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9-Shredders in 9-connected graphs

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Abstract. For a graph G, a subset S of $V(G)$ is called a shredder if $G - S$ consists of three or more components. We show that if G is a 9-connected graph of order at least 67, then the number of shredders of cardinality 9 of G is less than or equal to $(2|V(G)| - 9)/3$.

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§1. Introduction

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, we let $N_G(x)$ denote the set of vertices adjacent to x in G. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph induced by S in G, and $G - S$ denotes the subgraph obtained from G by deleting all vertices in S together with the edges incident with them; thus $G - S = \langle V(G) - S \rangle$.

As is introduced by Cheriyan and Thurimella in [1], a subset S of $V(G)$ is called a *shredder* if $G - S$ consists of three or more components. A shredder of cardinality k is referred to as a k-shredder. In [2; Theorem 1], it is proved that if $k \geq 5$ and G is a k-connected graph, then the number of k-shredders of G is less than $2|V(G)|/3$, and it is shown that for each fixed $k \geq 5$, the coefficient $2/3$ in the upper bound is best possible. For $k = 5$, it is shown in [3; Theorem 3] that if G is a 5-connected graph of order at least 135, then the number of 5-shredders of G is less than or equal to $(2|V(G)| - 10)/3$; for $k = 6$, it is shown in [7] that if G is a 6-connected graph of order at least 325, then the number of 6-shredders of G is less than or equal to $(2|V(G)| - 9)/3$; for $k = 7$, it is shown in [5] that if G is a 7-connected graph of order at least 42, then the number of 7-shredders of G is less than or equal to $(2|V(G)| - 8)/3$; for

$k = 8$, it is shown in [6] that if G is a 8-connected graph of order at least 177, then the number of 8-shredders of G is less than or equal to $(2|V(G)| - 10)/3$. It is also shown that each of these four bounds is attained by infinitely many graphs. For $k \geq 11$, it is shown in [3; Theorem 1] that if G is a k-connected graph of order at least $10k$, then the number of k-shredders of G is less than or equal to $(2|V(G)| - 6)/3$, and the upper bound $(2|V(G)| - 6)/3$ is believed to be best possible. If this bound is in fact best possible for $k \geq 11$, then the cases where $k = 9$ and $k = 10$ will be the only cases for which the best possible bound has not been obtained (for results concerning the case where $1 \leq k \leq 4$, the reader is referred to [4] and [2; Theorem 2]). In this paper, we take up the case where $k = 9$.

We have the following theorem.

Theorem 1. *Let G be a 9-connected graph of order at least 67. Then the number of 9-shredders of G is less than or equal to*

$$(2|V(G)| - 9)/3.$$

We here construct an infinite family of graphs G which attain the bound $(2|V(G)| - 9)/3$ in the Theorem. Let $m \geq 10$. Define an auxiliary graph H_m of order m by letting

$$\begin{aligned} V(H_m) &= \{v_i | 1 \leq i \leq m\}, \\ E(H_m) &= \{v_i v_{i+4} | 1 \leq i \leq m-4\} \\ &\cup \{v_1v_2,\ v_1v_3,\ v_1v_4,\ v_2v_3,\ v_2v_5,\ v_3v_4\} \\ &\cup \{v_m v_{m-1},\ v_m v_{m-2},\ v_m v_{m-3},\ v_{m-1}v_{m-2},\ v_{m-1}v_{m-4},\ v_{m-2}v_{m-3}\}. \end{aligned}$$

We define a graph G_m of order $3m - 6$ by adding $m - 6$ vertices to the so-called lexicographic product of H_m and the null graph of order 2. More precisely, we let

$$V(G_m) = \{x_{i,j} | 1 \leq i \leq m,\ 1 \leq j \leq 2\} \cup \{\alpha_i | 4 \leq i \leq m-4\} \cup \{a\},$$

$$\begin{aligned} E(G_m) = &\{x_{i,j}x_{i+4,k} | 1 \leq i \leq m-4,\ 1 \leq j,k \leq 2\} \\ &\cup \{x_{i-1,j}\alpha_i,\ x_{i,j}\alpha_i,\ x_{i+1,j}\alpha_i,\ x_{i+2,j}\alpha_i | 4 \leq i \leq m-4,\ 1 \leq j \leq 2\} \\ &\cup \{a\alpha_i | 4 \leq i \leq m-4\} \\ &\cup \{ax_{i,j} | 1 \leq i \leq m \textit{ and } i \neq 3, 5, m-4, m-2,\ 1 \leq j \leq 2\} \\ &\cup \{x_{1,j}x_{2,k},\ x_{1,j}x_{3,k},\ x_{1,j}x_{4,k},\ x_{2,j}x_{3,k},\ x_{2,j}x_{5,k},\ x_{3,j}x_{4,k} | 1 \leq j,k \leq 2\} \\ &\cup \{x_{m-4,j}x_{m-1,k},\ x_{m-3,j}x_{m-2,k},\ x_{m-3,j}x_{m,k},\ x_{m-2,j}x_{m-1,k}, \\ &\quad x_{m-2,j}x_{m,k},\ x_{m-1,j}x_{m,k} | 1 \leq j,k \leq 2\}. \end{aligned}$$

Then, as we shall see below, G_m is 9-connected, and has $2m - 7$ 9-shredders.

$\{x_{i-4,1}, x_{i-4,2}, x_{i+4,1}, x_{i+4,2}, \alpha_{i-2}, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, a\}$ $(6 \leq i \leq m-5)$,

$\{x_{i-1,1}, x_{i-1,2}, x_{i,1}, x_{i,2}, x_{i+1,1}, x_{i+1,2}, x_{i+2,1}, x_{i+2,2}, a\}$ $(4 \leq i \leq m-4)$,

$\{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{9,1}, x_{9,2}, \alpha_4, \alpha_5, \alpha_6, \}$,
$\{x_{m-8,1}, x_{m-8,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, \alpha_{m-6}, \alpha_{m-5}, \alpha_{m-4}\}$,
$\{x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2}, x_{8,1}, x_{8,2}, \alpha_4, \alpha_5, a\}$,
$\{x_{m-7,1}, x_{m-7,2}, x_{m-4,1}, x_{m-4,2}, x_{m,1}, x_{m,2}, \alpha_{m-5}, \alpha_{m-4}, a\}$,
$\{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{4,1}, x_{4,2}, x_{7,1}, x_{7,2}\alpha_4\}$,
$\{x_{m-6,1}, x_{m-6,2}, x_{m-3,1}, x_{m-3,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}\alpha_{m-4}\}$,
$\{x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2}, x_{5,1}, x_{5,2}, x_{6,1}, x_{6,2}, a\}$,
$\{x_{m-5,1}, x_{m-5,2}, x_{m-4,1}, x_{m-4,2}, x_{m-1,1}, x_{m-1,2}, x_{m,1}, x_{m,2}, a\}$,
$\{x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}, x_{5,1}, x_{5,2}, a\}$,
$\{x_{m-4,1}, x_{m-4,2}, x_{m-3,1}, x_{m-3,2}, x_{m-2,1}, x_{m-2,2}, x_{m-1,1}, x_{m-1,2}, a\}$.

Thus the number of 9-shredders of G_m is $2m - 7 = (2(3m - 6) - 9)/3 = (2|V(G_m)| - 9)/3$.

When $m = 14$, we obtain the Figure 1.

For completeness, we include the proof of the assertion that G is 9-connected. The following property of H_m plays an important role in our proof.

Lemma 1.1. *Let $S \subseteq V(H_m)$ be a cutset of H_m such that $|S| \leq 3$. Then one of the following holds:*

(i) *there exist integers t, k, l with $2 \leq t \leq 5$, $k \equiv t \pmod 4$, $l \geq 2$ and $2 \leq k < k + 4l \leq m - 1$ such that $\{v_k, v_{k+4l}\} \subseteq S \cap \{v_{t+4p} | 0 \leq p \leq \frac{1}{4}(m - t - 1)\} \subseteq \{v_k, v_{k+4}, v_{k+8}, \cdots, v_{k+4l-4}, v_{k+4l}\}$ and $\{v_{k+4}, v_{k+8}, \cdots, v_{k+4l-4}\} - S \neq \phi$;*

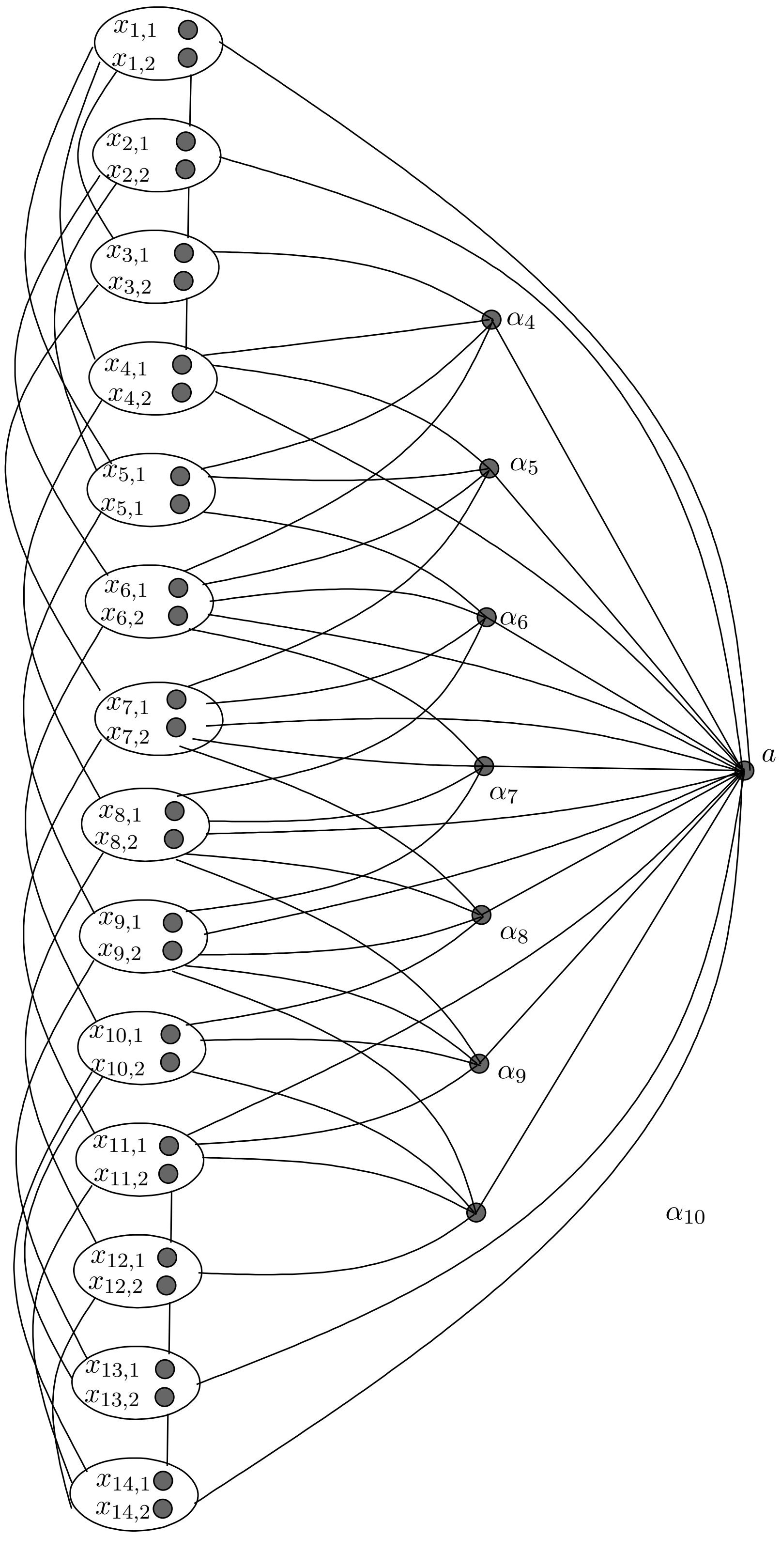
(ii) *$S = \{v_1, v_2, v_{5+4l}\}$ or $\{v_m, v_{m-1}, v_{m-4-4l}\}$ for some l with $1 \leq l \leq \frac{1}{4}(m - 6)$; or*

(iii) *$S = \{v_1, v_3, v_{4+4l}\}$ or $\{v_m, v_{m-2}, v_{m-3-4l}\}$ for some l with $1 \leq l \leq \frac{1}{4}(m - 5)$.*

Proof. For each r with $2 \leq r \leq 5$, set $V_r = \{v_{r+4p} | 0 \leq p \leq \frac{1}{4}(m - r - 1)\}$. Then $V(H_m) = V_2 \cup V_3 \cup V_4 \cup V_5 \cup \{v_1, v_m\}$ (disjoint union). Note that for each r, $\langle \{v_1\} \cup V_r \rangle$ and$\langle V_r \cup \{v_m\} \rangle$ are connected.

First we consider the case where there exists t with $2 \leq t \leq 5$ such that $|S \cap V_t| \geq 2$. If $|S \cap V_t| = 2$, write $S \cap V_t = \{v_k, v_{k+4l}\}$ $(l \geq 1)$; if $|S \cap V_t| = 3$, write $S \cap V_t = \{v_k, v_{k+4l'}, v_{k+4l}\}$ $(l > l'$ and $l \geq 1)$. We show that $l \geq 2$. Note that since $\langle \{v_1\} \cup V_r \rangle$ and $\langle V_r \cup \{v_m\} \rangle$ are connected for each $r \in \{2, 3, 4, 5\} - \{t\}$, $\langle V(H_m) - V_t - \{v_m\} \rangle$ and $\langle V(H_m) - V_t - \{v_1\} \rangle$ are connected.

By way of contradiction, suppose that $l = 1$. Then $S \cap V_t = \{v_k, v_{k+4}\}$. Assume for the moment that $S = \{v_k, v_{k+4}, v_1\}$. Then $V(H_m) - S = (V(H_m) - V_t - \{v_1\}) \cup \{v_{t+4p} | 0 \leq p \leq \frac{1}{4}(k - t - 4)\} \cup \{v_{k+4+4p} | 1 \leq p \leq \frac{1}{4}(m - k - 5)\}$. Since $\langle V(H_m) - V_t - \{v_1\} \rangle$ and $\langle \{v_{k+4+4p} | 1 \leq p \leq \frac{1}{4}(m - k - 5)\} \cup \{v_m\} \rangle$ are connected, $\langle (V(H_m) - V_t - \{v_1\}) \cup \{v_{k+4+4p} | 1 \leq p \leq \frac{1}{4}(m - k - 5)\} \rangle$ is connected.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α_4
α_5
α_6
α_7
α_8
α_9
α_{10}
a

Figure 1: $m = 14$

Since $\langle\{v_2, v_3, v_4, v_5\}\rangle$ and $\langle\{v_{t+4p} | 0 \leq p \leq \frac{1}{4}(k-t-4)\}\rangle$ are connected, this implies that $H_m - S$ is connected, which contradicts the assumption that S is a cutset. By symmetry, we also see that if $S = \{v_k, v_{k+4}, v_m\}$, then $H_m - S$ is connected, a contradiction. Finally if $S = \{v_k, v_{k+4}\}$ or $S = \{v_k, v_{k+4}, v_i\}$ with $v_i \in V(H_m) - V_t - \{v_1, v_m\}$, then it easily follows that $H_m - S$ is connected, a contradiction. Thus $l \geq 2$, as desired.

Now if $S \cap V_t = \{v_k, v_{k+4l}\}$, then (i) holds. Thus we may assume $S \cap V_t = \{v_k, v_{k+4l'}, v_{k+4l}\}$. If $l = 2$, then $l' = 1$ and $H_m - S$ is connected, a contradiction. Thus $l \geq 3$. Hence (i) holds.

Next we consider the case where $|S \cap V_r| \leq 1$ for each $2 \leq r \leq 5$. In this case, if $S \cap \{v_1, v_m\} = \phi$, then $H_m - S$ is clearly connected. Thus $S \cap \{v_1, v_m\} \neq \phi$. If $S \supseteq \{v_1, v_m\}$, then since $\langle\{v_2, v_3, v_4, v_5,\}\rangle$ and $\langle\{v_{m-1}, v_{m-2}, v_{m-3}, v_{m-4}\}\rangle$ are connected, $H_m - S$ is connected. Thus $|S \cap \{v_1, v_m\}| = 1$. By symmetry, we may assume $S \cap \{v_1, v_m\} = \{v_1\}$. If $|S \cap \{v_2, v_3, v_4, v_5\}| = 2$, then $H_m - S$ is connected. Thus $|S \cap \{v_2, v_3, v_4, v_5\}| \leq 1$. If $S \cap \{v_2, v_3, v_4, v_5,\} = \phi$, then since $\langle\{v_2, v_3, v_4, v_5\}\rangle$ is connected, $H_m - S$ is connected. Thus $|S \cap \{v_2, v_3, v_4, v_5\}| = 1$. Write $S \cap \{v_2, v_3, v_4, v_5\} = \{v_s\}$. Since $H_m - \{v_1, v_s\}$ is connected, we have $|S| = 3$. Write $S = \{v_1, v_s, v_i\}$. Then $6 \leq i \leq m-1$. Note that $v_i \notin V_s$ by assumption. If $s = 4$ or 5, then $\langle\{v_2, v_3, v_4, v_5\} - \{v_s\}\rangle$ is connected, and hence $H_m - S$ is connected. Thus $s = 2$ or 3. Note that $\{2, 3, 4, 5\} = \{s, 5-s, s+2, 7-s\}$. If $v_i \in V_{5-s} \cup V_{s+2}$, then since $v_{5-s}v_{s+2} \in E(H_m)$, $H_m - S$ is connected. Thus $v_i \in V_{7-s}$. Consequently (ii) or (iii) holds according as $s = 2$ or $s = 3$. This completes the proof of the lemma.

We also make use of the following lemma, which is easily verified.

Lemma 1.2. *Let G be a connected graph, and let $S \subseteq V(G)$ be a cutset with minimum cardinality. Let u, v be two vertices of G such that $N_G(u) = N_G(v)$. Then we have $\{u, v\} \subseteq S$ or $\{u, v\} \cap S = \emptyset$.*

Now let $G = G_m$, and set $A = \{\alpha_i | 4 \leq i \leq m-4\}$, $X_i = \{x_{i,1}, x_{i,2}\}$ $(1 \leq i \leq m)$, $Y_r = \cup_{0 \leq p \leq (m-r-1)/4} X_{r+4p}$ $(2 \leq r \leq 5)$, and $B = \cup_{1 \leq i \leq m} X_i$. Thus $B = Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup X_1 \cup X_m$ (disjoint union). Note that for each r $(2 \leq r \leq 5)$, $\langle X_1 \cup Y_r\rangle$ and $\langle Y_r \cup X_m\rangle$ are connected. Let $S \subseteq V(G)$ be a cutset of G with minimum cardinality and, by way of contradiction, suppose that $|S| \leq 8$.

Claim 1.1. $S \cap (\{a\} \cup A) \neq \emptyset$

Proof. Suppose that $S \cap (\{a\} \cup A) = \emptyset$. By the definition of G, $\langle\{a\} \cup A\rangle$ is connected and $N_G(x) \cap (\{a\} \cup A) \neq \emptyset$ for each $x \in B$. Hence $G - S$ is connected, which contradicts the assumption that S is a cutset of G.

Claim 1.2. *$\langle B-S\rangle$ is disconnected.*

Proof. Suppose that $\langle B-S\rangle$ is connected. Since $|S|\le 8$, it follows from Claim 1.1 that $|S\cap B|\le 7$. On the other hand, $|N_G(\alpha)\cap B|\ge 8$ for each $\alpha\in\{a\}\cup A$ by the definition of G. Hence $N_G(\alpha)\cap(B-S)\ne\emptyset$ for each $\alpha\in\{a\}\cup A$, which means that $G-S$ is connected, a contradiction.

Since $|S\cap B|\le 7$ by Claim 1.1, the following claim follows from Lemmas 1.1 and 1.2 and Claim 1.2.

Claim 1.3. *One of the following holds:*

(i) $|S\cap B|=4$ *or* 6, *and there exist integers* t,k,l *with* $2\le t\le 5$, $k\equiv t \pmod 4$, $l\ge 2$ *and* $2\le k<k+4l\le m-1$ *such that* $X_k\cup X_{k+4l}\subseteq S\cap Y_t\subseteq X_k\cup X_{k+4}\cup X_{k+8}\cup\cdots\cup X_{k+4l-4}\cup X_{k+4l}$ *and* $(X_{k+4}\cup X_{k+8}\cup\cdots\cup X_{k+4l-4})-S\ne\phi$;

(ii) $|S\cap B|=6$, *and* $S\cap B=X_1\cup X_2\cup X_{5+4l}$ *or* $X_m\cup X_{m-1}\cup X_{m-4-4l}$ *for some* l *with* $1\le l\le \frac{1}{4}(m-6)$; *or*

(iii) $|S\cap B|=6$, *and* $S\cap B=X_1\cup X_3\cup X_{4+4l}$ *or* $X_m\cup X_{m-2}\cup X_{m-3-4l}$ *for some* l *with* $1\le l\le \frac{1}{4}(m-5)$.

First we consider the case where (i) of Claim 1.3 holds. Let t,k,l be the integers as in Claim 1.3(i). Since $\langle X_1\cup Y_r\rangle$ and $\langle Y_r\cup X_m\rangle$ are connected for each $r\in\{2,3,4,5\}-\{t\}$, $\langle B-Y_t-X_m\rangle$ and $\langle B-Y_t-X_1\rangle$ are connected. Let $B_1=(X_{k+4}\cup X_{k+8}\cup\cdots\cup X_{k+4l-4})-S$ and $B_2=B-S-B_1$. By the condition that $S\cap Y_t\subseteq X_k\cup X_{k+4}\cup\cdots\cup X_{k+4l}$, we have $B_2=(B-Y_t-S)\cup(\cup_{0\le p\le(k-t-4)/4}X_{t+4p})\cup(\cup_{1\le p\le(m-k-4l-1)/4}X_{k+4l+4p})$.

Claim 1.4. *$\langle B_2\rangle$ is connected.*

Proof. Note that $|S\cap(B-Y_t)|\le 2$. Hence by Lemma 1.2, we have $S\cap(B-Y_t)=X_1$ and $B_2\supseteq X_m$, or $S\cap(B-Y_t)=X_m$ and $B_2\supseteq X_1$, or $B_2\supseteq X_1\cup X_m$. Assume first that $S\cap(B-Y_t)=X_1$ and $B_2\supseteq X_m$. Then $B_2=(B-Y_t-X_1)\cup(\cup_{0\le p\le(k-t-4)/4}X_{t+4p})\cup(\cup_{1\le p\le(m-k-4l-1)/4}X_{k+4l+4p})$. Since $\langle B-Y_t-X_1\rangle$ is connected and since $\langle(\cup_{1\le p\le(m-k-4l-1)/4}X_{k+4l+4p})\cup X_m\rangle$ is connected if $m-k-4l\ge 5$, we see that $\langle(B-Y_t-X_1)\cup(\cup_{1\le p\le(m-k-4l-1)/4}X_{k+4l+4p})\rangle$ is connected. Since $\langle X_2\cup X_3\cup X_4\cup X_5\rangle$ is connected and since $\langle\cup_{0\le p\le(k-t-4)/4}X_{t+4p}\rangle$ is connected if $k-t\ge 8$, this implies that $\langle B_2\rangle$ is connected. By symmetry, we also see that if $S\cap(B-Y_t)=X_m$ and $B_2\supseteq X_1$, then $\langle B_2\rangle$ is connected. Assume now that $B_2\supseteq X_1\cup X_m$. Since $|S\cap(B-Y_t-X_1-X_m)|\le 2$, $\langle B-Y_t-S\rangle$ is connected. Since $B_2=(B-Y_t-S)\cup(\cup_{0\le p\le(k-t-4)/4}X_{t+4p})\cup(\cup_{1\le p\le(m-k-4l-1)/4}X_{k+4l+4p})$, this implies that $\langle B_2\rangle$ is connected, as desired.

Claim 1.5. *$\langle B_2 \cup ((\{a\} \cup A) - S)\rangle$ is connected.*

Proof. Take $\alpha \in (\{a\} \cup A) - S$. If $\alpha \in A$, $|N_G(\alpha) \cap (B - Y_t)| = |N_G(\alpha) \cap B| - |N_G(\alpha) \cap Y_t| = 6$; if $\alpha = a$, $|N_G(\alpha) \cap (B - Y_t)| \geq |X_1 \cup X_m \cup ((X_2 \cup X_4) - X_t)| \geq 6$. Thus $|N_G(\alpha) \cap (B - Y_t)| \geq 6$. Since $|S \cap (B - Y_t)| \leq 2$ and $B - Y_t - S \subseteq B_2$, it follows that $N_G(\alpha) \cap B_2 \neq \phi$. Since $\alpha \in (\{a\} \cup A) - S$ is arbitrary, this together with Claim 1.4 implies that $\langle B_2 \cup ((\{a\} \cup A) - S)\rangle$ is connected.

Now take $x \in B_1$. Note that $x \in X_i$ for some i with $k+4 \leq i \leq k+4(l-1)$. Then $6 \leq i \leq m-5$, and hence $|N_G(x) \cap (\{a\} \cup A)| = 5$ by the definition of G. Since $|S \cap (\{a\} \cup A)| \leq 8 - |S \cap B| \leq 4$, it follows that $N_G(x) \cap ((\{a\} \cup A) - S) \neq \phi$. Since $x \in B_1$ is arbitrary, this together with Claim 1.5 implies that $G - S = \langle B_1 \cup B_2 \cup ((\{a\} \cup A) - S)\rangle$ is connected, which contradicts the assumption that S is a cutset of G.

Next we consider the case where (ii) or (iii) of Claim 1.3 holds. By symmetry, we may assume that $S \cap B = X_1 \cup X_2 \cup X_{5+4l}$ for some l with $1 \leq l \leq \frac{1}{4}(m-6)$ or $S \cap B = X_1 \cup X_3 \cup X_{4+4l}$ for some l with $1 \leq l \leq \frac{1}{4}(m-5)$. If $S \cap B = X_1 \cup X_2 \cup X_{5+4l}$, let $t = 5$ and $B_1 = \cup_{0 \leq p \leq l-1} X_{5+4p}$; if $S \cap B = X_1 \cup X_3 \cup X_{4+4l}$, let $t = 4$ and $B_1 = \cup_{0 \leq p \leq l-1} X_{4+4p}$. Also let $B_2 = B - S - B_1$. The following claim follows from the definition of G.

Claim 1.6. *$\langle B_2 \rangle$ is connected.*

Claim 1.7. *$\langle B_2 \cup ((\{a\} \cup A) - S)\rangle$ is connected.*

Proof. Take $\alpha \in (\{a\} \cup A) - S$. As in the proof of Claim 1.5, we obtain $|N_G(\alpha) \cap (B - Y_t)| \geq 6$. Since $|S \cap (B - Y_t)| \leq 4$, it follows that $N_G(\alpha) \cap B_2 \neq \phi$. Since $\alpha \in (\{a\} \cup A) - S$ is arbitrary, this together with Claim 1.6 implies that $\langle B_2 \cup ((\{a\} \cup A) - S)\rangle$ is connected.

Now take $x \in B_1$. Note that $x \in X_i$ for some i with $t \leq i \leq t+4(l-1)$. Then $4 \leq i \leq m-5$, and hence $|N_G(x) \cap (\{a\} \cup A)| \geq 3$ by the definition of G. Since $|S \cap (\{a\} \cap A)| \leq 8 - |S \cap B| = 2$, it follows that $N_G(x) \cap ((\{a\} \cup A) - S) \neq \phi$. Since $x \in B_1$ is arbitrary, this together with Claim 1.7 implies that $G - S$ is connected, which contradicts the assumption that S is a cutset of G.

This completes the proof of the assertion that G is 9-connected.

§2. Preliminary Result

Throughout the rest of this paper, let G be a 9-connected graph, and let $\mathscr{S}$ denote the set of 9-shredders of G. For each $S \in \mathscr{S}$, we define $\mathscr{K}(S)$,

$\mathscr{L}(S)$ and $L(S)$ as follows. Let $S \in \mathscr{S}$. We let $\mathscr{K}(S)$ denote the set of components of $G - S$. Write $\mathscr{K}(S) = \{H_1, \cdots, H_s\}$ ($s = |\mathscr{K}(S)|$). We may assume $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_s)|$ (any such labeling will do). Under this notation, we let $\mathscr{L}(S) = \mathscr{K}(S) - \{H_1\}$ and $L(S) = \cup_{2 \leq i \leq s} V(H_i)$; thus $L(S) = \cup_{C \in \mathscr{L}(S)} V(C)$. Now let $\mathscr{L} = \cup_{S \in \mathscr{S}} \mathscr{L}(S)$. A member F of $\mathscr{L}$ is said to *saturated* if there exists a subset $\mathscr{C}$ of $\mathscr{L} - \{F\}$ such that $V(F) = \cup_{C \in \mathscr{C}} V(C)$.

Let $S, T \in \mathscr{S}$ with $S \neq T$. We say that S *meshes* with T if S intersects with at least two members of $\mathscr{K}(T)$. It is easy to see that if S meshes with T, then T intersects with all members of $\mathscr{K}(S)$, and hence T meshes with S and S intersects with all members of $\mathscr{K}(T)$ (see [1; Lemma 4.3 (1)]).

The following two lemmas are proved in [4; Lemmas 2.1 and 3.1] (see also [2; Lemmas 3.2 and 3.4]).

Lemma 2.1. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S does not mesh with T. Then one of the following holds:*

(i) $L(S) \cap L(T) = \emptyset$, $(L(S) \cup L(T)) \cap (S \cup T) = \emptyset$, *and no edge of G joins a vertex in $L(S)$ and a vertex in $L(T)$;*

(ii) *there exists $C \in \mathscr{L}(S)$ such that $V(C) \supseteq L(T)$ (so $L(S) \supseteq L(T)$); or*

(iii) *there exists $D \in \mathscr{L}(T)$ such that $V(D) \supseteq L(S)$ (so $L(T) \supseteq L(S)$).*

Lemma 2.2. *Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that S meshes with T. Then the following hold.*

(i) $S \supseteq L(T)$ *or* $T \supseteq L(S)$.

(ii) $L(S) \cap L(T) = \emptyset$.

The following lemma is proved in [2; Lemma 3.6].

Lemma 2.3. *Let $F \in \mathscr{L}$, and suppose that F is saturated. Then $|V(F)| \geq 4$.*

The following lemmas are proved in [3; Lemmas 2.9 through 2.12].

Lemma 2.4. *Let $S \in \mathscr{S}$, and let $p = |\mathscr{L}(S)|$.*

Set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq L(S)\}$. $|\mathscr{T}| \leq (2|L(S)| - 2p + 3)/3 \leq (2|L(S)| - 1)/3$.

Lemma 2.5. *Let $X \subseteq V(G)$. Set $\mathscr{T} = \{T \in \mathscr{S} | L(T) \subseteq X\}$ and $\mathscr{L}_0 = \cup_{T \in \mathscr{T}} \mathscr{L}(T)$, and suppose that no component in $\mathscr{L}_0$ is saturated. Then $|\mathscr{T}| \leq |X|/2$.*

Lemma 2.6. *Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $L(S) \nsubseteq T$. Then $L(T) \subseteq S$ and $|L(T)| \leq 4$.*

Lemma 2.7. *Suppose that $|V(G)| \geq 19$. Let $S, T \in \mathscr{S}$, and suppose that S meshes with T, $L(S) \subseteq T$ and $L(T) \subseteq S$. Then $|L(S)| + |L(T)| \leq 9$.*

The following lemma follows from Lemmas 2.6 and 2.7.

Lemma 2.8. *Suppose that $|V(G)| \geq 19$. Let $S, T \in \mathscr{S}$, and suppose that S meshes with T and $|L(S)| \geq 5$. Then $L(T) \subseteq S$ and $|L(T)| \leq 4$.*

As an immediate corollary of Lemma 2.8, we obtain the following lemma.

Lemma 2.9. *Suppose that $|V(G)| \geq 19$. Let $S, T \in \mathscr{S}$ with $S \neq T$, and suppose that $|L(S)|, |L(T)| \geq 5$. Then S does not mesh with T.*

§3. Proof of the Theorem

We continue with the notation of the preceeding section, and prove the Theorem. Thus let $|V(G)| \geq 67$ and, by way of contradiction, suppose that

$$|\mathscr{S}| \geq (2|V(G)| - 8)/3. \tag{3.1}$$

We define an order relation $\leq$ in $\mathscr{S}$ as follows:

$$S \leq T \Longleftrightarrow L(S) \subseteq L(T) \ (S, T \in \mathscr{S}).$$

Let $S_1, \cdots, S_m$ be the maximal members of $\mathscr{S}$ with respect to the order relation $\leq$. We may assume $|L(S_1)| \geq \cdots \geq |L(S_m)|$. Let $p_i = |\mathscr{L}(S_i)|$ for each i, and let $W = V(G) - (L(S_1) \cup \cdots \cup L(S_m))$. Arguing as in [3; Claims 3.2 through 3.4], we obtain the following three claims. We include sketches of their proofs for the convenience of the reader.

Claim 3.1.

(i) $m + 2|W| \leq 8$.

(ii) $2p_1 + (m-1) + 2|W| \leq 11$.

Sketch of Proof. By (3.1) and Lemma 2.4, $(2|V(G)|-8)/3 \leq \sum_{1 \leq i \leq m} (2|L(S_i)| - 2p_i + 3)/3$, and hence $2(p_1 + \cdots + p_m) - 3m + 2|W| \leq 8$. Since $p_i \geq 2$ for all i, both (i) and (ii) follow from this.

Claim 3.2. *$|L(S_1)| \geq 5$.*

Sketch of Proof. If $|L(S_1)| \leq 4$, then by Claim 3.1 (i), $|V(G)| \leq 4m + |W| \leq 32$, which contradicts the assumption that $|V(G)| \geq 67$.

Claim 3.3. *$m \geq 2$ and $|L(S_2)| \geq 5$.*

Sketch of Proof. Suppose that $m = 1$ or $|L(S_2)| \leq 4$. Then by Claim 3.1 (ii), $|V(G) - L(S_1)| \leq 4(m-1) + |W| \leq 44 - 8p_1$, and hence $|V(G) - (S_1 \cup L(S_1))| \leq 35 - 8p_1$, which implies $|L(S_1)| \leq p_1(35 - 8p_1)$. Consequently $|V(G)| \leq p_1(35 - 8p_1) + 44 - 8p_1 \leq 66$ because $p_1 \geq 2$, which contradicts the assumption that $|V(G)| \geq 67$.

By Lemma 2.9, Claim 3.2 and Claim 3.3 imply that S_1 does not mesh with S_2. Since $L(S_1) \cap L(S_2) = \emptyset$ by the maximality of $L(S_1)$ and $L(S_2)$, $L(S_1) \cap S_2 = L(S_2) \cap S_1 = \emptyset$ by Lemma 2.1. Write $\mathscr{K}(S_1) - \mathscr{L}(S_1) = \{C_1\}$ and $\mathscr{K}(S_2) - \mathscr{L}(S_2) = \{C_2\}$; thus $C_1 = G - S_1 - L(S_1)$ and $C_2 = G - S_2 - L(S_2)$. We define $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_{1,1}$, $\mathscr{T}_{1,2}$, $\mathscr{T}_{1,3}$, $\mathscr{T}_{2,1}$, $\mathscr{T}_{2,2}$, $\mathscr{T}_{2,3}$ as follows:

$$\begin{aligned}
\mathscr{T}_1 &= \{T \in \mathscr{S} | L(T) \cap (S_1 \cup S_2) = \emptyset\}, \\
\mathscr{T}_2 &= \{T \in \mathscr{S} | L(T) \subseteq S_1 \cup S_2\}, \\
\mathscr{T}_{1,1} &= \{T \in \mathscr{S} | L(T) \subseteq L(S_1)\}, \\
\mathscr{T}_{1,2} &= \{T \in \mathscr{S} | L(T) \subseteq L(S_2)\}, \\
\mathscr{T}_{1,3} &= \{T \in \mathscr{S} | L(T) \subseteq V(C_1) \cap V(C_2)\}, \\
\mathscr{T}_{2,1} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 - S_2\}, \\
\mathscr{T}_{2,2} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_2 - S_1\}, \\
\mathscr{T}_{2,3} &= \{T \in \mathscr{T}_2 | L(T) \subseteq S_1 \cap S_2\}.
\end{aligned}$$

In view of the maximality of $L(S_1)$ and $L(S_2)$ and Claims 3.2 and 3.3, it follows from Lemmas 2.1 and 2.8 that $\mathscr{T}_1$ is the set of those members of $\mathscr{S}$ which mesh with neither S_1 nor S_2, and $\mathscr{T}_2$ is the set of those members of $\mathscr{S}$ which mesh with S_1 or S_2. Thus $\mathscr{S} = \mathscr{T}_1 \cup \mathscr{T}_2$ (disjoint union). Further by Lemma 2.1, $\mathscr{T}_1 = \mathscr{T}_{1,1} \cup \mathscr{T}_{1,2} \cup \mathscr{T}_{1,3}$ (disjoint union) and , by Lemma 2.8, $\mathscr{T}_2 = \mathscr{T}_{2,1} \cup \mathscr{T}_{2,2} \cup \mathscr{T}_{2,3}$ (disjoint union).

The following two claims immediately follow from Lemma 2.4 (see also [3; Claim 3.6]).

Claim 3.4. $|\mathscr{T}_{1,i}| \leq (2|L(S_i)| - 1)/3$ $(i = 1, 2)$.

Claim 3.5. $|\mathscr{T}_{1,3}| \leq 2|V(C_1) \cap V(C_2)|/3$.

Since $|L(T)| \leq 4$ for each $T \in \mathscr{T}_2$ by Lemma 2.8, the following claim follows from Lemmas 2.3 and 2.5 (see also [3; Claim 3.8]).

Claim 3.6.

(i) $|\mathscr{T}_{2,1}| \leq |S_1 - S_2|/2$.

(ii) $|\mathscr{T}_{2,2}| \leq |S_2 - S_1|/2$.

(iii) $|\mathscr{T}_{2,3}| \leq |S_1 \cap S_2|/2$.

Now it follows from Claims 3.4, 3.5 and 3.6 that

$$\begin{aligned}
|\mathscr{S}| &= |\mathscr{T}_1| + |\mathscr{T}_2| \\
&= |\mathscr{T}_{1,1}| + |\mathscr{T}_{1,2}| + |\mathscr{T}_{1,3}| + |\mathscr{T}_{2,1}| + |\mathscr{T}_{2,2}| + |\mathscr{T}_{2,3}| \\
&\leq (2|L(S_1| - 1)/3 + (2|L(S_2)| - 1)/3 + 2|V(C_1) \cap V(C_2)|/3 \\
&\quad + \lfloor |S_1 - S_2|/2 \rfloor + \lfloor |S_2 - S_1|/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor \\
&= (2(|L(S_1)| + |L(S_2)| + |V(C_1) \cap V(C_2)|) - 2)/3 \\
&\quad + 2\lfloor (7 - |S_1 \cap S_2|)/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor \\
&= (2(|V(G)| - |S_1 \cup S_2|) - 2)/3 + 2\lfloor (9 - |S_1 \cap S_2|)/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor \\
&= (2|V(G)| + 2|S_1 \cap S_2| - 38)/3 + 2\lfloor (9 - |S_1 \cap S_2|)/2 \rfloor + \lfloor |S_1 \cap S_2|/2 \rfloor.
\end{aligned}$$

Since $0 \leq |S_1 \cap S_2| \leq 8$, this implies that $|\mathscr{S}| \leq (2|V(G)| - 9)/3$, which contradicts (3.1). This completes the proof of the Theorem.

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