}}{\pi_{s(i)}}\right),$$

and

$$K\left(H^*, \pi\right) = \sum_{s=1}^{2}\sum_{i=1}^{r-1} H^*_{s(i)}\log\left(\frac{H^*_{s(i)}}{\pi_{s(i)}}\right).$$

Note that $K(\cdot,\cdot)$ is the Kullback-Leibler information. From (3.5), we obtain

$$K(H,\pi) = K(H^*,\pi) + K(H,H^*),$$

where

$$K(H,H^*) = \sum_{s=1}^{2}\sum_{i=1}^{r-1} H_{s(i)} \log\left(\frac{H_{s(i)}}{H^*_{s(i)}}\right).$$

Since π is fixed, we see

$$\min_H K(H,\pi) = K(H^*,\pi),$$

and then $\{H^*_{s(i)}\}$ uniquely minimize $K(H,\pi)$ (Darroch and Ratcliff, 1972). Thus we obtain

$$H_{s(i)} = H^*_{s(i)} \quad (s=1,2; i=1,\ldots,r-1). \tag{3.6}$$

Let $\{p^{**}_{i_1 i_2 i_3\cdots i_T} = p^*_{i_2 i_1 i_3 \cdots i_T}\}$, and let $H^{**}_{1(i)} = H^*_{2(i)}$ and $H^{**}_{2(i)} = H^*_{1(i)}$ $(i = 1,\ldots,r-1)$. Note that $\{\pi_{1(i)} = \pi_{2(i)}\}$. We see

$$\min_H K(H,\pi) = K(H^{**},\pi),$$

where

$$K(H^{**},\pi) = \sum_{s=1}^{2}\sum_{i=1}^{r-1} H^{**}_{s(i)} \log\left(\frac{H^{**}_{s(i)}}{\pi_{s(i)}}\right),$$

and then $\{H^{**}_{s(i)}\}$ uniquely minimize $K(H,\pi)$. Therefore, we obtain

$$H_{1(i)} = H^{**}_{1(i)} = H^*_{2(i)} \quad (i=1,\ldots,r-1), \tag{3.7}$$

and

$$H_{2(i)} = H^{**}_{2(i)} = H^*_{1(i)} \quad (i=1,\ldots,r-1).$$

Let $G_i^{(s)*}$ and $F_i^{(s)*}$ be $G_i^{(s)}$ and $F_i^{(s)}$ with $\{p_{i_1\cdots i_T}\}$ replaced by $\{p^*_{i_1\cdots i_T}\}$ $(s = 1,2)$, respectively. From (3.6) and (3.7), we see

$$H^*_{1(i)} = H^*_{2(i)} \quad (i=1,\ldots,r-1);$$

that is

$$G_i^{(1)*} = G_i^{(2)*} \quad (i=1,\ldots,r-1),$$

and we obtain

$$F_i^{(1)*} = F_i^{(2)*} \quad (i=1,\ldots,r-1).$$

In a similar way, considering the relationship between X_1 and X_t $(t = 3,\ldots,T)$, we obtain $F_i^{(1)*} = F_i^{(t)*}$ $(i = 1,\ldots,r-1)$. Thus the MH model holds. □

Note that a special case of this theorem with $m = 1$ is given by Tahata et al. (2007).

We also obtain the following theorem:

Theorem 2. *For a given m ($m = 1, \ldots, r-1$), the MH model holds if and only if the CL(m) and $\{$M-I(k)$\}$, $k = 1, \ldots, m$, models hold.*

The proof can be obtained in a similar manner to the proof of Theorem 1 by replacing $\{F_i^{(1)}\}$ and $\{F_i^{(2)}\}$ with $\{F_i^{c(1)}\}$ and $\{F_i^{c(2)}\}$, respectively. Note that a special case of this theorem with $m = 1$ is given by Tahata et al. (2007).

For a given integer k ($k = 2, \ldots, r-1$), consider a model defined by

$$\mu_k^{(1)} = \cdots = \mu_k^{(T)}, \tag{3.8}$$

where $\mu_k^{(t)} = E((X_t - E(X_t))^k)$ ($t = 1, \ldots, T$). When $k = 2$, this model indicates the marginal variance equality model. We shall refer to (3.8) as the k-th marginal moment equality II (M-II(k)) model.

On condition that all values of the variables are not identical, for a given k ($k = 2, \ldots, r-1$), consider the k-th conditional marginal moment equality II (CM-II(k)) model defined by

$$\mu_k^{c(1)} = \cdots = \mu_k^{c(T)},$$

where

$$\mu_k^{c(t)} = E[(X_t - \mu_1^{c(t)})^k | (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, r],$$

with

$$\mu_1^{c(t)} = E[X_t | (X_1, \ldots, X_T) \neq (s, \ldots, s),\ s = 1, \ldots, r].$$

We obtain the following remarks:

Remark 1. *For a given m ($m = 2, \ldots, r-1$), the MH model holds if and only if the L(m), ME and $\{$M-II(k)$\}$, $k = 2, \ldots, m$, models hold.*

Remark 2. *For a given m ($m = 2, \ldots, r-1$), the MH model holds if and only if the CL(m), ME and $\{$M-II(k)$\}$, $k = 2, \ldots, m$, models hold.*

Remark 3. *For a given m ($m = 2, \ldots, r-1$), the MH model holds if and only if the CL(m), ME and $\{$CM-II(k)$\}$, $k = 2, \ldots, m$, models hold.*

Because if both the ME and $\{$M-II(k)$\}$ models hold, the $\{$M-I(k)$\}$ models hold, the proof of Remarks 1 and 2 can be obtained. Also, because if both the ME and $\{$CM-II(k)$\}$ models hold, the $\{$M-I(k)$\}$ models hold, the proof of Remark 3 can be obtained. Note that special cases of Remarks 1, 2 and 3 with $m = 2$ are given by Kurakami et al. (2010).

§4. Goodness-of-fit test

Let $n_{i_1 \cdots i_T}$ denote the observed frequency in the $(i_1, \ldots, i_T)$ cell of the r^T table. Assume that a multinomial distribution is applied to the r^T table. The maximum likelihood estimates (MLEs) of expected frequencies under each model could be obtained by using the Newton-Raphson method in the log-likelihood equation. Each model can be tested for goodness-of-fit by, e.g., the likelihood ratio chi-squared statistic (denoted by G^2) with corresponding degrees of freedom (df). The test statistic G^2 for model M is given by

$$G^2(M) = 2 \sum_{i_1=1}^{r} \cdots \sum_{i_T=1}^{r} n_{i_1 \cdots i_T} \log \left(\frac{n_{i_1 \cdots i_T}}{\hat{m}_{i_1 \cdots i_T}} \right),$$

where $\hat{m}_{i_1 \cdots i_T}$ is the MLE of expected frequency $m_{i_1 \cdots i_T}$ under model M. The numbers of df for each model are given in Table 1. For Theorems 1 and 2, we note that the number of df for the MH model is equal to the sum of those for the L(m) (CL(m)) and {M-I(k)}, $k = 1, \ldots, m$, models. When we assume that a Poisson distribution is applied to the r^T table, same results can be obtained.

§5. Example

Consider the data in Table 2, taken directly from Clogg (1982), that is a three-way cross-classification of the indicators of satisfaction with life from the 1977 General Social Survey. The variables X_1, X_2 and X_3 mean L: satisfaction with hobbies, R: satisfaction with family, and C: satisfaction with residence, respectively. Each category means 1:"a fair amount, some, a little, or none"; 2:"quite a bit"; 3:"a great deal"; 4:"a very great deal", respectively. Table 3 gives the likelihood ratio chi-squared value for each model. From Table 3, each of the L(2) and CL(2) models fits these data well, and the others fit poorly.

Consider the hypothesis that the CL(1) model holds under the assumption that the CL(2) model holds; namely, the hypothesis that $\delta_1^{(2)*} = \delta_1^{(3)*} = 0$ under the assumption. Since $G^2(\mathrm{CL}(1)) - G^2(\mathrm{CL}(2)) = 6.90$ with two df, we reject the hypothesis at the 0.05 level. The MLEs of $\delta_0^{(2)*}, \delta_1^{(2)*}, \delta_0^{(3)*}$ and $\delta_1^{(3)*}$ under the CL(2) model are $\hat{\delta}_0^{(2)*} = 1.23, \hat{\delta}_1^{(2)*} = -0.08, \hat{\delta}_0^{(3)*} = -0.62$ and $\hat{\delta}_1^{(3)*} = 0.08$, respectively. The values of $\exp(\hat{\delta}_0^{(2)*} + i\hat{\delta}_1^{(2)*})$ for $i = 1, 2, 3$ are 3.16, 2.91 and 2.68, respectively. Also, the values of $\exp(\hat{\delta}_0^{(3)*} + i\hat{\delta}_1^{(3)*})$ for $i = 1, 2, 3$ are 0.59, 0.64 and 0.69, respectively. Thus we state that, for example, the odds that an observation will fall in "category 1" instead of in "not category 1", on condition that the satisfaction grades are not equal, is estimated to be 3.16 times higher in satisfaction with hobby than in satisfaction with family.

Since $\exp(\hat{\delta}_0^{(2)*} + i\hat{\delta}_1^{(2)*}) > 1$ and $\exp(\hat{\delta}_0^{(3)*} + i\hat{\delta}_1^{(3)*}) < 1$, when the satisfaction grades are not equal, we can estimate that satisfaction with family tends to be better than satisfaction with hobbies, and satisfaction with hobbies tends to be better than satisfaction with residence. In addition, from Remark 3, we can see that the poor fit of the MH model is caused by the influence of the poor fit of the ME and CM-II(2) models rather than the CL(2) model.

The MLEs of the $\delta_0^{(2)}, \delta_1^{(2)}, \delta_0^{(3)}$ and $\delta_1^{(3)}$ under the L(2) model are $\hat{\delta}_0^{(2)} = 0.89$, $\delta_1^{(2)} = -0.03, \hat{\delta}_0^{(3)} = -0.52$ and $\hat{\delta}_1^{(3)} = 0.08$, respectively. Under the L(2) model, we can obtain similar explanations to that under the CL(2) model, although the detail is omitted.

§6. Concluding Remarks

We have proposed the generalizations of the L and CL models. The proposed models are useful for seeing, in more details, the structure of the difference between two marginal cumulative logits (or the difference between two conditional marginal cumulative logits).

When the MH model fits the data poorly, Theorems and Remarks would be useful for seeing the reason for the poor fit of the MH model.

For Example in Section 5, the readers may be interested in (1) whether or not the probability distributions for grades of the satisfaction with hobbies (X_1), of the satisfaction with family (X_2), and of the satisfaction with residence (X_3) are homogeneous, or how those are not homogeneous, and also (2) whether or not the conditional probability distributions are homogeneous on condition that all grades of X_1, X_2 and X_3 are not equal, or how those are not homogeneous. For analyzing (1), the L(m) model (including the main diagonal probabilities $\{p_{ii\cdots i}\}$) and Theorem 1 are useful, and for analyzing (2), the CL(m) model (not including $\{p_{ii\cdots i}\}$) and Theorem 2 are useful.

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Table 1. Numbers of degrees of freedom for models applied to the r^T table.

Models	df
MH	$(T-1)(r-1)$
L(m)	$(T-1)(r-1-m)$
CL(m)	$(T-1)(r-1-m)$
M-I(k)	$T-1$
M-II(k)	$T-1$
CM-II(k)	$T-1$

Table 2. Three-way cross-classification of the indicators of satisfaction with life from the 1977 General Social Survey; taken directly from Clogg (1982). The upper and lower parenthesized values are the MLEs of the expected frequencies under the L(2) and CL(2) models, respectively.

L	R	$C=1$	$C=2$	$C=3$	$C=4$
1	1	76	14	15	4
		(75.50)	(14.33)	(14.53)	(4.05)
		(76.00)	(14.81)	(14.35)	(4.19)
1	2	32	17	7	3
		(34.51)	(18.95)	(7.35)	(3.30)
		(32.54)	(18.26)	(6.79)	(3.19)
1	3	64	23	28	15
		(62.63)	(23.19)	(26.74)	(14.94)
		(64.14)	(24.33)	(26.77)	(15.70)
1	4	41	11	27	16
		(41.64)	(11.52)	(26.73)	(16.55)
		(41.31)	(11.70)	(25.95)	(16.83)
2	1	15	2	7	4
		(13.78)	(1.89)	(6.28)	(3.74)
		(14.05)	(1.97)	(6.27)	(3.90)
2	2	27	20	9	5
		(26.75)	(20.42)	(8.70)	(5.04)
		(25.62)	(20.00)	(8.17)	(4.94)
2	3	57	31	24	15
		(51.63)	(28.86)	(21.25)	(13.81)
		(53.37)	(30.52)	(21.50)	(14.62)
2	4	27	9	22	16
		(25.31)	(8.68)	(20.14)	(15.26)
		(25.40)	(8.91)	(19.80)	(15.67)

Table 2 (continued).

L	R	$C=1$	$C=2$	$C=3$	$C=4$
3	1	13	6	13	5
		(13.54)	(6.45)	(13.20)	(5.31)
		(13.55)	(6.62)	(12.91)	(5.45)
3	2	12	13	10	6
		(13.63)	(15.29)	(11.04)	(6.96)
		(12.70)	(14.56)	(10.07)	(6.64)
3	3	46	32	75	20
		(47.18)	(33.86)	(74.98)	(20.90)
		(47.94)	(35.27)	(75.00)	(21.80)
3	4	54	26	58	55
		(57.58)	(28.64)	(60.22)	(59.80)
		(56.59)	(28.82)	(57.86)	(60.29)
4	1	7	6	7	6
		(6.70)	(5.91)	(6.54)	(5.84)
		(6.74)	(6.09)	(6.44)	(6.02)
4	2	7	2	3	6
		(7.25)	(2.14)	(3.03)	(6.33)
		(6.83)	(2.06)	(2.80)	(6.11)
4	3	12	11	31	15
		(11.31)	(10.67)	(28.55)	(14.39)
		(11.55)	(11.15)	(28.53)	(15.05)
4	4	52	36	80	101
		(50.80)	(36.23)	(76.27)	(100.46)
		(50.32)	(36.69)	(73.98)	(101.00)

Table 3. Likelihood ratio chi-squared values G^2 for each model applied to the data in Table 2.

Models	df	G^2
MH	6	398.24 *
L(1)	4	10.44 *
L(2)	2	4.89
CL(1)	4	11.26 *
CL(2)	2	4.36
M-I(1)	2	383.04 *
M-I(2)	2	384.41 *
M-II(2)	2	42.14 *
CM-II(2)	2	43.18 *

"$*$" means significant at the 0.05 level.

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