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Differential Coloring of Graphs

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Abstract. Let $G = (V, E)$ be a connected graph. For a set $S \subseteq V$, the *differential of* S, denoted by $\partial(S)$, is defined to be $\partial(S) = |B(S)| - |S|$, where $B(S) = N(S) \cap (V - S)$. A set $S \subseteq V$ is said to be a *positive differential set* if $\partial(S) \geq 0$. A partition $\{V_1, V_2, \ldots, V_k\}$ of V is said to be a *positive differential chromatic partition* of G if each V_i is both independent and positive differential. The minimum order of a positive differential chromatic partition of G is called the *differential chromatic number* of G and is denoted by $\chi_\partial(G)$. In this paper we initiate a study of this parameter.

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§1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [1]. All graphs in this paper are assumed to be connected with at least two vertices. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is $N(S) = \cup_{v \in S} N(u)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. The subgraph induced by S is denoted as $G[S]$.

Consider the game where you are allowed to buy as many tokens as you like at a cost of \$1 each. For example, suppose that you buy k tokens. You then place the tokens on some subset of k vertices of G. For each vertex of G which has no token on it, but is adjacent to a vertex with a token on it, you receive \$1 from the bank. Your objective is to maximize your profit, that is, the total value received from the bank minus the cost of the tokens bought. Notice that you do not receive any credit for the vertices in which you place a token.

The above game can be mathematically formulated as follows. For a set $S \subseteq V$, the *boundary* $B(S)$ of S is to be defined $B(S) = N(S) \cap (V - S)$ and then the differential $\partial(S)$ of S is to be defined $\partial(S) = |B(S)| - |S|$. Now, the *differential* of the graph G, denoted by $\partial(G)$, is defined to be $\partial(G) = max\{\partial(S) : S \subseteq V\}$. Thus, for the given graph G, maximizing the profit as mentioned in the game is equivalent to determining the value of $\partial(G)$.

The differential $\partial(S)$ of a set was introduced by Hedetniemi [4] and also was considered by Goddard and Henning [3] who denoted it by $\eta(S)$. Obviously, the differential of a set in a graph may be positive or negative or may be zero. However, since it has been proved in [5] that $\partial(G) \geq \Delta(G) - 1$, it follows that the differential of a graph without isolated vertices is either positive or zero. Hence one need not consider the sets of negative differential while determining $\partial(G)$ and so the sets of zero or positive differential play a vital role in the game. Motivated by this observation we take interest on such sets, that is, the sets of zero or positive differential and call these sets as positive differential set. (Here we use the term "*positive differential set*" in the sense that by the choice of those sets where the tokens bought are to be placed, one will not lose money, whether or not one gains money). However, our main focus in this paper is not on the study of the positive differential sets, but on the problem of partitioning V into independent positive differential sets, that is, positive differential sets in which no two vertices are adjacent. That is, we consider the partition of V into independent positive differential sets and we call this partition as *positive differential chromatic partition* and the minimum order of a positive differential chromatic partition of G is called the *differential chromatic number of G* and we denote it by $\chi_\partial(G)$. Also we use the term *χ_∂-partition* to denote a positive differential partition of order χ_∂. In this paper we initiate a study of this parameter.

§2. Differential Chromatic Number

In this section, we determine the value of the differential chromatic number for some families of graphs such as paths, cycles, complete graphs, wheels, complete multipartite graphs and complete binary trees.

Since it is straightforward to determine the value of the differential chromatic number for the above mentioned graphs, we just state them without proof.

Proposition 1. *(i) For the paths P_n and cycles C_n on n vertices,*

$$\chi_\partial(P_n) = \chi_\partial(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

(ii) If W_n is a wheel on n vertices, then

$$\chi_\partial(W_n) = \begin{cases} 4 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 2. *Let $G = K_{m_1,m_2,\ldots,m_k}$ be a complete k-partite graph with $m_1 \geq m_2 \geq \cdots \geq m_k$,$m_k \geq 1$,$k \geq 2$ and let $N = \sum_{i=2}^{k} m_i$. Then*

$$\chi_\partial(G) = \begin{cases} k + \left\lceil \dfrac{m_1 - N}{N} \right\rceil & \text{if } N < m_1, \\ k & \text{otherwise.} \end{cases}$$

Proof. Let G be a complete k-partite graph with k-partition $(X_1, X_2, \ldots, X_k)$. Let $|X_i| = m_i$, where $i = 1, 2, \ldots, k$. Clearly each X_i, where $2 \leq i \leq k$, is a positive differential set. Now, if $N \geq m_1$, then X_1 is also a positive differential set so that the k-partition $(X_1, X_2, \ldots, X_k)$ of G itself is a minimum positive differential chromatic partition of G. Hence $\chi_\partial(G) = k$. If $N < m_1$, let $X_1 = \{x_1, x_2, \ldots, x_{m_1}\}$. Then $\{\{x_1, x_2, \ldots, x_N\}$, $\{x_{N+1}, x_{N+2}, \ldots, x_{2N}\}$, $\{x_{2N+1}, x_{2N+2}, \ldots, x_{3N}\}$, $\{x_{3N+1}, x_{3N+2}, \ldots, x_{4N}\}$, $\ldots$, $\{x_{rN}, x_{rN+1}, x_{rN+2}, \ldots, x_{m_1}\}, X_2, X_3, \ldots, X_k\}$, where $r = \lceil \frac{m_1 - N}{N} \rceil$ is clearly a minimum positive differential chromatic partition of G. Hence $\chi_\partial(G) = k + \left\lceil \dfrac{m_1 - N}{N} \right\rceil$. □

Corollary 1. *For a complete graph K_n on n vertices, $\chi_\partial(K_n) = n$.*

Proposition 3. *If G is a complete binary tree, then $\chi_\partial(G) = 3$.*

Proof. Let G be a k-level complete binary tree, where $k \geq 1$. Let v_{ij} denote the j^{th} vertex in the i^{th} level. We consider the following cases.

Case 1. k is odd

Let
$$A = \{v_{kj} : \ j = 1, 2, 5, 6, 9, 10, \ldots, 2^k - 3, 2^k - 2\},$$
$$B = \{v_{k-1,j} : \ j = 2, 4, 6, \ldots, 2^k - 1\},$$
$$C = \{v_{mj} : \ 0 \leq m \leq k \text{ and } 1 \leq j \leq 2^m - 3, \text{where } m \text{ is odd}\},$$
$$V_1 = A \cup B \cup C,$$
$$V_2 = \{v_{mj} : \ 0 \leq m \leq k \text{ and } 1 \leq j \leq 2^m - 2, \text{where } m \text{ is even}\},$$
$$V_3 = V - \{V_1 \cup V_2\}.$$

Case 2. k is even

Let $A = \{v_{kj}:\ j = 1, 2, 5, 6, 9, 10, \ldots, 2^k - 4, 2^k - 3\}$,
$B = \{v_{k-1,j}:\ j = 2, 4, 6, \ldots, 2^k - 1\}$,
$V_1 = A \cup B$,
$C = \{v_{mj}:\ 0 \leq m \leq k \text{ and } 0 \leq j \leq 2^m - 3, \text{where } m \text{ is odd}\}$,
$D = \{v_{k,k-2}\}$,
$V_2 = C \cup D$,
$V_3 = V - \{V_1 \cup V_2\}$.

Thus in both the cases $\{V_1, V_2, V_3\}$ forms a positive differential chromatic partition of G and hence $\chi_\partial(G) \leq 3$. Now, since a complete binary tree is a bipartite graph with parts of unequal size, we have $\chi_\partial(G) \geq 3$ (see Theorem 1). Thus $\chi_\partial(G) = 3$. □

§3. Bounds on $\chi_\partial(G)$

In this section, we establish some bounds on χ_∂(G) in terms of order of the graph and some well known parameters such as chromatic number, independent domination number, induced paired domination number and packing number. Further we obtain an upper bound for trees in terms of the maximum degree. All graphs considered in this section are of order n.

Theorem 1. *For any graph G, we have $2 \leq \chi_\partial(G) \leq n$. Further, $\chi_\partial(G) = 2$ if and only if G is a bipartite graph with parts of equal size and $\chi_\partial(G) = n$ if and only if G is either a star or a complete graph.*

Proof. Obviously $2 \leq \chi_\partial(G) \leq n$. Suppose $\chi_\partial(G) = 2$. Let $\{V_1, V_2\}$ be a positive differential partition of G. Now, since V_1 is a positive differential set, we have $|N(V_1)| \geq |V_1|$ and since $|N(V_1)| \leq |V_2|$, it follows that $|V_2| \geq |V_1|$. Similarly $|V_1| \geq |V_2|$ and hence $|V_1| = |V_2|$. Also if G is a bipartite graph with parts of equal size, then $\chi_\partial(G) = 2$.

Now, suppose $\chi_\partial(G) = n$. Then G has no independent positive differential set with more than one vertex. If G is a star we are through. Otherwise we claim that G is complete. Suppose G has two non-adjacent vertices, say u and v. If x and y are neighbours of u and v respectively, then $x = y$, for otherwise $\{u, v\}$ will be an independent positive differential set in G. Further, since G is connected and is not a star, either u or v is adjacent to a vertex w other than x so that $\{u, v\}$ is again an independent positive differential set, which is a contradiction and hence G is complete. Conversely if G is either complete or a star then clearly $\chi_\partial(G) = n$. □

Theorem 2. *Let G be a non-complete graph. Then $\chi_\partial(G) = n-1$ if and only if G contains K_{n-1} as an induced subgraph, where $n \geq 4$.*

Proof. Suppose G contains K_{n-1} as an induced subgraph. Let $V = \{v_1, v_2, \ldots, v_n\}$. Since G is a non-complete graph, there exist two vertices v_i and v_j, where $1 \leq i, j \leq n$, which are not adjacent. Without loss of generality, let v_n and v_1 be non-adjacent. Then $\{\{v_1, v_n\}, \{v_2\}, \{v_3\}, \ldots, \{v_{n-1}\}\}$ is a positive differential chromatic partition of G so that $\chi_\partial(G) = n-1$.

Conversely suppose $\chi_\partial(G) = n-1$. Let $\{V_1, V_2, \ldots, V_{\chi_\partial(G)}\}$ be a χ_∂-partition of G. Then exactly one of these sets in this partition, say V_1, contains two vertices and all other sets have exactly one vertex. Let $V_1 = \{v_1, v_2\}$, $V_i = \{v_{i+1}\}$, for $i \geq 2$, and let $H = G[\{v_3, v_4, \ldots, v_n\}]$. We now prove the following claims.

Claim 1. Either v_1 or v_2 is adjacent to all the vertices in H.

Suppose there exist two vertices v_i and v_m $(i < m)$ in H such that v_i is not adjacent to v_1 and v_m is not adjacent to v_2. Then clearly $\{v_1, v_i\}$ and $\{v_2, v_m\}$ are positive differential sets so that $\chi_\partial(G) \leq n-2$, which is a contradiction. If $v_i = v_m$, then v_i is not adjacent to v_1 and v_2. Since G is connected, v_i is adjacent to some v_j in H. Now $\{v_i, v_1\}$ or $\{v_i, v_2\}$ forms a positive differential set, provided v_1 and v_2 have v_j as the only neighbor in H. Otherwise choose a vertex v_s not adjacent to v_j in H, so that $\{v_s, v_j\}$ forms a postive differential set. If v_j is adjacent to all the vertices in H and if $N(v_j)$ is independent, then G is a star. Then $\chi_\partial(G) = n$, which is a contradiction. Hence at least 2 neighbors say x and y of v_j in H are adjacent. Now $\{x, v_1\}$ forms a positive differential set. Hence $\chi_\partial(G) < n-1$ which is a contradiction.

Claim 2. H is complete.

Without loss of generality let us assume that v_1 is adjacent to all the vertices in H. Suppose H is not complete then there exists two vertices x, y in H, which are not adjacent. If $\deg x = \deg y = 1$, then since G is connected and v_1, v_2 are not adjacent, there exists $z \in H$ such that v_2 is adjacent to z. Hence $\{x, z\}$ is a positive differential set, which is a contradiction. If $\deg x$ or $\deg y > 1$, then there exists at least one vertex say w in H which is adjacent to either x or y. Hence $\{x, y\}$ forms a positive differential set, which is a contradiction. Hence, H is complete.

Now by claim 1 and claim 2 we see that the subgraph induced by $\{v_1, v_3, v_4, \ldots, v_n\}$ is K_{n-1}, as desired. □

Remark 1. *Since a χ_∂-partition of a graph G is a chromatic partition, it follows that $\chi(G) \leq \chi_\partial(G)$. Further, the difference between these parameters* *can be made as large as possible. For the star $K_{1,n-1}$, we have $\chi_\partial(G) = n$ and $\chi(G) = 2$.*

A subset $S \subseteq V$ is called a 2-*packing* if the distance between any two distinct vertices of S is at least three. The maximum cardinality of a 2-packing set of G is called the *packing number* of G and is denoted by $\rho(G)$. The following theorem relates χ_∂ with the packing number ρ.

Theorem 3. *For any graph G, we have $\chi_\partial(G) \leq n - \rho(G) + 1$.*

Proof. Let S be a 2-packing with $|S| = \rho(\text{G})$. Then obviously S is independent. Since G is connected it follows that every vertex in S has a neighbor in $V - S$. Also, by the definition of 2-packing no two vertices in S have a common neighbor so that S is a positive differential set in G. Hence the set S together with the singleton sets of vertices in $V-S$ form a positive differential chromatic partition of G so that $\chi_\partial(G) \leq n - \rho(\text{G}) + 1$. □

D.S. Studer *et al.* [6] introduced the concept of induced paired domination in graphs. A subset $S \subseteq V$ is said to be an *induced paired dominating set* if $G[S]$ is a perfect matching. The minimum cardinality of an induced paired dominating set of G is called the *induced paired domination number* of G and is denoted by $\gamma_{ipr}(G)$. In the next theorem we present an upper bound on χ_∂ in terms of $\gamma_{ipr}(\text{G})$.

Theorem 4. *If G is a graph for which $\gamma_{ipr}(G)$ is defined, then $\chi_\partial(G) \leq n - \gamma_{ipr}(G) + 2$.*

Proof. Let $S = \{x_1, x_2, \ldots, x_{\frac{\gamma_{ipr}}{2}}, y_1, y_2, \ldots, y_{\frac{\gamma_{ipr}}{2}}\}$ be a minimum induced paired dominating set of G. Then $S_1 = \{x_1, x_2, \ldots, x_{\frac{\gamma_{ipr}}{2}}\}$ and $S_2 = \{y_1, y_2, \ldots, y_{\frac{\gamma_{ipr}}{2}}\}$ are positive differential sets of G and hence $\{S_1, S_2\} \cup \{\{v\} : v \in V - S\}$ is a positive differential chromatic partition of G so that $\chi_\partial(G) \leq n - \gamma_{ipr}(\text{G}) + 2$. □

We now give an upper bound on $\chi_\partial(\text{G})$ in terms of independent domination number of a graph. A set of vertices S in a graph G is a *dominating set* if every vertex not in S is adjacent to at least one vertex in S. A dominating set which is also independent is called an *independent dominating set* and the minimum cardinality of an independent dominating set of G is called the *independent domination number* of G and is denoted by $i(G)$.

Theorem 5. *If G is a graph with $i(G) \leq n/2$, then $\chi_\partial(G) \leq n - i(G) + 1$.*

Proof. Let S be a minimum independent dominating set of G. Since $i(G) \leq n/2$, it follows $|N(S)| \geq n/2$ and so S is a positive differential set. Hence $\{S\} \cup \{\{v\} : v \in V - S\}$ is a positive differential chromatic partition of G so that $\chi_\partial(G) \leq n - i(G) + 1$. □

Corollary 2. *For any bipartite graph G, we have $\chi_\partial(G) \leq n - i(G) + 1$.*

Proof. Since $i(G) \leq n/2$ for any bipartite graph G, the result follows from the above theorem. □

A graph G is called *well-covered* if every maximal independent set is also maximum. That is $i(G) = \beta_0(G)$, where $\beta_0(G)$ is the independence number which is defined to be the maximum cardinality of an independent set of G.

Corollary 3. *If G is a well-covered graph, then $\chi_\partial(G) \leq n - i(G) + 1$.*

Proof. Since G is well-covered, we have $i(G) = \beta_0(G)$. Also it has been proved in [7] that, for a graph G, if $i(G) = \beta_0(G)$ then $i(G) \leq n/2$ and hence the result follows from Theorem 5. □

In [2], it has been proved that regular graphs on n vertices of degree at least $n - 7$ admit at least two disjoint independent dominating sets and hence the independent domination number is bounded by $n/2$ so that we have:

Corollary 4. *If G is a regular graph on n vertices of degree at least $n - 7$, then $\chi_\partial(G) \leq n - i(G) + 1$.*

§4. Realizability

A graph $G = (V, E)$ is called a split graph if the vertex set has a bipartition $V = X \cup Y$, where X is an independent set (no two vertices in X are adjacent) and $G[Y]$ is a complete graph (every pair of vertices are adjacent).

We have already observed in Theorem 1 that $2 \leq \chi_\partial(\text{G}) \leq n$ and also characterized the extremal graphs. We now show that the differential chromatic number $\chi_\partial(\text{G})$ can assume any value between 2 and n. That is,

Theorem 6. *Given positive integers k and n with $2 \leq k < n$, there exists a graph G on n vertices with $\chi_\partial(G) = k$.*

Proof. If $n - k$ is even, let G be a graph with vertex set $V = V_1 \cup V_2 \cup V_3$ with $|V_1| = |V_2| = r$, $|V_3| = n - 2r$, where $r = \frac{n-k+2}{2}$ such that $G[V_1 \cup V_2] = K_{r,r}$ and $G[V_i \cup V_3]$, $i = 1, 2$ is a split graph with bipartition (V_i, V_3), $i = 1, 2$, where V_i is an independent set and V_3 is complete. Suppose $n - k$ is odd and if $n = k + i$, $i = 1, 3$, let G be a split graph with bipartition (X, Y) where $|X| = i$ and $|Y| = k$, where Y is complete and X is an independent set. For the remaining cases, let G be the graph given in Figure 1. It can be easily verified that $\chi_\partial(G) = k$ in all the cases. □

It has been observed in section 3 that $\chi(\mathrm{G}) \leq \chi_\partial(\mathrm{G})$ and that the difference between these parameters can be made arbitrarily large. But in fact these two parameters can assume any arbitrary values which we now prove in the following theorem.

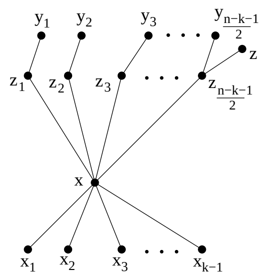

Figure 1:

Theorem 7. *For any two positive integers a and b with $a \leq b$, where $a \geq 2$, there exists a graph G such that $\chi(G) = a$ and $\chi_\partial(G) = b$.*

Proof. Let a and b be two positive integers with $a \leq b$. We now construct the graph G with $\chi_\partial(G) = b$ and $\chi(G) = a$ as follows.

Consider the complete graph K_a with $V(K_a) = \{u_1, u_2, \ldots, u_a\}$. At each vertex u_i, $1 \leq i \leq a$, attach b pendant edges, say $u_i u_i^j$, $1 \leq j \leq b$, (By a *pendant edge* we mean an edge which is incident to a vertex of degree one). Again attach $k = (a-2)(a+b-2) + 2(b-a)$ pendant edges at the vertex u_1^1 (for convenience we choose u_1^1 and in fact this may be any u_i^j). Let S be the set of these k newly added vertices. Further join each vertex of S to a vertex on K_a except u_1, say u_2. For $a = 4$ and $b = 5$, the graph G is given in Figure 2.

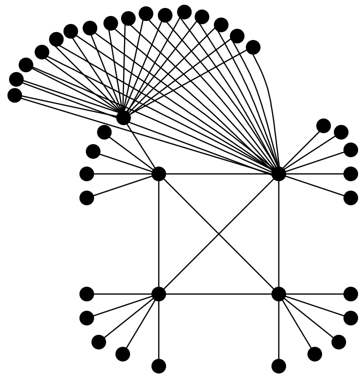

Figure 2:

Now let

$$S_1 = \{u_2, u_1^1, u_1^2, \ldots, u_1^b\} \cup \left(\bigcup_{i=3}^{a} \left(\bigcup_{j=1}^{b} \{u_i^j\} \right) \right),$$

$$S_2 = \{u_1, u_2^1, u_2^2, \ldots, u_2^b\}$$

and

$$S_3 = \{u_3\} \cup S_3',$$

where S_3' is the set consisting of any $a+b-2$ vertices in S. Also, for $i = 4, 5, \ldots, a$, let $S_i = \{u_i\} \cup S_i'$, where S_i' is the set consisting of any $a+b-2$ vertices in S but not in the set $\cup_{j=3}^{i-1} S_j'$. Now, let $w_1, w_2, \ldots, w_{2(b-a)}$ be the vertices in S which are not yet included in the above defined sets and let $A_i = \{w_{2i-1}, w_{2i}\}$, for all $i = 1, 2, \ldots, b-a$. Now, clearly the sets S_i's and A_i's are independent positive differential so that $\{S_1, S_2, \ldots, S_a, A_1, A_2, \ldots, A_{b-a}\}$ is a positive differential chromatic partition of minimum order and hence $\chi_\partial(G) = b$. Also clearly $\chi(G) = a$. □

§5. Split Graphs with $\chi_\partial(G) = 3$

In section 3, we have observed that the value of the differential chromatic number of a graph is at least two and also characterized the extremal graphs. In the following theorem we characterize split graphs for which $\chi_\partial = 3$.

Theorem 8. *Let G be a split graph with bipartition (X, Y), where X is independent and Y is complete and* $\deg(x) \le |Y| - 1$ *for all $x \in X$. Let $u_1 \in V$ with* $\deg u_1 = \Delta$, *where Δ is the maximum degree of G. Then $\chi_\partial(G) = 3$ if and only if the following conditions hold.*

(i) $|Y| \le 3$.

(ii) When $|Y| = 2$, $\deg u_1 = \deg u_2 + 1$, *where* $Y = \{u_1, u_2\}$.

(iii) When $|Y| = 3$, $\deg u_1 \le \deg u_2 + \deg u_3$ *where* $Y = \{u_1, u_2, u_3\}$, *further* $|N(u_1) \cap N(u_2)| \le \deg u_3 - 1$ *and* $|N(u_1) \cap N(u_3)| \le \deg u_2 - 1$.

Proof. Suppose $\chi_\partial(G) = 3$. Since Y is complete, $|Y| \le 3$. If $|Y| = 2$, let $Y = \{u_1, u_2\}$ and $\deg u_2 = m$. We now claim that $\Delta = m + 1$. Suppose not. Then either $\Delta = m$ or $\Delta > m + 1$. Let $N(u_2) = \{u_1, v_1, v_2, \ldots, v_{m-1}\}$. If $\Delta = m$, then $\{\{u_1, v_1, v_2, \ldots, v_{m-1}\}, S \cup \{u_2\}\}$, where $S = X - N(u_2)$, forms a positive differential chromatic partition of G and hence $\chi_\partial(G) = 2$, which is a contradiction. If $\Delta > m+1$, let $N(u_1) = \{u_2, w_1, w_2, \ldots, w_{\Delta-1}\}$. Then clearly $\{\{u_2, w_1, w_2, \ldots, w_{m-1}\}, \{u_1, v_1, v_2, \ldots, v_{m-1}\}, \{w_m\}, \{w_{m+1}\}, \ldots, \{w_{\Delta-1}\}\}$ form a positive differential chromatic partition of G and hence $\chi_\partial(G) > 3$, which is a contradiction. Hence $\deg u_1 = \deg u_2 + 1$.

Suppose $|Y| = 3$. Let $Y = \{u_1, u_2, u_3\}$ with $\deg u_1 = m_1$, $\deg u_2 = m_2$, $\deg u_3 = m_3$ and $m_1 \leq m_2 \leq m_3$. Now we claim that $\deg u_1 \leq \deg u_2 + \deg u_3$. If $\deg u_1 > \deg u_2 + \deg u_3$, then there exists at least one member of $N(u_1)$, which does not belong to $\cup_{i=1}^{3} S_i$ where S_i is the positive differential set containing u_i $(1 \leq i \leq 3)$ which implies that $\chi_\partial(G) > 3$, which is a contradiction. We claim that $|N(u_1) \cap N(u_2)| \leq \deg u_3 - 1$. If not, then at least one member of $N(u_1) \cap N(u_2)$ does not belong to $\cup_{i=1}^{3} S_i$, which implies that $\chi_\partial(G) > 3$, which is a contradiction. Hence $|N(u_1) \cap N(u_2)| \leq \deg u_3 - 1$ and similarly $|N(u_1) \cap N(u_3)| \leq \deg u_2 - 1$.

Conversely, suppose G satisfies the given conditions. Now $\chi_\partial(G) = 2$ only when $\deg u_1 = \deg u_2$ and $N(u_1) \cap N(u_2) = \phi$ which is not possible by the condition given in the theorem and hence $\chi_\partial(G) \geq 3$. Let

$$\begin{aligned} \ell_1 &= |N(u_1) \cap N(u_2)| \\ \ell_2 &= |N(u_2) \cap N(u_3)| \\ \ell_3 &= |N(u_3) \cap N(u_1)| \\ \text{and} \qquad S_1 &= \{u_1\} \cup (N(u_2) \cap N(u_3)) \cup S_1' \cup S_2', \end{aligned}$$

where S_1' is the set consisting of the vertices which are adjacent to u_3 only, and S_2' is the set consisting of $m_1 - \ell_1 - |S_1'| - 1$ vertices which are adjacent to u_2 only. Let $S_2 = \{u_3\} \cup [N(u_1) \cap N(u_2)] \cup S_1'' \cup S_2''$ where S_1'' is the set consisting of vertices which are adjacent to u_2 only (which are not in S_2') and S_2'' consists of $m_3 - \ell_3 - |S_1''| - 2$ vertices which are adjacent to u_1 only. Let $S_3 = \{u_2\} \cup [N(u_1) \cap N(u_3)] \cup S_1'''$ where S_1''' is the set consisting of vertices which are adjacent to u_1 only (which are not in S_2''). Then $\{S_1, S_2, S_3\}$ is a positive differential chromatic partition of G. Hence $\chi_\partial(G) = 3$. □

§6. Differential Chromatic Number of Product Graphs

In this section we determine the value of differential chromatic number for some product graphs such as $P_n \times C_m$, $C_n \times C_m$, $P_m \times P_n$ and $K_m \times K_n$.

Theorem 9.

$$\chi_\partial(P_n \times C_m) = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 3 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Let the vertices of $P_n \times C_m$ be $\{v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}\}$.

Case 1. m is even

If n is odd

Let $V_1 = \{v_{ij}\colon\ 1 \le i \le n \text{ and } 1 \le j \le m-1, \text{where both } i \text{ and } j \text{ are odd}\}$

If n is even

Let $V_1 = \{v_{ij}\colon\ 1 \le i \le n-1 \text{ and } 1 \le j \le m-1, \text{where both } i \text{ and } j \text{ are odd}\}$

and $V_2 = V - V_1$. Then $\{V_1, V_2\}$ is a positive differential chromatic partition of G. Hence $\chi_\partial(G) = 2$.

Case 2. m is odd and n is even

Let $A = \{v_{ij}\colon\ 1 \le i \le n-1 \text{ and } 1 \le j \le m-2, \text{where both } i \text{ and } j \text{ are odd}\}$,
$B = \{v_{ij}\colon\ 2 \le i \le n \text{ and } 2 \le j \le m-1, \text{where both } i \text{ and } j \text{ are even}\}$,
$C = \{v_{im}\colon\ 2 \le i \le n, \text{ where } i \text{ is even}\}$,
$D = \{v_{ij}\colon\ 1 \le i \le n-1 \text{ and } 2 \le j \le m-1, \text{where } i \text{ is odd and } j \text{ is even}\}$,
$V_1 = A \cup B$,
$V_2 = C \cup D$,
$V_3 = V - \{V_1, V_2\}$.

Case 3. Both m and n are odd

Let $A = \{v_{ij}\colon\ 1 \le i \le n \text{ and } 1 \le j \le m-2, \text{where both } i \text{ and } j \text{ are odd}\}$,
$B = \{v_{ij}\colon\ 1 \le i \le n-1 \text{ and } 2 \le j \le m-1, \text{where both } i \text{ and } j \text{ are even}\}$,
$C = \{v_{im}\colon\ 2 \le i \le n-1, \text{ where } i \text{ is even}\}$,
$D = \{v_{ij}\colon\ 1 \le i \le n \text{ and } 2 \le j \le m-1, \text{where } i \text{ is odd and } j \text{ is even}\}$,
$V_1 = A \cup B$,
$V_2 = C \cup D$,
$V_3 = V - \{V_1, V_2\}$.

Then $\{V_1, V_2, V_3\}$ is a positive chromatic differential partition of $P_n \times C_m$, so that $\chi_\partial(P_n \times C_m) \le 3$. Further, since m is odd it follows that $P_n \times C_m$ contains an odd cycle so that $\chi_\partial(P_n \times C_m) \ge 3$. Thus $\chi_\partial(P_n \times C_m) = 3$. □

Theorem 10.

$$\chi_\partial(C_n \times C_m) = \begin{cases} 2 & \text{if both } m \text{ and } n \text{ are even,} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let the vertices of $C_n \times C_m$ be $\{v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}\}$.

Case 1. m and n are odd

Let $A = \{v_{1j}\colon\ 1 \le j \le m-2, \text{where } j \text{ is odd}\}$,
$B = \{v_{ij}\colon\ 2 \le i \le n-1 \text{ and } 2 \le j \le m-1, \text{where both } i \text{ and } j \text{ are even}\}$,
$C = \{v_{ij}\colon\ 3 \le i \le n-2 \text{ and } 1 \le j \le m, \text{where both } i \text{ and } j \text{ are odd}\}$,
$D = \{v_{nj}\colon\ 3 \le j \le m, \text{where } j \text{ is odd}\}$,
$V_1 = A \cup B \cup C \cup D$,
$V_2 = \{v_{1n}, v_{21}, v_{n1}\} \cup \{v_{im}\colon\ 4 \le i \le n-1, \text{where } i \text{ is even}\}$,
$V_3 = V - \{V_1 \cup V_2\}$.

Case 2. m is even and n is odd

Let $A = \{v_{ij}\colon\ 1 \le i \le n-2 \text{ and } 1 \le j \le m-1, \text{where both } i \text{ and } j \text{ are odd}\}$,
$B = \{v_{ij}\colon\ 2 \le i \le n-1 \text{ and } 2 \le j \le m, \text{where both } i \text{ and } j \text{ are even}\}$,
$C = \{v_{ij}\colon\ 2 \le i \le n-1 \text{ and } 1 \le j \le m-1, \text{where } i \text{ is even and } j \text{ is odd}\}$,
$D = \{v_{ij}\colon\ 3 \le i \le n \text{ and } 2 \le j \le m, \text{where } i \text{ is odd and } j \text{ is even}\}$,
$V_1 = A \cup B$,
$V_2 = C \cup D$,
$V_3 = V - \{V_1 \cup V_2\}$.

Case 3. m is odd and n is even

Let $A = \{v_{ij}\colon\ 1 \le i \le n-1 \text{ and } 1 \le j \le m, \text{where both } i \text{ and } j \text{ are odd}\}$,
$B = \{v_{ij}\colon\ 2 \le i \le n \text{ and } 2 \le j \le m-1, \text{where both } i \text{ and } j \text{ are even}\}$,
$V_1 = A \cup B$,
$V_2 = \{v_{im}\colon\ 2 \le i \le n, \text{where } i \text{ is even}\}$,
$V_3 = V - \{V_1 \cup V_2\}$.

Then in all the above cases, $\{V_1, V_2, V_3\}$ is a positive differential chromatic partition of $C_n \times C_m$ and since, in each case $C_n \times C_m$ contains an odd cycle it follows that $\chi_\partial(C_n \times C_m) = 3$.

Case 4. m and n are even

Let $V_1 = \{v_{ij}\colon\ 1 \le i \le n \text{ and } 1 \le j \le m-1, \text{where } j \text{ is odd}\}$,
$V_2 = V - V_1$.

Then $\{V_1, V_2\}$ is a positive differential chromatic partition of $C_n \times C_m$ and hence $\chi_\partial(G) = 2$. □

Theorem 11.

$$\chi_\partial(P_m \times P_n) = \begin{cases} 3 & \text{if both } m \text{ and } n \text{ are odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $G = P_m \times P_n$. Clearly G is a bipartite graph. When both m and n are odd, the number of vertices in G is odd. Hence G is a bipartite graph with bipartition (X, Y) and $||X| - |Y|| = 1$. Then clearly $\chi_\partial(G) = 3$. In all other cases the number of vertices in G is even. Hence G is a bipartite graph with equal parts. Hence by Theorem 1, $\chi_\partial(G) = 2$. □

§7. Conclusion and Scope

Graph coloring theory is an important branch of discrete mathematics and is of interest for its applications in several areas. Several variations of graph colorings such as edge coloring, total coloring, acyclic coloring, list coloring, equitable coloring, star chromatic number, subchromatic number and game chromatic number have been investigated by several authors. In this sequence we have introduced another type of coloring namely differential coloring and we have just initiated a study of this new colouring parameter. However, there is an abundant scope for further research on this topic. Here we list some interesting problems for further investigation.

1. Characterize graphs for which $\chi_\partial(G) = 3$.
2. Characterize trees for which $\chi_\partial(G) = \Delta + 1$.
3. Characterize graphs for which
   (i) $\chi_\partial(G) = n - \rho(G) + 1$.
   (ii) $\chi_\partial(G) = n - \gamma_{ipr}(G) + 2$.
4. Characterize graphs with $i(G) \leq n/2$ for which $\chi_\partial(G) = n - i(G) + 1$.
5. Characterize graphs for which $\chi_\partial(G) = \chi(G)$.

For any graph theoretic parameter the effect of removal of a vertex or an edge on the parameter is of practical importance. Hence one can investigate the criticality concepts and the effect of an edge removal and vertex removal as done in the case of chromatic number.

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