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On a class of K-contact manifolds

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Abstract. The object of this paper is to study K-contact manifolds with quasi-conformal curvature tensor. We characterised K-contact manifolds satisfying certain curvature conditions on the quasi-conformal curvature tensor.

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§1. Introduction

Let (M^n, g), $(n = 2m+1)$ be a contact Riemannian manifold with contact form η, associated vector field ξ, $(1,1)$- tensor field ϕ and associated Riemannian metric g. If ξ is a Killing vector field, then M^n is called a K-contact manifold ([2], [15]). K-contact manifolds have been studied by several authors such as S. Tanno [18], [19], [20], S.Sasaki [16], D. E. Blair [2], Y. Hatakeyama, Y. Ogawa and S. Tanno [11], M. C. Chaki and D. Ghosh [5], U. C. De and S. Biswas [7] and many others.

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [22]. According to them a quasi-conformal curvature tensor was given by

$$(1.1)\qquad \begin{aligned}\tilde{C}(X,Y)Z \;=\;& aR(X,Y)Z \\ &+b[S(Y,Z)X - S(X,Z)Y \\ &+g(Y,Z)QX - g(X,Z)QY] \\ &-\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y],\end{aligned}$$

where a and b are constants and R, S, Q and r are the Riemannian curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and scalar curvature of the manifold respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the form

$$\begin{aligned} \tilde{C}(X,Y)Z &= R(X,Y)Z \\ &\quad -\frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &\quad +g(Y,Z)QX - g(X,Z)QY] \\ &\quad +\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z, \end{aligned} \tag{1.2}$$

where $C(X,Y)Z$ is the conformal curvature tensor ([10]). Thus the conformal curvature tensor C is a particular case of the tensor $\tilde{C}$. A manifold (M^n, g), $(n > 3)$ is called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C} = 0$. It is known ([1]) that the quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a \neq 0$, $b \neq 0$. Since they give no restriction for manifold if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$. Recently De and Matsuyama [6] studied quasi-conformally flat manifold. Also quasi-conformal curvature tensor have been studied by Özgür and De [13], De and Gazi [8] and many others. A Riemannian manifold satisfying $R(X,Y).R = 0$ is called semisymmetric ([17]), where $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. In an anologous way we define quasi-conformally semisymmetric manifold. A K-contact manifold is said to be quasi-conformally semisymmetric if $R(X,Y).\tilde{C} = 0$, where $\tilde{C}$ is the quasi-conformal curvature tensor.
The paper organised as follows:
After preliminaries in section 3, we first prove that a quasi-conformally flat K-contact manifold is an η-Einstein manifold. As a consequence of this we obtain that a quasi-conformally flat K-contact manifold is Sasakian. Section 4 deals with the study of a K-contact manifold satisfying $div\tilde{C} = 0$ and we prove that such a K-contact manifold is also Sasakian. Section 5 is devoted to the study of a K-contact Einstein (or η-Einstein) quasi-conformally semisymmetric manifold. In section 6 we prove that a ξ-quasi-conformally flat K-contact manifold is an η-Einstein manifold. Finally some applications are given.

§2. Preliminaries

By a contact manifold we mean an $n = (2m+1)$- dimensional differentiable manifold M^n which carries a global 1-form η, there exists a unique vector field

ξ, called the characteristic vector field such that, $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. A Riemannian metric g on M^n is said to be an associated metric if there exists a $(1, 1)$ tensor field ϕ such that

$$d\eta(X, Y) = g(X, \phi Y), \;\; \eta(X) = g(X, \xi), \;\; \phi^2 = -I + \eta \otimes \xi. \tag{2.1}$$

From these equations we have

$$\phi\xi = 0, \;\; \eta\phi = 0, \;\; g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

The manifold M equipped with the contact structure (ϕ, ξ, η, g) is called a contact metric manifold. A contact metric structure is said to be normal (Sasakian) if the almost complex structure J on $M \times R$ defined by, $J(X, f\frac{d}{dt})$ $= (\phi X - f\xi, \eta(X)\frac{d}{dt})$, f being a function on M^n, is integrable. A contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.3}$$

Every Sasakian manifold is K-contact, but the converse need not be true, except in dimension three ([12]). K-contact metric manifold are not too well known, because there is no such a simple expression for the curvature tensor as in the case of Sasakian manifold. For details we refer to ([2], [3], [15]).

Besides the above relations in K-contact manifold the following relations hold ([2], [3], [15]):

$$\nabla_X \xi = -\phi X. \tag{2.4}$$

$$g(R(\xi, X)Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.5}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi. \tag{2.6}$$

$$S(X, \xi) = (n - 1)\eta(X). \tag{2.7}$$

$$(\nabla_X \phi) = R(\xi, X)Y. \tag{2.8}$$

for any vector fields X, Y.

Further since ξ is a Killing vector field, S and r remains invariant under it, i.e.,

$$L_\xi S = 0 \quad and \quad L_\xi r = 0, \tag{2.9}$$

where L denotes the Lie-derivation.

Again a K-contact manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant and η-Einstein if the Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are smooth functions on M. It is well known ([12]) that in a K-contact manifold a and b are constants. Also it is known that every manifold of constant curvature is an Einstein manifold. The converse is only true for dimension three. Again a compact Einstein K-contact manifold is Sasakian ([4]).

A Riemannian or semi-Riemannian manifold is said to be semi-symmetric ([9]) if $R(X,Y).R = 0$, where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies $R(X,Y).\tilde{C} = 0$, where $\tilde{C}$ is the quasi-conformal curvature tensor, then the manifold is said to be quasi-conformally semi-symmetric manifold.

§3. Quasi-conformally flat K-contact manifolds

In 1967 Tanno [20] proved that a conformally flat K-contact manifold is of constant curvature +1 and Sasakian. In this section we consider quasi-conformally flat K-contact manifold. If a K-contact manifold $(M^n, \phi, \xi, \eta, g)$ is quasi-conformally flat, then from (1.1) we get

$$\begin{aligned} g(R(X,Y)Z,W) = {} & \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ & +S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ & +\frac{r}{an}(\frac{a}{n-1} + 2b)[g(Y,Z)g(X,W) \\ & -g(X,Z)g(Y,W)]. \end{aligned} \tag{3.1}$$

Now putting $X = Z = \xi$ in (3.1) we obtain

$$
\begin{aligned}
g(R(\xi, Y)\xi, W) \quad = \quad & \frac{b}{a}[S(\xi, \xi)g(Y, W) \\
& -S(Y, \xi)g(\xi, W) + S(Y, W)g(\xi, \xi) \\
& -S(\xi, W)g(Y, \xi)] + \frac{r}{an}(\frac{a}{n-1} + 2b) \\
& [g(Y, \xi)g(\xi, W) - g(\xi, \xi)g(Y, W)].
\end{aligned} \tag{3.2}
$$

Now using (2.1), (2.2), (2.5) and (2.7) it follows from (3.2) that

$$S(Y, W) = Ag(Y, W) + B\eta(Y)\eta(W), \tag{3.3}$$

where A and B are given by

$$A = -\frac{a}{b} + \frac{r}{nb}(\frac{a}{n-1} + 2b) - (n-1). \tag{3.4}$$

$$B = \frac{a}{b} - \frac{r}{nb}(\frac{a}{n-1} + 2b) + 2(n-1). \tag{3.5}$$

It follows from (3.4) and (3.5) that $A + B = (n-1)$.

In view of the relation (3.3) we state the following:

Proposition 3.1. *A quasi-conformally flat K-contact manifold is an η-Einstein manifold.*

Putting $Y = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, in (3.3) and taking summation over i, $1 \leq i \leq n$, we get

$$r = An + B. \tag{3.6}$$

Now with the help of (3.4) and (3.5) the equation (3.6) gives

$$[(n-2) + \frac{a}{b}][\frac{r}{n} + (1-n)] = 0, \tag{3.7}$$

Hence either

$$b = \frac{a}{2-n} \tag{3.8}$$

or

$$r = n(n-1). \tag{3.9}$$

If $b = \frac{a}{2-n}$, then putting this into (1.1) we get

$$\tilde{C}(X,Y)Z = aC(X,Y)Z, \tag{3.10}$$

where $C(X,Y)Z$ denotes the Weyl conformal curvature tensor. So the quasi-conformally flatness and conformally flatness are equivalent in this case. A conformally flat K-contact manifold (M^n, g) $(n \geq 5)$ is of constant curvature ([20]). But a manifold of constant curvature is conformally flat. Hence a K-contact manifold is conformally flat if and only if it is locally isometric with a unit sphere $S^n(1)$. So in this case, M^n is locally isometric with a unit sphere.

If $r = n(n-1)$, then from (3.3), (3.4) and (3.5) we obtain

$$S(Y,Z) = (n-1)g(Y,Z). \tag{3.11}$$

This implies that M^n is an Einstein manifold. Putting (3.9) and (3.11) into (3.1) we obtain

$$R(X,Y,Z,W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W). \tag{3.12}$$

Thus M^n is of constant curvature $+1$. Hence it is locally isometric with a unit sphere $S^n(1)$. If M^n is locally isometric with a unit sphere $S^n(1)$, it is easy to see that M^n is quasi-conformally flat. This leads to the following theorem:

Theorem 3.1. *Let (M^n, g) $(n \geq 5)$ be a K-contact manifold. Then M^n is quasi-conformally flat if and only if it is locally isometric with a unit sphere $S^n(1)$.*

It is known ([14]) that if a contact metric manifold M^n is of constant curvature c and dimention≥ 5, then $c = 1$ and the structure is Sasakian.

Since the manifold under consideration is of constant curvature $+1$, therefore by the above mentioned result we get the following:

Corollary 3.1. *A quasi-conformally flat K-contact manifold M^n $(n \geq 5)$ is Sasakian.*

§4. K-contact manifold satisfying $div\tilde{C} = 0$

This section deals with a K-contact Riemannian manifold satisfying

$$div\tilde{C} = 0, \tag{4.1}$$

where *div* denotes the divergence of the quasi-conformal curvature tensor $\tilde{C}$.

Differentiating (1.1) covariantly along U, we obtain

$$\begin{aligned}(\nabla_U\tilde{C})(X,Y)Z &= a(\nabla_U R)(X,Y)Z \\ &\quad +b[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y \\ &\quad +g(Y,Z)(\nabla_U Q)X - g(X,Z)(\nabla_U Q)Y] \\ &\quad -\frac{[a+2(n-1)b]dr(U)}{n(n-1)}[g(Y,Z)(X) \\ &\quad -g(X,Z)(Y)].\end{aligned} \tag{4.2}$$

Contraction of (4.2) yields

$$\begin{aligned}(div\tilde{C})(X,Y)Z &= (a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ &\quad -\frac{a-(n-1)(n-2)b}{n(n-1)}[g(Y,Z)dr(X) \\ &\quad -g(X,Z)dr(Y)].\end{aligned} \tag{4.3}$$

From (4.1) and (4.3) it follows that

$$\begin{aligned}&(a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ &= \frac{a-(n-1)(n-2)b}{n(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)].\end{aligned} \tag{4.4}$$

From (2.9) we get

$$(\nabla_\xi S)(Y,Z) = -S(\nabla_Y\xi, Z) - S(\nabla_Z\xi, Y). \tag{4.5}$$

Putting $X = \xi$ in (4.4) and then using (4.5) and $dr(\xi) = 0$ we get

$$\begin{aligned}(a+b)[S(\nabla_Y\xi,Z) &+ S(\nabla_Z\xi,Y) + (\nabla_Y S)(\xi,Z)] \\ &= \frac{a-(n-1)(n-2)b}{n(n-1)}\eta(Z)dr(Y).\end{aligned} \tag{4.6}$$

From (2.7) we have

$$(\nabla_Y S)(\xi,Z) = (n-1)(\nabla_Y\eta)(Z) - S(\nabla_Y\xi, Z). \tag{4.7}$$

Again using the relation $(\nabla_Y \eta)(Z) = g(\nabla_Y \xi, Z)$ in (4.7) we obtain

$$(\nabla_Y S)(\xi, Z) = (n-1)g(\nabla_Y \xi, Z) - S(\nabla_Y \xi, Z). \tag{4.8}$$

Using (4.8) in (4.6) we get

$$\begin{aligned}(a+b)[(n-1)g(\nabla_Y \xi, Z) + S(\nabla_Z \xi, Y)] \\ = \frac{a-(n-1)(n-2)b}{n(n-1)}\eta(Z)dr(Y).\end{aligned} \tag{4.9}$$

In view of (2.4) we obtain from (4.9)

$$\begin{aligned}-(a+b)[(n-1)g(\phi Y, Z) \quad + \quad S(\phi Z, Y)] \\ = \frac{a-(n-1)(n-2)b}{n(n-1)}\eta(Z)dr(Y).\end{aligned} \tag{4.10}$$

Replacing Z by ϕZ in (4.10) and using (2.1) we get

$$(a+b)[S(Y, Z) - (n-1)g(Y, Z)] = 0, \tag{4.11}$$

which implies

$$S(Y, Z) = (n-1)g(Y, Z), \tag{4.12}$$

provided $a + b \neq 0$. Hence (4.12) follows that

$$QY = (n-1)Y. \tag{4.13}$$

Hence in view of (4.12) and (4.13) we get from (1.1)

$$\tilde{C}(X, Y)Z = a[R(X, Y)Z + g(X, Z)Y - g(Y, Z)X]. \tag{4.14}$$

From (4.12) we get the following:

Theorem 4.1. *A K-contact manifold with divergence free quasi-conformal curvature tensor is an Einstein manifold provided $a + b \neq 0$.*

Since a compact K-contact Einstein manifold is Sasakian ([4]), hence we obtain the following:

Corollary 4.1. *A compact K-contact manifold with divergence free quasi-conformal curvature tensor is Sasakian.*

If a K-contact manifold (M^n, g) $(n \geq 5)$ is quasi-conformally symmetric, then it satisfies $div\tilde{C} = 0$. Hence the relation (4.14) holds from which it follows that the manifold is locally symmetric. Again it is known ([20]) that a locally symmetric K-contact manifold is of constant curvature +1 and Sasakian. Hence we state the following:

Corollary 4.2. *A quasi-conformally symmetric K-contact manifold (M^n, g) $(n \geq 5)$ is of constant curvature +1 and Sasakian.*

§5. K-contact manifold satisfying $R(X,Y).\tilde{C} = 0$

To solve this problem we consider following two cases:
Case i) The manifold is Einstein.
Case ii) The manifold is η-Einstein.

Case i) In this case we have,

$$S(X,Y) = \lambda g(X,Y), \tag{5.1}$$

where λ is a constant. Putting $X = Y = \xi$ in (5.1) we get by virtue of (2.1), (2.2) and (2.7) that $\lambda = n - 1$. Hence (5.1) reduces to

$$S(X,Y) = (n-1)g(X,Y), \tag{5.2}$$

which yields

$$QX = (n-1)X \tag{5.3}$$

and

$$r = n(n-1). \tag{5.4}$$

Now from (1.1), using (5.2), (5.3) and (5.4), we get

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z + [2b(n-1) - \frac{ar}{n(n-1)} \\ &\quad - \frac{2br}{n}][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{5.5}$$

Therefore we get

$$R.\tilde{C} = aR.R. \tag{5.6}$$

From (5.6) it follows that on a K-contact Einstein manifold the quasi-conformally semi-symmetry and semi-symmetry are equivalent, since by assumption $a \neq 0$.

It is known ([21]) that a semi-symmetric K-contact manifold is of constant curvature +1 and Sasakian. Hence an Einstein quasi-conformally semi-symmetric K-contact manifold is of constant curvature +1 and Sasakian.

Case ii) In this case we have,

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{5.7}$$

where α and β are scalars. Putting $X = Y = \xi$ in (5.7) we get

$$(n - 1) = \alpha + \beta. \tag{5.8}$$

Let $\{e_i\}$ $(i = 1, 2, ..., n)$ be the orthonormal basis of the tangent space at each point of the manifold M. Then putting e_i in the place of X and Y of (5.7) and summing up over 1 to n we get

$$r = \alpha n + \beta. \tag{5.9}$$

Solving (5.8) and (5.9) we have

$$\alpha = \frac{r}{n-1} - 1, \quad \beta = n - \frac{r}{n-1}. \tag{5.10}$$

Again (5.7) yields

$$QX = \alpha X + \beta\eta(X)\xi. \tag{5.11}$$

Then using (5.7), (5.10) and (5.11) in (1.1) we get

$$\begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z \\ &+[2b(\frac{r}{n-1} - 1) - \frac{r}{n}(\frac{a}{n-1} + 2b)] \times \\ &\quad [g(Y, Z)X - g(X, Z)Y] \\ &+b(n - \frac{r}{n-1})[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] \\ &+b(n - \frac{r}{n-1})[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{5.12}$$

Then from (5.12) we have

$$\grave{\tilde{C}}(X,Y,Z,W) = a\grave{R}(X,Y,Z,W) + [2b(\frac{r}{n-1}-1) - \frac{r}{n}(\frac{a}{n-1}+2b)] \times [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b(n-\frac{r}{n-1})[g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)] + b(n-\frac{r}{n-1})[\eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W)], \tag{5.13}$$

where

$$\grave{\tilde{C}}(X,Y,Z,W) = g(\tilde{C}(X,Y)Z,W) \text{ and } \grave{R}(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Since r, g, $\eta(X)$, $\eta(Y)$ and $\eta(Z)$ all are scalars, then (5.13) yields

$$R(X,Y).\grave{\tilde{C}} = aR(X,Y).\grave{R}. \tag{5.14}$$

Thus from (5.14) we have quasi-conformally semi-symmetry and semi-symmetry are equivalent since by assumption $a \neq 0$.

It is well known ([21]) that a semi-symmetric K-contact manifold is of constant cvurvature +1 and Sasakian.

Therefore from the above discussions we can state the following theorem:

Theorem 5.1. *A quasi-conformally semi-symmetric K-contact Einstein (or η-Einstein) manifold is a manifold of constant curvature* $+1$ *and Sasakian.*

Since a manifold of constant curvature +1 is locally isometric with a unit sphere, hence we have:

Corollary 5.1. *A quasi-conformally semi-symmetric K-contact Einstein (or η-Einstein) manifold is locally isometric with a unit sphere.*

§6. ξ-quasi-conformally flat K-contact manifold

ξ-conformally flat K-contact manifolds have been studied by Zhen, Cabrerizo, L. M. Fernandez and M. Fernandez [23]. Here we study ξ-quasi-conformally flat K-contact manifold.

Definition 6.1. A K-contact manifold is said to be ξ-quasi-conformally flat ([23]) if $\tilde{C}(X,Y)\xi = 0$.

Let us assume that the manifold M^n is ξ- quasi-conformally flat. Then using $\tilde{C}(X,Y)\xi = 0$ in (1.1) we get

$$\begin{aligned} (6.1) \qquad aR(X,Y)\xi &+ b[(n-1)\eta(Y)X - (n-1)\eta(X)Y + \eta(Y)QX \\ &- \eta(X)QY] - \frac{r}{n}(\frac{a}{n-1} + 2b)[\eta(Y)X - \eta(X)Y] \\ &= 0. \end{aligned}$$

Putting $X = \xi$ in (6.1) and using (2.6) and $\eta(\xi) = 1$ we get

$$\begin{aligned} (6.2) \qquad a(-Y + \eta(Y)\xi) &+ b[(n-1)(\eta(Y)\xi - Y) + \eta(Y)Q\xi - QY] \\ &- \frac{r}{n}(\frac{a}{n-1} + 2b)[\eta(Y)\xi - Y] = 0. \end{aligned}$$

i.e,

$$(6.3) \qquad S(Y,W) = Ag(Y,W) + B\eta(Y)\eta(W),$$

where A and B are given by

$$(6.4) \qquad A = -\frac{a}{b} + \frac{r}{nb}(\frac{a}{n-1} + 2b) - (n-1)$$

and

$$(6.5) \qquad B = \frac{a}{b} - \frac{r}{nb}(\frac{a}{n-1} + 2b) + 2(n-1).$$

In view of (6.3) we state the following:

Theorem 6.1. *A ξ-quasi-conformally flat K-contact manifold is an η-Einstein manifold.*

Let us assume that there exist two functions L and M on M^n such that

$$(6.6) \qquad (\nabla_X Q)Y - (\nabla_Y Q)X = LX + MY,$$

for X, $Y \in T(M)$.

From (6.3) we have

$$QX = AX + B\eta(X)\xi, \tag{6.7}$$

where A and B are given by (6.4) and (6.5) respectively. Thus we have

$$\begin{aligned}(\nabla_X Q)Y - (\nabla_Y Q)X &= (XA)Y - (YA)X + (XB)\eta(Y)\xi \\ &\quad -(YB)\eta(X)\xi - B\eta(Y)\phi X \\ &\quad +B\eta(X)\phi Y - 2Bg(\phi X, Y)\xi.\end{aligned} \tag{6.8}$$

Replacing X by ϕX and Y by ϕY in (6.8), we get

$$\begin{aligned}(\nabla_{\phi X} Q)\phi Y - (\nabla_{\phi Y} Q)\phi X &= (\phi XA)\phi Y - (\phi YA)\phi X \\ &\quad -2Bg(\phi^2 X, \phi Y)\xi.\end{aligned} \tag{6.9}$$

From (6.6) and (6.9) we obtain

$$(L + (\phi YA))\phi X + (M - (\phi XA))\phi Y = -2Bg(\phi^2 X, \phi Y)\xi,$$

which shows that

$$-2Bg(\phi^2 X, \phi Y) = 0. \tag{6.10}$$

Replacing X by ϕY in (6.10) we have

$$2Bg(\phi Y, \phi Y) = 0. \tag{6.11}$$

Hence from (6.11), it follows that $B = 0$. Therefore from (6.7), we get $QX = AX$. Then from (2.7) we obtain $QX = (n-1)X$.
Therefore we have the following:

Corollary 6.1. *Let M^n be a ξ-quasi-conformally flat K-contact manifold. If there exist functions L and M on M^n such that*

$$(\nabla_X Q)Y - (\nabla_Y Q)X = LX + MY,$$

for X, $Y \in T(M)$, then

$$QX = (n-1)X. \tag{6.12}$$

From Corollary 6.1 we have the following applications:

Corollary 6.2. *A quasi-conformally flat K-contact manifold is of constant curvature* $+1$ *and Sasakian.*

Proof: Let us suppose that a K-contact manifold is quasi-conformally flat. Then from (4.4) it follows that

$$(\nabla_X Q)(Y) - (\nabla_Y Q)(X) = \frac{[a-(n-1)(n-2)b]}{(a+b)n(n-1)} [Y dr(X) - X dr(Y)]. \tag{6.13}$$

Hence by the above Corollary 6.1 we obtain the manifold is an Einstein manifold. Using (6.12) and (6.13) in (3.1) we obtain

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{6.14}$$

Therefore the quasi-conformally flat K-contact manifold is of constant curvature +1 and Sasakian. This proves the Corollary.

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References

[1] Amur, K. and Maralabhavi, Y. B., On quasi-conformally flat spaces, Tensor, N. S. 31(1977), 194-198.

[2] Blair, D. E., Contact manifold in Riemannian geometry, Lecture Notes on Mathematics, 509, Springer-Verlag, Berlin, 1976.

[3] Blair, D. E., Riemannian geometry of contact and symplectic manifolds, Progress in Math., Vol. 203, Berlin.

[4] Boyer, C. P. and Galicki, K., Einstein manifold and contact geometry, Proc. Amer. Math. Soc., 129(2001), 2419-2430.

[5] Chaki, M. C. and Ghosh, D., On a type of K-contact Riemannian manifold, J. Australian Math. Soc., 13(1972), 447-450.

[6] De, U. C. and Matsuyama, Y., On quasi-conformally flat manifolds, SUT. J. Math., Vol.42, No. 2(2006), 295-303.

[7] De, U. C. and Biswas, S., On K-contact η-Einstein manifolds, Bull. Math. Soc.Sc. Math. Roumanie. Tome. 48(96), No. 3, 2005, 295-301.

[8] De, U. C. and Gazi, A., Sasakian manifold with quasi confomal curvature tensor, Bull. Korean. Math. Soc. 45(2008), No.2, 313-319.

[9] Deszez, R., On pseudosymmetric spaces, Bull. Belg. Math. Soc. Seules A, 44(1992), 1-34.

[10] Eisenhart, L. P., Riemannian geometry, Princeton Univ. Press(1949).

[11] Hatakeyama, Y., Ogawa, Y. and Tanno, S., Some properties of manifolds with almost contact structures, Tohoku Math. J., 15(1963), 42-48.

[12] June, J. B. and Kim, U. K., On 3-dimensional almost contact metric manifold, Kyungpook Math. J., 34(1994), No. 2, 293-301.

[13] *Özgür*, C. and De, U. C., On the quasi-conformal curvature tensor of a Kenmotsu manifold, Mathematica Pannonica., 17/2(2006), 221-228.

[14] Olszak, Z., On contact metric manifolds, Tohoku. Math. J., 31(1979), 247-253.

[15] Sasaki. S., Lecture note on almost contact manifold, Part-1, Tohoku University, 1965.

[16] Sasaki, S., Almost contact manifolds, Lecture notes, Tohoku Univ. Vol. 1, 1965; Vol. 2, 1967; Vol. 3, 1968.

[17] Szab*ó*, Z. I., Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$, II. Global version, Geom. Dedicata, 19(1985), 65-108.

[18] Tanno, S., A conformal transformation of a certain contact Riemannian manifolds, Tohoku Math. J., 18(1966), 270-273.

[19] Tanno, S., A remark on transformations of a K-contact manifolds, Tohoku Math. J., 16(1964), 173-175.

[20] Tanno, S., Locally symmetric K-contact manifolds, Proc. Japan Acad., 43(1967), 581-583.

[21] Tanno, S., Isometric immersions of Sasakian manifolds in spheres, Kodai Math. Sem. Rep. 21(1969), 448-458.

[22] Yano, K. and Sawaki, S., Riemannian manifolds admitting a conformal transformation group, J. Diff. Geom. 2(1968), 161-184.

[23] Zhen, G., Cabrerizo, J. L., Fernandez, L. M. and Fernandez, M., On ξ-conformally flat K-contact metric manifolds, Indian J. Pure Appl. Math, 28(1997), 725-734.

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