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HOMOCLINIC ORBITS FOR 3–DIMENSIONAL SYSTEMS

Ilham ELI

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Abstract. Suppose a dynamical system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}; \mu), \mathbf{x} \in \mathbf{R}^s, \mu \in \mathbf{R}^m$, has a hyperbolic saddle at $\mathbf{x} = \mathbf{0}$ with a homoclinic loop, for $\mu = \mu^0$. When μ varies from μ^0, the loop will be destroyed in general. For $s = 2$, Perko proved that, if μ varies on an $(m-1)$ dimensional hypersurface, then the system remains to admit homoclinic orbit. We consider here the same problem for $s = 3$. The result is: if μ varies on an $(m-2)$ hypersurface, then the system remains to admit homoclinic orbit.

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§1. Introduction

Consider a 3-dimensional dynamical system

$$
(1) \qquad \begin{cases} \dfrac{dx_1}{dt} = F_1(x_1, x_2, x_3; \mu), \\ \dfrac{dx_2}{dt} = F_2(x_1, x_2, x_3; \mu), \\ \dfrac{dx_3}{dt} = F_3(x_1, x_2, x_3; \mu), \end{cases}
$$

$$
F_j(0, 0, 0; \mu) = \mathbf{0}, j = 1, 2, 3,
$$

in which $\mu \in \mathbf{R}^m$, $m \geq 3$. F_j are supposed to be of C^2-class with respect to both $\mathbf{x} = {}^t(x_1, x_2, x_3)$ and $\mu = {}^t(\mu_1, ..., \mu_m)$.

Suppose that, for $\mu = \mu^0$, (1) has a hyperbolic saddle at (0,0,0) with a homoclinic loop $\Gamma : \mathbf{x} = \gamma(t)$. When μ varies from μ^0, the loop will be destroyed in general. For standard exposition of these facts, see [2]. For 2-dimensional systems of C^∞ or C^ω class, Perko [4] proved that, if μ varies on an $(m-1)$ dimensional hypersurface, then the system remains to admit homoclinic orbit. We consider here 3-dimensional case.

Now we suppose that, for $\mu = \mu^0, \mathbf{F} = {}^t(F_1, F_2, F_3)$ is expanded at (0,0,0) as follows:

$$\text{(2)} \qquad \mathbf{F}(\mathbf{x}, \mu^0) = \Lambda\mathbf{x} + \Phi^0(\mathbf{x}), \quad \Phi^0(\mathbf{x}) = O(|\mathbf{x}|^2),$$

in which

$$\Lambda = \begin{pmatrix} \lambda_1 & \epsilon & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\lambda_j \ \text{ are real and } \ \lambda_1 \le \lambda_2 < 0 < \lambda_3, \qquad \epsilon = 0 \ \text{ if } \ \lambda_1 \ne \lambda_2.$$

Now we put

$$\text{(3)} \qquad I_j = \int_{-\infty}^{\infty} \exp[-\int_0^t (\nabla\mathbf{F} - {}^tDF)(\gamma(s))ds] \left\{ \mathbf{F} \times \frac{\partial \mathbf{F}}{\partial \mu_j} \right\} (\gamma(t))dt$$

$$= {}^t(I_{j1}, I_{j2}, I_{j3}),$$

for $j = 1, ..., m$, in which we assume $\mu = \mu^0$.

We will prove the following theorem:

Theorem. *Suppose that, for $\mu = \mu^0$, (1) has a hyperbolic saddle at $(0,0,0)$ with a homoclinic loop Γ, and $\mathbf{F} = {}^t(F_1, F_2, F_3)$ is expanded at $(0,0,0)$ as shown in (2), with the following condition (Λ):*

$$\lambda_3 > \lambda_2 - \lambda_1.$$

Further, suppose that

$$\text{(4)} \qquad \begin{vmatrix} I_{11} & I_{21} \\ I_{12} & I_{22} \end{vmatrix} \ne 0$$

Then there are $\delta > 0$ and two functions h_1, h_2 of $(\mu_3, ..., \mu_m)$ defined for $|\mu_3 - {\mu_3}^0| + ... + |\mu_m - {\mu_m}^0| < \delta$ such that, when $\mu = (\mu_1, \mu_2, \mu_3, ..., \mu_m)$ varies satisfying $\mu_j = h_j(\mu_3, ..., \mu_m)$, $j = 1, 2$, then (1) remains to admit homoclinic loop at $(0,0,0)$.

This is a 3-dimensional generalization of a theorem of Perko [4]. Generalizations to higher dimensional case will be further topics.

§2. Proof of the Theorem

For simplicity, we write $\mathbf{F}(\mathbf{x},\mu^0)$ as $\mathbf{F}_0(\mathbf{x})$, $\partial\mathbf{F}(\mathbf{x},\mu^0)/\partial\mu_j$ as $\partial\mathbf{F}_0(\mathbf{x})/\partial\mu_j$.

By taking suitable coordinates, we can assume that the local stable manifold S_0 and local unstable manifold U_0 are

$$S_0 : x_3 = 0 \quad \text{and} \quad U_0 : x_1 = x_2 = 0,$$

respectively, and that the condition (4) holds still. Then we have, in (2),

$$\Phi_1^0(0,0,x_3) = \Phi_2^0(0,0,x_3) = \Phi_3^0(x_1,x_2,0) = 0.$$

For general μ, we write stable manifold and unstable manifold as M_μ^S and M_μ^U, respectively. By the stable manifold theorem [3], these manifolds are C^2 continuous with respect to μ.

We assume that $\gamma(0) = \mathbf{x}_0 \in S_0$. Let Π be a plane crossing with Γ at $\mathbf{x}_0$. Take a point $\mathbf{b} \in U_0 \cap \Gamma$. Let $\mathbf{a}_\mu \in M_\mu^S$ and $\mathbf{b}_\mu \in M_\mu^U$ be points such that they depends on μ as C^2-class functions, and $\mathbf{a}_{\mu_0} = \mathbf{x}_0$, $\mathbf{b}_{\mu_0} = \mathbf{b}$.

Now let $\phi(t,\xi,\mu)$, $\xi \in \mathbf{R}^3$, denote the solution of (1) which satisfies the initial condition $\phi(0,\xi,\mu) = \xi$. Let τ^U be the time such that $\phi(\tau^U,\mathbf{b},\mu^0) = \mathbf{x}_0$, and τ_μ^S, τ_μ^U be the times such that $\phi(\tau_\mu^S,\mathbf{a}_\mu,\mu) \in \Pi$, $\phi(\tau_\mu^U,\mathbf{b}_\mu,\mu) \in \Pi$. The following lemma is proved easily, as in Perko [4].

Lemma 1. *Under the hypothese of Theorem, we can take τ_μ^S and τ_μ^U so that $\tau_\mu^S \to 0$ and $\tau_\mu^U \to \tau^U$ as $\mu \to \mu^0$.*

Write $\phi(t+\tau_\mu^S,\mathbf{a}_\mu,\mu)$ as $\mathbf{x}^S(t,\mu)$ and $\phi(t+\tau_\mu^U,\mathbf{b}_\mu,\mu)$ as $\mathbf{x}^U(t,\mu)$. Put

$$\mathbf{x}^S(0,\mu) = \mathbf{x}_0^S(\mu) \;\; \textit{and} \;\; \mathbf{x}^U(0,\mu) = \mathbf{x}_0^U(\mu),$$

$$\mathbf{d}(\mu) = \mathbf{x}_0^U(\mu) - \mathbf{x}_0^S(\mu). \tag{5}$$

If $\mathbf{d}(\mu) = \mathbf{0}$, then $\mathbf{x}^S(t,\mu) = \mathbf{x}^U(t,\mu)$ represents a homoclinic loop. Write $\mathbf{x}^S(t,\mu)$ or $\mathbf{x}^U(t,\mu)$ simply as $\mathbf{x}(t,\mu)$, and

$$\xi_k(t,\mu) = \frac{\partial\mathbf{x}(t,\mu)}{\partial\mu_k},$$

$$\rho_k(t,\mu) = \xi_k(t,\mu) \times \mathbf{F}(\mathbf{x}(t,\mu),\mu),$$

then

$$\frac{d\xi_k}{dt} = D\mathbf{F}(\mathbf{x}(t,\mu),\mu)\xi_k + \frac{\partial\mathbf{F}(\mathbf{x}(t,\mu),\mu)}{\partial\mu_{k_j}}$$

and

$$\frac{d\rho_k}{dt} = (\nabla \mathbf{F} - {}^t D\mathbf{F})(\mathbf{x}(t,\mu),\mu)\rho_k + \frac{\partial \mathbf{F}}{\partial \mu_k}\times\mathbf{F}(\mathbf{x}(t,\mu),\mu). \tag{6}$$

To see (6), writing ξ_k and ρ_k simply as ξ and ρ, respectively, and differentiating ρ by t,

$$\frac{d\rho}{dt} = \frac{d\xi}{dt}\times\mathbf{F} + \xi\times\frac{d\mathbf{F}}{dt}$$
$$= ((D\mathbf{F})\xi + \frac{\partial \mathbf{F}}{\partial \mu_k})\times\mathbf{F} + \xi\times((D\mathbf{F})\mathbf{F}).$$

Let $D\mathbf{F} = (a_{ij})$. Then the first component of $\{((D\mathbf{F})\xi)\times\mathbf{F} + \xi\times((D\mathbf{F})\mathbf{F})\}$ is, by an easy calculation,

$$\begin{vmatrix} \Sigma a_{2j}\xi_j & F_2 \\ \Sigma a_{3j}\xi_j & F_3 \end{vmatrix} + \begin{vmatrix} \xi_2 & \Sigma a_{2j}F_j \\ \xi_3 & \Sigma a_{3j}F_j \end{vmatrix}$$
$$= (a_{22} + a_{33})(\xi\times\mathbf{F})_1 - a_{21}(\xi\times\mathbf{F})_2 - a_{31}(\xi\times\mathbf{F})_3.$$

The second and third components are obtained similarly, and we have

$$((D\mathbf{F})\xi)\times\mathbf{F} + \xi\times((D\mathbf{F})\mathbf{F}) = (\nabla\mathbf{F} - {}^t D\mathbf{F})(\xi\times\mathbf{F}),$$

which shows (6). Write

$$\nabla\mathbf{F} - {}^t D\mathbf{F} = \mathrm{H}, \qquad \mathrm{H}(\mu = \mu_0) = \mathrm{H}_0.$$

Then (6) can be written as

$$\frac{d\rho_k}{dt} = \mathrm{H}\rho_k + \frac{\partial \mathbf{F}}{\partial \mu_k}\times\mathbf{F}. \tag{6'}$$

For $\rho_k = \rho_k^S$ with $\mu = \mu^0$ we have, solving the first order linear differential equation (6'),

$$\left[\exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds]\rho_k^S(t,\mu^0)\right]_{t_0}^{t_1}$$
$$= \int_{t_0}^{t_1} \exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds]\left\{\frac{\partial \mathbf{F}}{\partial \mu_k}\times\mathbf{F}\right\}(\gamma(t))dt.$$

Letting $t_0 = 0$, $t_1 \to \infty$, we get

$$\lim_{t\to\infty}\left\{\exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds]\rho_k^S(t,\mu^0)\right\} - \rho_k^S(0,\mu^0)$$

$$= \int_0^{\infty} \exp[-\int_0^{\infty} \mathrm{H}_0(\gamma(s))ds] \left\{ \frac{\partial \mathbf{F}}{\partial \mu_k} \times \mathbf{F} \right\} (\gamma(t))dt.$$

Similarly we have

$$\lim_{t\to-\infty} \left\{ \exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds] \rho_k^U(t,\mu^0) \right\} - \rho_k^U(0,\mu^0)$$

$$= \int_0^{-\infty} \exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds] \left\{ \frac{\partial \mathbf{F}}{\partial \mu_k} \times \mathbf{F} \right\} (\gamma(t))dt.$$

By the condition (Λ), we obtain that

(7) the first and second components of

$$\exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds] \rho_k^S(t,\mu_0) \quad \text{tend to 0 as } t \to \infty,$$

and that

(7') $$\lim_{t\to-\infty} \exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds] \rho_k^U(t,\rho^0) \quad = \mathbf{0},$$

respectively, which will be shown later. Then we get

(8) $$\rho_k^U(0,\mu^0) - \rho_k^S(0,\mu^0)$$

$$= [\frac{\partial \mathbf{x}^U(0,\mu^0)}{\partial \mu_k} - \frac{\partial \mathbf{x}^S(0,\mu^0)}{\partial \mu_k}] \times \mathbf{F}_0(\mathbf{x}_0)$$

$$= \frac{\partial \mathbf{d}(\mu_0)}{\partial \mu_k} \times \mathbf{F}_0(\mathbf{x}_0)$$

$$= \int_{-\infty}^{\infty} \exp[-\int_0^t \mathrm{H}_0(\gamma(s))ds] \left\{ \frac{\partial \mathbf{F}}{\partial \mu_k} \times \mathbf{F} \right\} (\gamma(t))dt + \begin{pmatrix} 0 \\ 0 \\ c_k \end{pmatrix}$$

$$= I_k + \begin{pmatrix} 0 \\ 0 \\ c_k \end{pmatrix}.$$

Since there holds, for vectors $\mathbf{A}, \mathbf{B}, \mathbf{F}$,

$$(\mathbf{A}\times\mathbf{F})\times(\mathbf{B}\times\mathbf{F}) = ((\mathbf{A}\times\mathbf{B})\cdot\mathbf{F})\mathbf{F},$$

the third components of

$$\mathbf{I}_1 \times \mathbf{I}_2 \quad \text{and} \quad (\left\{ \frac{\partial \mathbf{d}(\mu^0)}{\partial \mu_1} \times \frac{\partial \mathbf{d}(\mu^0)}{\partial \mu_2} \right\} \cdot \mathbf{F}_0(\mathbf{x}_0))\mathbf{F}_0(\mathbf{x}_0)$$

coincide. If (4) holds, then $[\partial\mathbf{d}(\mu^0)/\partial\mu_1]\times[\partial\mathbf{d}(\mu^0)/\partial\mu_2] \neq 0$. Therefore, we may take, for example, that

$$\begin{vmatrix} \partial d_1(\mu^0)/\partial\mu_1 & \partial d_1(\mu^0)/\partial\mu_2 \\ \partial d_2(\mu^0)/\partial\mu_1 & \partial d_2(\mu^0)/\partial\mu_2 \end{vmatrix} \neq 0.$$

Then, by the implicite function theorem, there are two functions h_1, h_2 of $(\mu_3, ..., \mu_m)$, defined for $|\mu_3 - \mu_3^0| + ... + |\mu_m - \mu_m^0| < \delta$ with sufficiently small $\delta > 0$, such that, when $\mu = (\mu_1, \mu_2, \mu_3, ..., \mu_m)$ varies satisfying $\mu_j = h_j(\mu_3, ..., \mu_m)$, $j = 1, 2$, then $d_1(\mu) = d_2(\mu) = 0$. Since $\mathbf{d}(\mu)$ moves on the plane Π, we obtain that $\mathbf{d}(\mu) = \mathbf{0}$, which proves the existence of homoclinic loop.

It remains to prove (7) and (7').

On the local stable manifold for $\mu = \mu^0$, we have $x_3 = 0$ and

$$\Phi_3^0(x_1, x_2, 0) = 0, \quad \frac{\partial\Phi_3^0(x_1, x_2, 0)}{\partial x_1} = \frac{\partial\Phi_3^0(x_1, x_2, 0)}{\partial x_2} = 0.$$

Then (1,3) and (2,3) elements h_{13}^0 and h_{23}^0 of H_0 are zero. As $x_1^S(t, \mu^0) = \exp[\lambda_1 t](a + o(1))$, $x_2^S(t, \mu^0) = \exp[\lambda_2 t](b + o(1))$, we get, when $t \to \infty$,

$$\mathrm{H}_0 = \begin{pmatrix} \lambda_2+\lambda_3 & & \\ & \lambda_3+\lambda_1 & \\ & & \lambda_1+\lambda_2 \end{pmatrix} + \begin{pmatrix} O(\exp[\lambda_2 t]) & 0 \\ O(\exp[\lambda_2 t]) & 0 \\ O(\exp[\lambda_2 t]) & \end{pmatrix},$$

$$-\int_0^t \mathrm{H}_0 ds = \begin{pmatrix} -(\lambda_2+\lambda_3)t & & \\ & -(\lambda_3+\lambda_1)t & \\ & & -(\lambda_1+\lambda_2)t \end{pmatrix} + \begin{pmatrix} O(1) & 0 \\ O(1) & 0 \\ O(1) & \end{pmatrix},$$

hence

$$\exp[-\int_0^t \mathrm{H}_0 ds] = \begin{pmatrix} O(\exp[-(\lambda_3+\lambda_1)t]) & 0 \\ O(\exp[-(\lambda_3+\lambda_1)t]) & 0 \\ O(\exp[-(\lambda_1+\lambda_2)t]) & \end{pmatrix}.$$

As $\rho_k^S(t) = O(\exp[\lambda_2 t])$, we have

$$\exp[-\int_0^t \mathrm{H}_0 ds]\rho_k^S(t) = \begin{pmatrix} O(\exp[(-\lambda_1+\lambda_2-\lambda_3)t]) \\ O(\exp[(-\lambda_1+\lambda_2-\lambda_3)t]) \\ O(\exp[-\lambda_1 t]) \end{pmatrix}.$$

Since $-\lambda_1 + \lambda_2 - \lambda_3 < 0$ by (Λ), the first and second elements of the right side tend to 0 as $t \to \infty$, which proves (7).

Next, as $t \to -\infty$, we have

$$\exp[-\int_0^t \mathrm{H}_0 ds] = O(\exp[-(\lambda_2 + \lambda_3)t]) + O(1),$$

$$\rho_k^U(t) = O(\exp[\lambda_3 t]).$$

$$\exp[-\int_0^t \mathrm{H}_0 ds]\rho_k^U(t) = O(\exp[-\lambda_2 t]) + O(\exp[\lambda_3 t]).$$

Since $\lambda_2 < 0 < \lambda_3$, the right side tends to $\mathbf{0}$ as $t \to \infty$, which proves (7').

REMARK. When $\mathbf{F}$ is expanded at (0,0,0) as follows:

$$\mathbf{F}(\mathbf{x}, \mu^0) = \Lambda\mathbf{x} + O(|\mathbf{x}|^2),$$

$$\Lambda = \begin{pmatrix} \lambda_1 & -\nu & 0 \\ \nu & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$$\lambda_j, \quad \nu \quad \text{are real, and} \quad \lambda_1 < 0 < \lambda_3,$$

then we can obtain also a similar result as above.

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Ilham Eli
Department of Mathematics, Graduate School of Science and Technology, Chiba University
1-33 Yayoi-cho, Inage-ku, Chiba 263, Japan