

## Generalized finite operators and orthogonality

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**Abstract.** In this paper we prove that a spectraloid operator is finite, we present some generalized finite operators and we give a new class of finite operators. Also, the orthogonality of some operators is studied.

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### §1. Introduction

Let  $H$  be a separable infinite dimensional complex Hilbert space, and let  $\mathcal{L}(H)$  denote the algebra of all bounded linear operators on  $H$ . For  $A, B \in \mathcal{L}(H)$ , the generalized derivation  $\delta_{A,B} : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$  is defined by

$$\delta_{A,B}(X) = AX - XB.$$

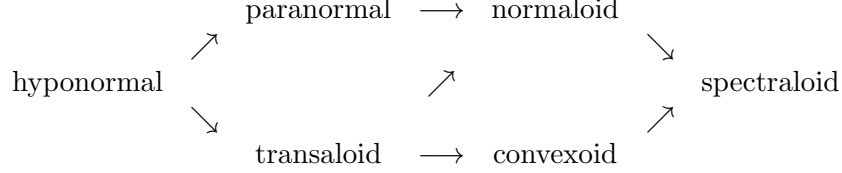
We denote  $\delta_{A,A}$  by  $\delta_A$ . Let  $E$  be a complex Banach space. We say [1] that  $b \in E$  is orthogonal to  $a \in E$  if for all complex  $\lambda$  there holds  $\|a + \lambda b\| \geq \|a\|$ . An operator  $A \in \mathcal{L}(H)$  is called *finite* by J. P. Williams [12] if  $\|AX - XA - I\| \geq 1$  for all  $X \in \mathcal{L}(H)$ , i.e. the range of  $\delta_A$  is orthogonal to the identity operator. The pair  $(A, B) \in \mathcal{L}(H) \times \mathcal{L}(H)$  is said to be generalized finite operators [7] if  $\|AX - XB - I\| \geq 1$  for all  $X \in \mathcal{L}(H)$ .  $\mathcal{F}(H)$  and  $\mathcal{GF}(H)$  denote the class of finite operators and the class of generalized finite operators respectively.

For  $A \in \mathcal{L}(H)$  the set  $W(A) = \{(Ax, x) : x \in H \text{ and } \|x\| = 1\}$  is called the numerical range of  $A$ .

In the following we will denote the spectrum, the point spectrum, the approximate spectrum and the approximate reducing spectrum of  $A \in \mathcal{L}(H)$  by  $\sigma(A)$ ,  $\sigma_p(A)$ ,  $\sigma_a(A)$  and  $\sigma_{ar}(A)$  respectively.

An operator  $A \in \mathcal{L}(H)$  is said to be spectraloid if  $\omega(A) = r(A)$ , where  $r(A)$  (resp.  $\omega(A)$ ) is the spectral radius (resp. numerical radius) of  $A$ , convexoid

if  $\overline{W(A)} = \text{co}\sigma(A)$ , where  $\text{co}\sigma(A)$  is the convex hull of  $\sigma(A)$ , and transaloid if  $r((A - \lambda I)^{-1}) = \|(A - \lambda I)^{-1}\|$  for all  $\lambda \notin \sigma(A)$ . We have the following inclusions:



A bounded linear operator  $A$  is in the class  $\mathcal{Y}_\alpha$  for certain  $\alpha \geq 1$  if there exists a positive number  $k_\alpha$  such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 ((A - \lambda I)^*(A - \lambda I)), \text{ for all } \lambda \in \mathbb{C}.$$

It is known that  $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$  for each  $\alpha, \beta$  such as  $1 \leq \alpha \leq \beta$  [11], where  $\mathcal{Y} = \cup_{\alpha \geq 1} \mathcal{Y}_\alpha$ .

In this paper we prove that a spectraloid operator is finite and that the operator of the form  $A + K$  is also finite, where  $A$  is convexoid and  $K$  is compact. We present some generalized finite operators and we give a new class of finite operators. Also we study the orthogonality of certain operators.

## §2. Preliminaries

**Lemma 1.** *Let  $A \in \mathcal{L}(H)$ . If  $\sigma_{ar}(A)$  is not empty, then  $A$  is finite.*

*Proof.* Let  $\lambda \in \sigma_{ar}(A)$  and  $\{x_n\}$  be a normalized sequence such that  $(A - \lambda I)x_n \rightarrow 0$  and  $(A - \lambda I)^*x_n \rightarrow 0$ . If  $X \in \mathcal{L}(H)$ , then we have

$$\begin{aligned}
 \|AX - XA - I\| &= \|(A - \lambda I)X - X(A - \lambda I) - I\| \\
 &\geq |\langle (A - \lambda I)Xx_n, x_n \rangle - \langle X(A - \lambda I)x_n, x_n \rangle - 1|.
 \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain  $\|AX - XA - I\| \geq 1$ .  $\square$

**Lemma 2.** *Let  $A \in \mathcal{L}(H)$ . If  $\mathcal{R}eA \geq 0$ , then  $\sigma_a(A) \subset \sigma_{ar}(A)$ .*

*Proof.* For  $\lambda \in \sigma_a(A)$ , there exists a sequence  $\{x_n\}$  such that  $(A - \lambda I)x_n \rightarrow 0$ , and then

$$B = \mathcal{R}e(A - \lambda I) = \frac{1}{2}[(A - \lambda I) + (A - \lambda I)^*]$$

satisfies  $\langle Bx_n, x_n \rangle \rightarrow 0$ . Since  $B \geq 0$ , it results that  $Bx_n \rightarrow 0$ , i.e.,

$$\frac{1}{2}[(A - \lambda I)x_n + (A - \lambda I)^*x_n] \rightarrow 0.$$

Since  $(A - \lambda I)x_n \rightarrow 0$ , we have  $(A - \lambda I)^*x_n \rightarrow 0$ .  $\square$

**Lemma 3.** For  $A \in \mathcal{L}(H)$ ,  $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$ .

*Proof.* By the transformation  $A \mapsto \alpha A + \beta$  the hypothesis  $\lambda \in \partial W(A) \cap \sigma(A)$  can be replaced by  $0 \in \partial W(A) \cap \sigma(A)$  with  $\operatorname{Re} A \geq 0$ . Since  $0 \in \partial \sigma(A) \subset \sigma_a(A)$ , it results from the previous lemmas that  $0 \in \sigma_{ar}(A)$ , hence  $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$ .  $\square$

### §3. Main results

**Theorem 1.** Let  $A \in \mathcal{L}(H)$  be convexoid. Then  $A$  is finite.

*Proof.* If  $A$  is convexoid, then  $\overline{W(A)} = \operatorname{co}\sigma(A)$ . Hence  $\partial W(A) \cap \sigma(A) \neq \emptyset$ . It follows immediately from the previous lemmas that  $A$  is finite.  $\square$

**Remark 1.** It is known that transaloid operators are convexoid operators, and then  $\mathcal{F}(H)$  contains all the transaloid operators.

**Theorem 2.** Let  $A \in \mathcal{L}(H)$  be spectraloid. Then  $A$  is finite.

*Proof.* We have  $\omega(A) = r(A)$ . This implies that there exists  $\lambda \in \sigma(A) \subset \overline{W(A)}$  such that  $|\lambda| = \omega(A)$ , hence  $\lambda \in \partial W(A)$ , then  $\partial W(A) \cap \sigma(A) \neq \emptyset$ , which implies that  $A \in \mathcal{F}(H)$ .  $\square$

As a consequence of the previous theorem we obtain:

**Corollary 1.** The following operators are finite.

- (1) Hyponormal operators,
- (2) Transaloid operators,
- (3) Paranormal operators,
- (4) Normaloid operators.

**Lemma 4.** [9] For  $A \in \mathcal{L}(H)$ , the following holds

$$\overline{W(A)} = \operatorname{co}\sigma(A) \iff \forall \lambda \notin \operatorname{co}\sigma(A) : \left\| (A - \lambda I)^{-1} \right\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(A))]^{-1}.$$

Hence a convexoid element on a  $C^*$ -algebra  $\mathcal{A}$ , may be defined as an element  $a \in \mathcal{A}$  satisfying

$$\forall \lambda \notin \operatorname{co}\sigma(a) : \left\| (a - \lambda e)^{-1} \right\| \leq [\operatorname{dist}(\lambda, \operatorname{co}\sigma(a))]^{-1}.$$

**Theorem 3.** Let  $\mathcal{A}$  be a  $C^*$ -algebra and let  $a$  be a convexoid element on  $\mathcal{A}$ . Then  $a$  is finite.

*Proof.* It is known [6, p. 97] that there exist a Hilbert space  $H$  and a \*-isometric homomorphism  $\varphi$  ( $\varphi : \mathcal{A} \rightarrow \mathcal{L}(H)$ ). Then  $\varphi(a)$  is convexoid. Since  $\varphi$  is isometric it results from Theorem 1 that  $a$  is finite.  $\square$

**Corollary 2.** *Let  $A \in \mathcal{L}(H)$  be convexoid. Then  $T = A + K$  is finite, where  $K$  is a compact operator.*

*Proof.* Since the Calkin algebra  $\mathcal{L}(H)/\mathcal{K}(H)$  is a  $C^*$ -algebra (where  $\mathcal{K}(H)$  is the set of compact operators),  $[A] = \{A + K : K \in \mathcal{K}(H)\}$  is convexoid. Hence it follows from Theorem 3  $[A]$  is finite and we have, for all  $X \in \mathcal{L}(H)$

$$\begin{aligned} \|TX - XT - I\| &= \|[TX - XT - I]\| \\ &= \|[T][X] - [X][T] - [I]\| \\ &= \|[A][X] - [X][A] - [I]\| \\ &\geq 1. \end{aligned}$$

$\square$

**Lemma 5.** *For  $A, T \in \mathcal{L}(H)$ , if  $A \in \mathcal{Y}$  and  $T$  is a normal operator such as  $AT = TA$ , then for all  $\lambda \in \sigma_p(T)$*

$$\|AX - XA - T\| \geq |\lambda|, \text{ for all } X \in \mathcal{L}(H).$$

*Proof.* Let  $\lambda \in \sigma_p(T)$  and  $M_\lambda$  be the eigenspace associated with  $\lambda$ . Since  $AT = TA$ , we have  $AT^* = T^*A$  by the Fuglede's theorem [4]. Hence  $M_\lambda$  reduces both  $A$  and  $T$ . According to the decomposition  $H = M_\lambda \oplus M_\lambda^\perp$ , we can write  $A, T$  and  $X \in \mathcal{L}(H)$  as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, T = \begin{bmatrix} \lambda & 0 \\ 0 & T_2 \end{bmatrix} \text{ and } X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Since the restriction of a class  $\mathcal{Y}$  operator to a reduced subspace is a class  $\mathcal{Y}$  operator and since  $\mathcal{Y} \subset \mathcal{F}(H)$  [2], we have

$$\begin{aligned} \|AX - XA - T\| &= \left\| \begin{bmatrix} A_1X_1 - X_1A_1 - \lambda & * \\ * & * \end{bmatrix} \right\| \\ &\geq \|A_1X_1 - X_1A_1 - \lambda\| \\ &\geq |\lambda| \left\| A_1\left(\frac{X_1}{\lambda}\right) - \left(\frac{X_1}{\lambda}\right)A_1 - I \right\| \\ &\geq |\lambda|. \end{aligned}$$

$\square$

In the sequel, we need the Berberian technique, and it allows us to construct a Hilbert space which contains a given Hilbert space  $H$  on which we could speak about "approached eigenvectors" and those as regarded as eigenvectors.

**Proposition 1** (Berberian technique). *Let  $H$  be a complex Hilbert space, then there exist a Hilbert space  $\widehat{H} \supset H$  and an  $*$ -isometric homomorphism  $\varphi : \mathcal{L}(H) \rightarrow \mathcal{L}(\widehat{H})$  ( $A \mapsto \widehat{A}$ ) preserving the order, i.e. for all  $A, B \in \mathcal{L}(H)$  and for all  $\alpha, \beta \in \mathbb{C}$  we have:*

$$(1) \widehat{A^*} = \widehat{A}^*,$$

$$(2) \widehat{I} = I,$$

$$(3) \widehat{\alpha A + \beta B} = \alpha \widehat{A} + \beta \widehat{B},$$

$$(4) \widehat{AB} = \widehat{A}\widehat{B},$$

$$(5) \|\widehat{A}\| = \|A\|,$$

$$(6) \sigma(\widehat{A}) = \sigma(A), \sigma_p(\widehat{A}) = \sigma_a(\widehat{A}) = \sigma_a(A),$$

$$(7) \text{ If } A \text{ is positive, then } \widehat{A} \text{ is positive and } \widehat{A^\alpha} = \widehat{A}^\alpha \text{ for all } \alpha > 0.$$

**Theorem 4.** *Let  $A \in \mathcal{Y}$ . Then for every normal operator  $T$  such that  $AT = TA$ , we have*

$$\|AX - XA - T\| \geq \|T\|, \text{ for all } X \in \mathcal{L}(H).$$

*Proof.* Let  $\lambda \in \sigma(T) = \sigma_a(T)$  [5]. Then it follows from Proposition 1 that  $\widehat{T}$  is normal,  $\widehat{A} \in \mathcal{Y}$ ,  $\widehat{AT} = \widehat{T}\widehat{A}$  and  $\lambda \in \sigma_p(\widehat{T})$ . By applying Lemma 5, we get

$$\|AX - XA - T\| = \|\widehat{A}\widehat{X} - \widehat{X}\widehat{A} - \widehat{T}\| \geq |\lambda|,$$

for all  $X \in \mathcal{L}(H)$ . Hence

$$\|AX - XA - T\| \geq \sup_{\lambda \in \sigma(\widehat{T})} |\lambda| = r(\widehat{T}) = \|\widehat{T}\| = \|T\|,$$

for all  $X \in \mathcal{L}(H)$ . □

**Theorem 5.** *Let  $A, B \in \mathcal{L}(H)$ . If  $A, B^* \in \mathcal{Y}$ , then*

$$\|AX - XB - T\| \geq \|T\|,$$

for all  $X \in \mathcal{L}(H)$  and for all  $T \in \ker \delta_{A,B}$ .

*Proof.* Let  $T \in \ker \delta_{A,B}$ . Then  $T \in \ker \delta_{A^*,B^*}$  [11, Theorem 2]. Therefore,  $ATT^* = TBT^* = TT^*A$ . Since  $A \in \mathcal{Y}$ ,  $TT^*$  is normal and  $A(TT^*) = (TT^*)A$ , the previous theorem implies that

$$\begin{aligned} \|T\|^2 = \|TT^*\| &\leq \|TT^* - (AXT^* - XT^*A)\| \\ &= \|TT^* - (AXT^* - XBT^*)\| \\ &\leq \|T^*\| \|T - (AX - XB)\|. \end{aligned}$$

Thus

$$\|AX - XB - T\| \geq \|T\|.$$

□

**Theorem 6.** Let  $A, B \in \mathcal{L}(H)$  be  $A = \bigoplus_{i=1}^n A_i$ ,  $B = \bigoplus_{i=1}^n B_i$ . If there exists  $j \leq n$  such that  $(A_j, B_j) \in \mathcal{GF}(H_j)$ , then  $(A, B) \in \mathcal{GF}(H)$ .

*Proof.* Let  $j \leq n$  such that  $(A_j, B_j) \in \mathcal{GF}(H_j)$ . Then for all  $X \in \mathcal{L}(H)$

$$\begin{aligned} \|AX - XB - I\| &= \left\| \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & A_j X_{jj} - X_{jj} B_j - I_j & * & * & * & * \\ * & * & * & * & * & . & * & * \\ * & * & * & * & * & * & . & * \\ * & * & * & * & * & * & * & * \end{pmatrix} \right\| \\ &\geq \|A_j X_{jj} - X_{jj} B_j - I_j\| \\ &\geq 1. \end{aligned}$$

□

**Proposition 2.** For  $(A, B) \in \mathcal{GF}(H)$ , the following assertions hold:

- (1)  $(\alpha A + \beta, \alpha B + \beta) \in \mathcal{GF}(H)$ , for each  $\alpha, \beta \in \mathbb{C}$ .
- (2)  $(A^{-1}, B^{-1}) \in \mathcal{GF}(H)$ , if  $A$  and  $B$  are invertible.
- (3)  $(R, T) \in \mathcal{GF}(H)$ , if  $R$  and  $T$  are simultaneously unitarily equivalent to  $A$  and  $B$  respectively.
- (4)  $(B^*, A^*) \in \mathcal{GF}(H)$ .
- (5)  $(A^{2^m}, B^{2^m}) \in \mathcal{GF}(H)$ , for all  $m \in \mathbb{N}$ .
- (6)  $\sigma(A) \cap \sigma(B) \neq \emptyset$ .

*Proof.* (1) If  $(A, B) \in \mathcal{GF}(H)$  then [7, Theorem 18] there exists a state  $f$  on  $\mathcal{L}(H)$  such that  $f(AX) = f(XB)$  for all  $X \in \mathcal{L}(H)$ . As a consequence of the linearity of  $f$ ,

$$\forall \alpha, \beta \in \mathbb{C} : f((\alpha A + \beta)X) = f(X(\alpha B + \beta))$$

for all  $X \in \mathcal{L}(H)$ .

(2) Let  $f$  be a state on  $\mathcal{L}(H)$  such that  $f(AX) = f(XB)$  for all  $X \in \mathcal{L}(H)$ . Then we have

$$f(A^{-1}X) = f((A^{-1}XB^{-1})B) = f(A(A^{-1}XB^{-1})) = f(XB^{-1}),$$

for all  $X \in \mathcal{L}(H)$ .

(3) Let  $U$  be a unitary operator. Then by [7, Theorem 18] we have

$$\begin{aligned} (A, B) \in \mathcal{GF}(H) &\iff 0 \in \overline{W(AX - XB)}, \forall X \in \mathcal{L}(H) \\ &\iff 0 \in \overline{W(U^*(AX - XB)U)}, \forall X \in \mathcal{L}(H) \\ &\iff 0 \in \overline{W(U^*(AUU^*X - XU U^*B)U)}, \forall X \in \mathcal{L}(H) \\ &\iff 0 \in \overline{W((U^*AU)Y - Y(U^*BU))}, \forall Y \in \mathcal{L}(H) \\ &\iff (U^*AU, U^*BU) \in \mathcal{GF}(H). \end{aligned}$$

(4) Let  $f$  be a state on  $\mathcal{L}(H)$  such that  $f(AX) = f(XB)$  for all  $X \in \mathcal{L}(H)$ . Then we have

$$\begin{aligned} f^*(B^*X) &= \overline{f(B^*X)^*} = \overline{f(X^*B)} \\ &= \overline{f(AX^*)} = \overline{f(XA^*)^*} \\ &= f^*(XA^*), \end{aligned}$$

for all  $X \in \mathcal{L}(H)$ . Since the adjoint of a state is a state, we have  $(B^*, A^*) \in \mathcal{GF}(H)$ .

(5) Let  $f$  be a state on  $\mathcal{L}(H)$  such that  $f(AX - XB) = 0$  for all  $X \in \mathcal{L}(H)$ . By recurrence we have:

For  $m = 0$ ,  $(A^{2^0}, B^{2^0}) = (A, B) \in \mathcal{GF}(H)$ . Suppose that, for all  $m \in \mathbb{N}$ , there exists a state  $f$  on  $\mathcal{L}(H)$  such that

$$f(A^{2^m}X - XB^{2^m}) = 0, \text{ for all } X \in \mathcal{L}(H).$$

Then

$$f(A^{2^m}(A^{2^m}X) - (A^{2^m}X)B^{2^m}) = 0 \text{ and } f(A^{2^m}(XB^{2^m}) - (XB^{2^m})B^{2^m}) = 0,$$

hence

$$f(A^{2^{m+1}}X - XB^{2^{m+1}}) = 0.$$

(6) Suppose that  $\sigma(A) \cap \sigma(B) = \phi$ . In [10] M. Rosenblum proved that  $\sigma(\delta_{A,B}) \subset \sigma(A) - \sigma(B)$ , and then  $\delta_{A,B}$  is invertible, hence there exists  $X \in \mathcal{L}(H)$  for which  $\|\delta_{A,B}(X) - I\| < 1$ .  $\square$

**Theorem 7.** *Let  $A, B \in \mathcal{L}(H)$ . If there exist a normed sequence  $(f_n)_{n \geq 1}$  in  $H$  and a scalar  $\lambda$  such that*

$$\|(A - \lambda I)^* f_n\| \longrightarrow 0 \text{ and } \|(B - \lambda I) f_n\| \longrightarrow 0,$$

*then  $(A, B) \in \mathcal{GF}(H)$ .*

*Proof.* If  $X \in \mathcal{L}(H)$ . Then

$$\begin{aligned} \|AX - XB - I\| &= \|(A - \lambda I)X - X(B - \lambda I) - I\| \\ &\geq |[(A - \lambda I)X - X(B - \lambda I) - I] f_n, f_n| \\ &= |(X f_n, (A - \lambda I)^* f_n) - ((B - \lambda I) f_n, X^* f_n) - 1|. \end{aligned}$$

By passage to the limit, we get  $\|AX - XB - I\| \geq 1$ , for all  $X \in \mathcal{L}(H)$ .  $\square$

**Corollary 3.** *Let  $A \in \mathcal{L}(H)$ . Then, for all  $\lambda \in \sigma_a(A)$  and for all  $C \in \mathcal{L}(H)$ ,*

$$((A - \lambda I)^*, C(A - \lambda I)) \in \mathcal{GF}(H).$$

*Proof.* Let  $\lambda \in \sigma_a(A)$ , then there exists a normed sequence  $(f_n)_{n \geq 1}$  in  $H$  such that  $\|(A - \lambda I) f_n\| \longrightarrow 0$ . If  $T = A - \lambda I$  and  $R = CT$  with  $C \in \mathcal{L}(H)$ , then

$$\|[(T - 0)^*]^* f_n\| = \|(A - \lambda I) f_n\| \longrightarrow 0$$

and

$$\|(R - 0) f_n\| = \|C(A - \lambda I) f_n\| \longrightarrow 0,$$

hence

$$((A - \lambda I)^*, C(A - \lambda I)) = (T^*, R) \in \mathcal{GF}(H).$$

$\square$

**Corollary 4.** *For all  $A \in \mathcal{L}(H)$ , there exists  $B \in \mathcal{L}(H)$  for which  $(A, B)$  is a generalized finite operator.*

*Proof.* We say that the approximate spectrum is never empty. Let  $\lambda \in \sigma_a(A^*)$ , hence it follows from the previous corollary that

$$((A^* - \lambda I)^*, C(A^* - \lambda I)) = ((A - \bar{\lambda}I), C(A^* - \lambda I)) \in \mathcal{GF}(H),$$

for all  $C \in \mathcal{L}(H)$ , and by applying (1) of Proposition 2 we get

$$(A, B) \in \mathcal{GF}(H),$$

where  $B = C(A^* - \lambda I) + \bar{\lambda}I$ .  $\square$



**Corollary 5.**  $\mathcal{F}(H)$  contains the following class:

$$S(H) = \{A \in \mathcal{L}(H) : A - \bar{\lambda}I = C(A^* - \lambda I) \text{ with } \lambda \in \sigma_a(A^*) \text{ and } C \in \mathcal{L}(H)\}.$$

*Proof.* It follows from the previous corollary that, if  $A \in S(H)$ , then  $(A, A) \in \mathcal{GF}(H)$  i.e.  $A \in \mathcal{F}(H)$ .  $\square$

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