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On conformal curvature tensor of (ϵ)-para Sasakian manifolds

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Abstract. In this article, we study (ϵ)-para Sasakian manifolds with conformal curvature tensor. In this context, we consider conformally flat, quasi-conformally flat and quasi-conformally semi-symmetric (ϵ)-para Sasakian manifolds. It is proved that (ϵ)-para Sasakian manifold $M^n(n > 3)$ is conformally flat if and only if it is locally isometric to a pseudo hyperbolic space $H^n_\nu(1)$ or to a pseudo sphere $S^n_\nu(1)$.

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§1. Introduction

In Riemannian geometry, one of the basic interest is curvature properties and to what extent these determine the manifold itself. One of the important curvature properties is conformal flatness.

The Weyl conformal curvature tensor C of type (1,3) of an n-dimensional $(n > 3)$ Riemannian manifold is given by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where R, S, Q and r are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator and the scalar curvature of the manifold respectively. If C vanishes identically, then the manifold is called conformally flat. As a generalization of the notion of conformal flatness, K.Yano and S.Sawaki [11] introduced the notion of the quasi-conformal curvature tensor $\widetilde{C}$, which is defined as

$$\begin{aligned} \widetilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ + g(Y,Z)QX - g(X,Z)QY] \\ -\frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.2}$$

where a and b are constants. If $a = 1$ and $b = -\frac{1}{n-2}$, then $\widetilde{C} = C$. Thus the conformal curvature tensor C is a particular case of the tensor $\widetilde{C}$.

An n-dimensional ($n > 3$) manifold is called quasi-conformally flat if $\widetilde{C} = 0$ identically. Such a quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a = 0$ and $b \neq 0$ [1]. Several geometers ([1],[2],[3],[5],[9]) studied contact metric manifolds with quasi-conformal curvature tensor.

On the other hand, in [10], authors introduced the notion of an (ϵ)-para Sasakian manifold by associating a semi-Riemannian metric to an almost paracontact structure and gave several examples to ensure the existence of such manifolds.

In present article, we study (ϵ)-para Sasakian manifolds with conformal and quasi-conformal curvature tensor. In section 2, we review some basic formulae and definitions for (ϵ)-para Sasakian manifolds. Section 3 is devoted to conformally flat (ϵ)-para Sasakian manifolds. In section 4, we study quasi-conformally flat and quasi-conformally semi-symmetric (ϵ)-para Sasakian manifolds. It is proved that such manifolds are η-Einstein.

§2. (ϵ)-para Sasakian manifolds

An n-dimensional differentiable manifold M is called an (ϵ)-para Sasakian manifold [10], if it admits a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g on M satisfying

$$\phi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\circ\phi = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \phi)Y = -g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X, \tag{2.4}$$

for all $X, Y, Z \in T(M)$, where $T(M)$ denotes the Lie algebra of vector fields on M and ϵ is 1 or -1 according as ξ is spacelike or timelike.

In particular, if the metric g is positive definite, i.e. $\epsilon = 1$, then M is usual para Sasakian manifold ([7], [8]). If g is Lorentzian, i.e. $\epsilon = -1$ and ξ is replaced by $-\xi$, then M is a Lorentzian para Sasakian manifold [4].

Now, from (2.3), we have

$$g(X, \phi Y) = g(\phi X, Y), \quad g(X, \xi) = \epsilon\eta(X), \tag{2.5}$$

for all $X, Y \in T(M)$. From (2.5), we obtain

$$g(\xi, \xi) = \epsilon, \tag{2.6}$$

which shows that the structure vector field ξ is never lightlike. The fundamental (0,2) symmetric tensor of an (ϵ)-para Sasakian structure is defined by $\Phi(X, Y) = g(X, \phi Y)$, for all $X, Y \in T(M)$. For an (ϵ)-para Sasakian manifold [6], we have

$$\nabla\xi = \epsilon\phi, \tag{2.7}$$

$$\Phi(X, Y) = g(\phi X, Y) = \epsilon g(\nabla_X \xi, Y) = (\nabla_X \eta)Y. \tag{2.8}$$

An (ϵ)- para Sasakian manifold is called η-Einstein if its Ricci tensor S satisfies the condition

$$S(X, Y) = Ag(X, Y) + B\eta(X)\eta(Y), \tag{2.9}$$

where A and B are smooth functions and X, Y are any vector fields on M. If $B = 0$, then it becomes Einstein manifold.

For an n-dimensional (ϵ)-para Sasakian manifold, the Riemannian curvature tensor R and Ricci tensor S satisfy the following relations [6]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(X, \xi)\xi = \eta(X)\xi - X, \tag{2.11}$$

$$R(\xi, X)Y = -\epsilon g(X, Y)\xi + \eta(Y)X, \tag{2.12}$$

$$R(X, Y, U, \xi) = -\eta(X)g(Y, U) + \eta(Y)g(X, U), \tag{2.13}$$

$$\eta(R(X, Y)U) = -\epsilon\eta(X)g(Y, U) + \epsilon\eta(Y)g(X, U), \tag{2.14}$$

$$S(X, \xi) = -(n-1)\eta(X). \tag{2.15}$$

§3. Conformally flat (ϵ)-para Sasakian manifolds

Lemma 1. *Let R be the Riemannian curvature tensor of an (ϵ)-para Sasakian manifold M, then*

$$(3.1)\quad R(X,Y)\phi Z - \phi R(X,Y)Z = \epsilon\{g(X,Z)\phi Y - g(Y,Z)\phi X + g(X,\phi Z)Y - g(Y,\phi Z)X\} + 2\epsilon\{g(Z,\phi X)\eta(Y)\xi + g(Y,\phi X)\eta(Z)\xi - g(Z,\phi Y)\eta(X)\xi - g(X,\phi Y)\eta(Z)\xi\} + 2\{\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y\}.$$

Proof. By using (2.4), we have

$$(3.2)\quad \nabla_X \phi Y = \phi\nabla_X Y - g(X,Y)\xi - \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

In view of (3.2), we obtain

$$(3.3)\quad R(X,Y)\phi Z = \nabla_X\{\phi\nabla_Y Z - g(Y,Z)\xi - \epsilon\eta(Z)Y + 2\epsilon\eta(Y)\eta(Z)\xi\} - \nabla_Y\{\phi\nabla_X Z - g(X,Z)\xi - \epsilon\eta(Z)X + 2\epsilon\eta(X)\eta(Z)\xi\} - \left\{\phi\nabla_{[X,Y]}Z - g([X,Y],Z)\xi - \epsilon\eta(Z)[X,Y] + 2\epsilon\eta([X,Y])\eta(Z)\xi\right\}.$$

On simplification, the equation (3.3) becomes

$$R(X,Y)\phi Z = \phi R(X,Y)Z + \epsilon\{g(X,Z)\phi Y - g(Y,Z)\phi X + g(Z,\phi Y)X - g(Z,\phi X)Y\} + 2\epsilon\{g(Z,\phi X)\eta(Y)\xi + g(Y,\phi X)\eta(Z)\xi - g(Z,\phi Y)\eta(X)\xi - g(X,\phi Y)\eta(Z)\xi\} + 2\{\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y\},$$

which implies (3.1).

Proposition 2. *Let M be an n-dimensional ($n > 3$) (ϵ)-para Sasakian manifold. If M is conformally flat, then M is a space of constant curvature $-\epsilon$.*

Proof. Let us suppose that M is conformally flat, then the Riemannian curvature tensor is given by

$$R(X,Y)Z = \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(X,Z)Y - g(Y,Z)X\}. \tag{3.4}$$

By (3.4) and by putting $X = Z = \xi$ and $W = X$ in $g(R(X,Y)Z,W)$, we obtain

$$(n-1)S(X,Y) = \{(n-1)\epsilon + r\}\, g(X,Y) - \epsilon\{r + \epsilon n(n-1)\}\,\eta(X)\eta(Y). \tag{3.5}$$

Moreover, using (3.4) and (3.5), we have

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z = {} & \frac{1}{(n-1)(n-2)}\{2(n-1)\epsilon + r\}\{g(Y,\phi Z)X \\ & - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y\} \\ & + \{r + \epsilon n(n-1)\}\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi \\ & + \epsilon\eta(X)\eta(Z)\phi Y - \epsilon\eta(Y)\eta(Z)\phi X\}. \end{aligned} \tag{3.6}$$

Now, using (3.1) in (3.6), we obtain

$$\begin{aligned} & \{r + \epsilon n(n-1)\}\{g(X,\phi Z)Y - g(Y,\phi Z)X\} \\ & + \{\epsilon(n-1)(n-4) - r\}\{g(X,Z)\phi Y - g(Y,Z)\phi X\} \\ & - \{\epsilon(n-1)(3n-4) + r\}\{g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi\} \\ & + \{(n-1)(3n-4) + r\epsilon\}\{\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y\} \\ & + 2\{\epsilon(n-1)(n-2)\}\{g(Y,\phi X)\eta(Z)\xi - g(X,\phi Y)\eta(Z)\xi\} = 0. \end{aligned} \tag{3.7}$$

Putting $X = e_1$, $Y = e_2$ and $Z = \phi e_2$ in (3.7), where $\{\xi, e_1, \phi e_1, ..., e_n, \phi e_n\}$ is an orthonormal basis of T_pM at $p \in M$, we get

$$r = -\epsilon n(n-1). \tag{3.8}$$

So, using (3.8) in (3.5), we have

$$S(X,Y) = -\epsilon(n-1)g(X,Y). \tag{3.9}$$

Thus, M is an Einstein manifold.

Now, from (3.9), we have

$$QX = -\epsilon(n-1)X. \tag{3.10}$$

By substituting (3.8), (3.9) and (3.10) in (3.4), we get

$$R(X,Y)Z = -\epsilon\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.11}$$

which shows that M is a space of constant curvature $-\epsilon$.

Thus, in view of corollary (5.13) [10], we can state the following:

Corollary 1. *Let M be an n-dimensional $(n > 3)$ (ϵ)-para Sasakian manifold. Then following statements are equivalent:*
(I). M is conformally flat.
(II). M is symmetric.
(III). M is semi-symmetric.
(IV). M satisfies $R(\xi,X).R = 0$.

An n-dimensional $(n > 3)$ conformally flat (ϵ)-para Sasakian manifold is of constant curvature $-\epsilon$. Also a manifold of constant curvature is conformally flat. Thus an (ϵ)-para Sasakian manifold is conformally flat if and only if it is a space of constant curvature $-\epsilon$. Consequently in this case M is locally isometric to a pseudo hyperbolic space $H^n_\nu(1)$ or pseudo sphere $S^n_\nu(1)$ according as ξ is spacelike or timelike [10].

Thus we have the following:

Proposition 3. *Let M be an n-dimensional $(n > 3)$ (ϵ)-para Sasakian manifold. Then it is conformally flat if and only if it is locally isometric to a pseudo hyperbolic space $H^n_\nu(1)$ or to a pseudo sphere $S^n_\nu(1)$.*

§4. (ϵ)-para Sasakian manifolds with quasi-conformal curvature tensor

Let $M(\phi,\xi,\eta,g,\epsilon)$ be an n-dimensional quasi-conformally semi-symmetric (ϵ)-para Sasakian manifold, i.e.

$$R(X,Y).\widetilde{C} = 0, \tag{4.1}$$

which implies that

$$\begin{aligned} R(X,Y)\widetilde{C}(U,V)W - &\widetilde{C}(R(X,Y)U,V)W \\ - \widetilde{C}(U,R(X,Y)V)W &- \widetilde{C}(U,V)R(X,Y)W = 0. \end{aligned} \tag{4.2}$$

Putting $X = \xi$ in (4.2), we obtain

$$R(\xi,Y)\widetilde{C}(U,V)W - \widetilde{C}(R(\xi,Y)U,V)W \tag{4.3}$$

$$- \widetilde{C}(U, R(\xi, Y)V)W - \widetilde{C}(U, V)R(\xi, Y)W = 0.$$

In view of (2.12), the above expression gives

$$\begin{aligned} -\acute{\widetilde{C}}(U, V, W, Y) + g(Y, U)\eta(\widetilde{C}(\xi, V)W) \\ + g(Y, V)\eta(\widetilde{C}(U, \xi)W) + g(Y, W)\eta(\widetilde{C}(U, V)\xi) \\ + \epsilon\eta(Y)\eta(\widetilde{C}(U, V)W) - \epsilon\eta(U)\eta(\widetilde{C}(Y, V)W) \\ - \epsilon\eta(V)\eta(\widetilde{C}(U, Y)W) - \epsilon\eta(W)\eta(\widetilde{C}(U, V)Y) = 0, \end{aligned} \tag{4.4}$$

where $\acute{\widetilde{C}}(U, V, W, Y) = g(\widetilde{C}(U, V)W, Y)$.

Now, from equation (1.2), we have

$$\begin{aligned} \eta(\widetilde{C}(X, Y)Z) = a\eta(R(X, Y)Z) \\ + b[S(Y, Z)\eta(X) - S(X, Z)\eta(Y) \\ + g(Y, Z)\eta(QX) - g(X, Z)\eta(QY)] \\ - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \end{aligned} \tag{4.5}$$

Taking $X = Y$ in the above equation, we get

$$\eta(\widetilde{C}(X, X)Z) = 0. \tag{4.6}$$

Again, putting $Z = \xi$ in equation (4.5) and using (2.10), (2.15), we obtain

$$\eta(\widetilde{C}(X, Y)\xi) = 0. \tag{4.7}$$

By substituting $Y = U$ in equation (4.4) and using (4.6), (4.7), we get

$$\begin{aligned} -\acute{\widetilde{C}}(U, V, W, U) + g(U, U)\eta(\widetilde{C}(\xi, V)W) \\ + g(U, V)\eta(\widetilde{C}(U, \xi)W) - \epsilon\eta(W)\eta(\widetilde{C}(U, V)U) = 0. \end{aligned} \tag{4.8}$$

Now, putting $U = e_i$ in (4.8) and taking summation over $i, 1 \leq i \leq n$, where $\{e_1, e_2, ..., e_{n-1}, e_n = \xi\}$ is an orthonormal basis of tangent space at each point of the manifold, we obtain

$$\begin{aligned} -\sum_{i=1}^{n} \acute{\widetilde{C}}(e_i, V, W, e_i) + (n-1+\epsilon)\eta(\widetilde{C}(\xi, V)W) \\ + \sum_{i=1}^{n} g(e_i, V)\eta(\widetilde{C}(e_i, \xi)W) - \epsilon\eta(W)\sum_{i=1}^{n}\eta(\widetilde{C}(e_i, V)e_i) = 0. \end{aligned} \tag{4.9}$$

Moreover, from (2.14), we get

$$\sum_{i=1}^{n} \eta(R(e_i, V)e_i) = \epsilon(n-1)\eta(V), \tag{4.10}$$

$$\eta(R(\xi, V)W) = -\epsilon g(V, W)+\eta(V)\eta(W) \tag{4.11}$$

and

$$\sum_{i=1}^{n} g(e_i, V)\eta(R(e_i, \xi)W) = -\eta(V)\eta(W)+\epsilon g(V, W). \tag{4.12}$$

Now, from (1.2), we have

$$\begin{aligned} \acute{\tilde{C}}(U, V, W, Y) = a\acute{R}(U, V, W, Y) \\ + b[S(V, W)g(U, Y) - S(U, W)g(V, Y) \\ + g(V, W)g(QU, Y) - g(U, W)g(QV, Y)] \\ - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(V, W)g(U, Y) - g(U, W)g(V, Y)]. \end{aligned} \tag{4.13}$$

Putting $U = Y = e_i$ in (4.13) and taking summation over $i, 1 \leq i \leq n$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \acute{\tilde{C}}(e_i, V, W, e_i) = \{a + b(n - 3 + \epsilon)\} S(V, W) \\ + \left\{br - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)(n - 2 + \epsilon)\right\} g(V, W). \end{aligned} \tag{4.14}$$

Again, by using (2.5), (2.15) and (4.10), the equation (4.5) implies

$$\begin{aligned} \sum_{i=1}^{n} \eta(\widetilde{C}(e_i, V)e_i) = \{a\epsilon(n-1)+b\epsilon(n-1)^2-b\epsilon(n-1) \\ - br + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)(n - 1)\}\eta(V). \end{aligned} \tag{4.15}$$

In view of (2.5), (2.15), (4.12) and (4.5), we get

$$\begin{aligned} \sum_{i=1}^{n} g(e_i, V)\eta(\widetilde{C}(e_i, \xi)W) = -bS(V, W) \\ +\left\{b\epsilon(n - 1) + a\epsilon + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right\} g(V, W) \end{aligned} \tag{4.16}$$

$$-\left\{2b(n-1)+a+\frac{r\epsilon}{n}\left(\frac{a}{n-1}+2b\right)\right\}\eta(V)\eta(W)$$

and

$$\sum_{i=1}^{n} g(e_i,V)\eta(\widetilde{C}(e_i,\xi)W) = -\eta(\widetilde{C}(\xi,V)W). \tag{4.17}$$

Therefore, by using (4.14), (4.15), (4.16), (4.17) in equation (4.9), we obtain

$$S(V,W) = \frac{1}{(a-b)}[-\{b(r+(n-1))+\epsilon(n-2)(a+b(n-1))+a\}\,g(V,W) \tag{4.18}$$

$$+\{b(n-1)(n-2)+b\epsilon(r+(n-1))+\epsilon(a+b(n-1))-a\}\,\eta(V)\eta(W)],$$

which can be written as

$$S(V,W) = Ag(V,W)+B\eta(V)\eta(W), \tag{4.19}$$

where

$$A = \frac{1}{(a-b)}[-\{b(r+(n-1))+\epsilon(n-2)(a+b(n-1))+a\}] \tag{4.20}$$

and

$$B = \frac{1}{(a-b)}[\{b(n-1)(n-2)+b\epsilon(r+(n-1))+\epsilon(a+b(n-1))-a\}]. \tag{4.21}$$

Hence, we can state the following:

Proposition 4. *A quasi-conformally semi symmetric (ϵ)-para Sasakian manifold is an η-Einstein manifold.*

As a quasi-conformally flat (ϵ)-para Sasakian manifold is quasi-conformally semi-symmetric, so we can state the following corollary:

Corollary 2. *A quasi-conformally flat (ϵ)-para Sasakian manifold is an η-Einstein manifold.*

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