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GLOBAL STABILITY OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. An initial value problem for an impulsive system of differential-difference equations is considered. By means of piecewise continuous auxiliary functions which are modifications of classical Lyapunov's functions, some sufficient conditions for global stability of the zero solution of such problems are presented. The discontinuity of these auxiliary functions corresponds to the fact that solutions of the systems under consideration are piecewise continuous functions.

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§1. Introduction

Impulsive differential-difference equations are obtained by the natural combination of the impulsive ordinary differential equations without delay and the differential-difference equations without impulses. They are adequate mathematical models of numerous processes and phenomena in science and technology that are characterized by a change of their state by jumps and by a dependence of the process at each moment of time on its pre-history. The impulsive systems of differential and differential-difference equations are rich in mathematical problems as compared to the corresponding theory of the systems without impulses. That is why in the last years these systems have been subjected to intensive developments by many authors (see [1]–[8], [10] and references therein).

One of the most important aspects of the qualitative theory of such equations is the stability theory. A quite general method of investigation of the stability of the solutions of impulsive differential-difference equations is the Lyapunov direct method [1], [4]–[13]. The application of this method to the investigation of stability of the solutions of equations of the type considered requires the use of a class of piecewise continuous auxiliary functions which are modification to the classical Lyapunov's functions. Moreover, the technique in the application essentially depends on the choice of minimal subsets of a suitable space of piecewise continuous functions, by the elements of which the derivatives of Lyapunov's functions are estimated [9], [11]–[13].

In the present paper some sufficient conditions for the global stability are presented for the solutions of nonlinear systems of impulsive differential-difference equations when the impulsive action takes place at prescribed and fixed moments.

§2. Preliminary notes and definitions

Let $\boldsymbol{R}^+ = [0,\infty)$, $\boldsymbol{R}^n$ be the n-dimensional Euclidean space with elements $x = \operatorname{col}(x_1,\dots,x_n)$ and the norm $|x| = \left(\sum_{k=1}^n x_k^2\right)^{1/2}$, $h > 0$, $t_0 \in \boldsymbol{R}$, $\varphi_0 \in C[[t_0-h,t_0],\boldsymbol{R}^n]$.

Consider the initial value problem

$$\dot{x}(t) = f(t,x(t),x(t-h)), \quad t \neq \tau_k, \quad t > t_0, \tag{1}$$

$$x(t) = \varphi_0(t), \quad t \in [t_0-h,t_0], \tag{2}$$

$$\Delta x(\tau_k) = x(\tau_k+0) - x(\tau_k-0) = I_k(x(\tau_k)),\ \tau_k > t_0,\ k = 1,2,\dots \tag{3}$$

where $f\colon (t_0,\infty)\times\boldsymbol{R}^n\times\boldsymbol{R}^n \to \boldsymbol{R}^n$, $I_k\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$, $k = 1,2,\dots$,

$$t_0 \equiv \tau_0 < \tau_1 < \dots \tau_k < \tau_{k+1} < \dots$$

Introduce the following notations:

$$G_k = \{(t,x) \in [t_0,\infty)\times\boldsymbol{R}^n\colon\ \tau_{k-1} < t < \tau_k\}, \quad k = 1,2,\dots,$$

$$G = \cup_{k=1}^\infty G_k, \qquad C_0 = C\left[[t_0-h,t_0],\boldsymbol{R}^n\right],$$

$\|\varphi\| = \max_{t\in[t_0-h,t_0]}|\varphi(t)|$ is the norm of the function $\varphi \in C_0$, $\mathcal{K} = \{a \in C[\boldsymbol{R}^+,\boldsymbol{R}^+]\colon\ a(u)$ is strictly increasing with respect to u and $a(0) = 0\}$, $B_\varepsilon(t_0,C_0) = \{\varphi \in C_0\colon\ \|\varphi\| < \varepsilon\}$, $\varepsilon = \text{const} > 0$, $x(t) = x(t;t_0,\varphi_0)$ is a solution, if any, of the problem (1), (2), (3), $J^+(t_0,\varphi_0)$ is the maximal interval of the type $[t_0,\beta)$ in which the solution $x(t;t_0,\varphi_0)$ is defined, $\tau_l^h = \tau_l + h$, $l = 0,1,2,\dots$.

We shall give a description of the solution $x(t) = x(t; t_0, \varphi_0)$ of the problem (1), (2), (3).

1. For $t_0 - h \le t \le t_0$ the solution $x(t)$ coincides with the initial function $\varphi_0 \in C_0$.

2. Let

$$\left\{t_i\right\}_{i=1}^{\infty} = \left\{\tau_k\right\}_{k=1}^{\infty} \cup \left\{\tau_l^h\right\}_{l=0}^{\infty}$$

and $t_1 \le t_2 \le \ldots \le t_s \le t_{s+1} \le \ldots$.

It is possible that

$$\left\{\tau_k\right\}_{k=1}^{\infty} \cap \left\{\tau_l^h\right\}_{l=0}^{\infty} \ne \emptyset$$

in general, i.e., $\tau_k = \tau_l^h$ for some positive integers k and l.

2.1. For $t_0 < t \le t_1$ the solution $x(t)$ coincides with the solution of the problem (1), (2) without impulses (3).

2.2. For $t_i < t \le t_{i+1}$, $i = 1, 2, \ldots$, one of the following three cases may occur:

a) If $t_i \in \left\{\tau_k\right\}_{k=1}^{\infty} \setminus \left\{\tau_l^h\right\}_{l=0}^{\infty}$ and $t_i = \tau_k$ for some positive integer k, then the solution of the problem (1), (2), (3) coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h)), \tag{4}$$

$$y(\tau_k) = x(\tau_k) + I_k(x(\tau_k)). \tag{5}$$

b) If $t_i \in \left\{\tau_l^h\right\}_{l=0}^{\infty} \setminus \left\{\tau_k\right\}_{k=1}^{\infty}$, then $x(t)$ coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h+0)), \tag{6}$$

$$y(t_i) = x(t_i). \tag{7}$$

c) If $t_i \in \left\{\tau_k\right\}_{k=1}^{\infty} \cap \left\{\tau_l^h\right\}_{l=0}^{\infty}$ and $t_i = \tau_k$ for some positive integer k, then the solution $x(t)$ of the problem (1), (2), (3) coincides with the solution of the problem (5), (6).

3. The function $x(t)$ is piecewise continuous on $J^+(t_0, \varphi_0)$, continuous from the left at the points $\tau_1, \tau_2, \ldots \in J^+(t_0, \varphi_0)$ and

$$x(\tau_k + 0) = x(\tau_k) + I_k(x(\tau_k)), \qquad \tau_k \in J^+(t_0, \varphi_0).$$

We shall use the following definitions of stability and global stability of the zero solution of the problem (1), (2), (3).

Definition 1([7]). The zero solution $x(t) \equiv 0$ of the problem (1), (2), (3) is said to be

a) *stable* if

$$(\forall \varepsilon > 0)\ (\forall t_0 \in \boldsymbol{R})\ (\exists \delta = \delta(t_0, \varepsilon) > 0);$$

$$(\forall \varphi_0 \in B_\delta(t_0, C_0))\ (\forall t \in J^+(t_0, \varphi_0)) \Longrightarrow$$

$$|x(t; t_0, \varphi_0)| < \varepsilon.$$

b) *uniformly stable* if

$$(\forall \varepsilon > 0)\ (\exists \delta = \delta(\varepsilon) > 0);\ \ (\forall t_0 \in \boldsymbol{R})$$

$$(\forall \varphi_0 \in B_\delta(t_0, C_0))\ (\forall t \in J^+(t_0, \varphi_0)) \Longrightarrow$$

$$|x(t; t_0, \varphi_0)| < \varepsilon.$$

c) *globally equiattractive* if

$$(\forall t_0 \in \boldsymbol{R})\ (\forall \alpha > 0)\ (\forall \varepsilon > 0)$$

$$(\exists \sigma = \sigma(t_0, \alpha, \varepsilon) > 0);\ \ (\forall \varphi_0 \in B_\alpha(t_0, C_0))$$

$$(\forall t \geq t_0 + \sigma,\ t \in J^+(t_0, \varphi_0)) \Longrightarrow$$

$$|x(t; t_0, \varphi_0)| < \varepsilon.$$

d) *uniformly globally attractive* if the number σ in c) does not depend on $t_0 \in \boldsymbol{R}$.

e) *globally equiasymptotically stable* if it is stable and globally equiattractive.

f) *uniformly globally asymptotically stable* if it is uniformly stable, uniformly globally attractive and

$$(\forall \alpha > 0)\ (\exists \beta = \beta(\alpha) > 0);\ \ (\forall t_0 \in \boldsymbol{R})$$

$$(\forall \varphi_0 \in B_\alpha(t_0, C_0))\ (\forall t \in J^+(t_0, \varphi_0)) \Longrightarrow$$

$$|x(t; t_0, \varphi_0)| < \beta.$$

g) *exponentially globally asymptotically stable* if there exists a constant $c > 0$ such that

$$(\forall \alpha > 0)\ (\exists k = k(\alpha) > 0);\ \ (\forall t_0 \in \boldsymbol{R})$$

$$(\forall \varphi_0 \in B_\alpha(t_0, C_0))\ (\forall t \in J^+(t_0, \varphi_0)) \Longrightarrow$$

$$|x(t; t_0, \varphi_0)| \leq k(\alpha) \|\varphi_0\| \exp[-c(t - t_0)].$$

In the following considerations we shall use a class of piecewise continuous functions which are modifications to Lyapunov's functions [1], [4]–[7], [10].

Definition 2. We say that the function $V\colon [t_0,\infty)\times \boldsymbol{R}^n \to \boldsymbol{R}^+$ belongs to the class $\mathcal{V}_0$ if

1. The function V is continuous in G and $V(t,0)=0$ for $t\in[t_0,\infty)$.
2. The function V is Lipschitz continuous with respect to its second argument $x\in\boldsymbol{R}^n$ in each of the sets G_k, $k=1,2,\ldots$.
3. For each $k=1,2,\ldots$ and $x_0\in\boldsymbol{R}^n$ there exist the finite limits

$$V(\tau_k-0,x_0)=\lim_{\substack{(t,x)\in G_k\\(t,x)\to(\tau_k,x_0)}} V(t,x),\quad V(\tau_k+0,x_0)=\lim_{\substack{(t,x)\in G_{k+1}\\(t,x)\to(\tau_k,x_0)}} V(t,x).$$

4. The following equality holds

$$V(\tau_k-0,x_0)=V(\tau_k,x_0),\qquad k=1,2,\ldots,\quad x_0\in\boldsymbol{R}^n.$$

Furthermore we shall also use the classes of functions ([1], [9]–[13])

$PC[[t_0,\infty),\boldsymbol{R}^n] = \Big\{x\colon[t_0,\infty)\to\boldsymbol{R}^n$: $x(t)$ is piecewise continuous with points of discontinuity of the first kind (i.e., left and right limits exist there, and they are finite) τ_1, τ_2, $\ldots$ in the interval $[t_0,\infty)$ at which it is continuous from the left$\Big\}$, and

$$\Omega_1=\Big\{x\in PC[[t_0,\infty),\boldsymbol{R}^n]:\; V(s,x(s))\le V(t,x(t)),\ t-h\le s\le t,\ t\ge t_0,\ V\in\mathcal{V}_0\Big\}.$$

Let $V\in\mathcal{V}_0$ and $x\in PC[[t_0,\infty),\boldsymbol{R}^n]$. Let $t\ne\tau_k$, $k=1,2,\ldots$. Introduce the function

$$D_-V(t,x(t))=\liminf_{\sigma\to-0}\sigma^{-1}\Big[V(t+\sigma,x(t)+\sigma f(t,x(t),x(t-h)))-V(t,x(t))\Big].$$

Introduce the following conditions:

H1. $f\in C[(t_0,\infty)\times\boldsymbol{R}^n\times\boldsymbol{R}^n,\boldsymbol{R}^n]$.

H2. $f(t,0,0)=0$ for $t\in(t_0,\infty)$.

H3. The function f is Lipschitz continuous with respect to its second and third arguments in $(t_0,\infty)\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ uniformly on $t\in(t_0,\infty)$.

H4. $I_k\in C[\boldsymbol{R}^n,\boldsymbol{R}^n]$, $k=1,2,\ldots$.

H5. $I_k(0)=0$, $k=1,2,\ldots$.

H6. $t_0\equiv\tau_0<\tau_1<\ldots<\tau_k<\tau_{k+1}<\ldots$.

H7. $\lim\limits_{k\to\infty}\tau_k=\infty$.

We shall use the following lemmas:

Lemma 1. *Let the conditions* H1–H7 *hold. Then* $J^+(t_0, \varphi_0) = [t_0, \infty)$.

Proof. Since by the conditions H1–H5 the solution $x(t) = x(t; t_0, \varphi_0)$ of the problem (1), (2), (3), $\varphi_0 \in C_0$ uniquely exists in each one of the intervals $(\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \ldots$, then from the conditions H6, H7 we conclude that it can be continued for each $t \geq t_0$. □

Lemma 2([13]). *Let the following conditions hold:*

1. $v \in C[[t-0, \infty) \times \boldsymbol{R}^n, \boldsymbol{R}^+]$ *and* v *is locally Lipschitz continuous with respect to its second argument* z *in* $[t_0, \infty) \times \boldsymbol{R}^n$.

2. $g^* \in C[(t_0, \infty) \times \boldsymbol{R}^+, \boldsymbol{R}^+]$ *and the maximal solution* $u(t; t_0, u_0)$ *of the scalar differential equation*

$$\dot{u}(t) = g^*(t, u(t)), \quad u(t_0) = u_0 \geq 0$$

is defined in the interval $[t_0, \omega)$.

3. *The solution* $z(t) = z(t; t_0, \varphi_0)$ *of the problem* (1), (2) *for which* $v(t_0, \varphi_0(t_0)) \leq u_0$ *is defined in the interval* $[t_0, \omega)$.

4. *The inequality*

$$D_- v(t, z(t)) \leq g^*(t, v(t, z(t)))$$

is valid for each $t \geq t_0$ *and any function* $z \in C[[t_0, \infty), \boldsymbol{R}^n]$ *for which*

$$v(s, z(s)) \leq v(t, z(t)), \qquad s \in [t_0, t],$$

where

$$D_- v(t, z(t)) = \liminf_{\sigma \to -0} \sigma^{-1} \Big[v(t + \sigma, z(t) + \sigma f(t, z(t), z(t-h))) - v(t, z(t)) \Big].$$

Then

$$v(t, z(t; t_0, \varphi_0)) \leq u(t; t_0, u_0), \qquad t \in [t_0, \omega).$$

Lemma 3. *Let the following conditions hold:*

1. *Conditions* H1–H7 *are met.*
2. $g \in PC[(t_0, \infty) \times \boldsymbol{R}^+, \boldsymbol{R}^+]$.
3. $B_k \in C[\boldsymbol{R}^+, \boldsymbol{R}^+]$, $k = 1, 2, \ldots$.
4. *The maximal solution* $r(t; t_0, u_0)$ *of the impulsive problem*

$$\begin{cases} \dot{u}(t) = g(t, u(t)), \quad t > t_0, \quad t \neq \tau_k, \quad k = 1, 2, \ldots, \\ u(t_0 + 0) = u_0 \geq 0, \\ \Delta u(\tau_k) = B_k(u(\tau_k)), \quad \tau_k > t_0, \quad k = 1, 2, \ldots \end{cases} \tag{8}$$

is defined in the interval $[t_0, \infty)$.

5. *The functions* $\psi_k \colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$, $\psi_k(u) = u + B_k(u)$ *are nondecreasing with respect to* u.

6. *The function* $V \in \mathcal{V}_0$ *is such that the inequalities*

$$\begin{cases} D_-V(t, x(t)) \le g(t, V(t, x(t))), \quad t \ne \tau_k, \quad k = 1, 2, \dots, \\ V(t+0, x(t) + I_k(x(t))) \le \psi_k(V(t, x(t))), \quad t = \tau_k, \quad k = 1, 2, \dots \end{cases} \tag{9}$$

are valid for each $t \in [t_0, \infty)$ *and* $x \in \Omega_1$.

7. $u_0 \ge V(t_0, \varphi_0(t_0))$.

Then

$$V(t, x(t; t_0, \varphi_0)) \le r(t; t_0, u_0), \qquad t \in [t_0, \infty). \tag{10}$$

Proof. From Lemma 1 it follows that $J^+(t_0, \varphi_0) = [t_0, \infty)$.

Since in the interval $(\tau_k, \tau_{k+1}]$, $k = 1, 2, \dots$, $x(t)$ coincides with the solution of the problem (1), (5), we conclude that for $\tau_k < t \le \tau_{k+1}$ the function $x(t)$ satisfies the integral equation

$$x(t) = x(\tau_k) + I_k(x(\tau_k)) + \int_{\tau_k}^{t} f(s, x(s), x(s-h))ds.$$

On the other hand, the maximal solution of the problem (8) is defined by the equality

$$r(t; t_0, u_0) = \begin{cases} r_0(t; t_0, u_0^+), \quad t_0 < t \le \tau_1, \\ r_1(t; \tau_1, u_1^+), \quad \tau_1 < t \le \tau_2, \\ \vdots \\ r_k(t; \tau_k, u_k^+), \quad \tau_k < t \le \tau_{k+1}, \\ \vdots \end{cases}$$

where $r_k(t; \tau_k, u_k^+)$ is the maximal solution of the equation $\dot{u} = g(t, u)$ without impulses, that is defined in the interval $(\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$, for which $u_k^+ = \psi_k(r_{k-1}(\tau_k; \tau_{k-1}, u_{k-1}^+))$, $k = 1, 2, \dots$ and $u_0^+ = u_0$.

Let $t \in (t_0, \tau_1]$. From Lemma 2 it follows that

$$V(t, x(t; t_0, \varphi_0)) \le r(t; t_0, u_0),$$

i.e., inequality (10) is valid for $t \in (t_0, \tau_1]$.

Assume that (10) is satisfied for $(\tau_{k-1}, \tau_k]$, $k > 1$. Then, making use of (9) and the fact that the function ψ_k is nondecreasing, we get

$$V(\tau_k + 0, x(\tau_k + 0; t_0, \varphi_0)) \le \psi_k(V(\tau_k, x(\tau_k; t_0, \varphi_0)))$$
$$\le \psi_k(r(\tau_k; t_0, u_0)) = \psi_k(r_{k-1}(\tau_k; \tau_{k-1}, u_{k-1}^+)) = u_k^+.$$

We apply again Lemma 2 for $t \in (\tau_k, \tau_{k+1}]$ and obtain

$$V(t, x(t; t_0, \varphi_0)) \le r_k(t; \tau_k, u_k^+) = r(t; t_0, u_0),$$

i.e., inequality (10) is also satisfied for $t \in (\tau_k, \tau_{k+1}]$. The proof is completed by induction. □

Corollary 1. *Let the following conditions hold:*
1. *Conditions* H1–H7 *are met.*
2. *The function $V \in \mathcal{V}_0$ is such that the inequalities*

$$D_- V(t, x(t)) \le 0, \quad t \ne \tau_k, \quad k = 1, 2, \dots,$$
$$V(t + 0, x(t) + I_k(x(t))) \le V(t, x(t)), \quad t = \tau_k, \quad k = 1, 2, \dots$$

are valid for each $t \ge t_0$ and $x \in \Omega_1$.

Then
$$V(t, x(t; t_0, \varphi_0)) \le V(t_0 + 0, \varphi_0(t_0)), \qquad t \in [t_0, \infty).$$

§3. Main results

Theorem 1. *Let the following conditions hold:*
1. *Conditions* H1–H7 *are met.*
2. *The functions $V \in \mathcal{V}_0$ and $a \in \mathcal{K}$ are such that*

$$a(|x(t)|) \le V(t, x(t)), \quad t \in [t_0, \infty), \quad x \in PC[[t_0, \infty), \boldsymbol{R}^n]. \tag{11}$$

3. *The inequalities*

$$D_- V(t, x(t)) \le -cV(t, x(t)), \quad t \ne \tau_k, \quad k = 1, 2, \dots, \tag{12}$$
$$V(t + 0, x(t) + I_k(x(t))) \le V(t, x(t)), \quad t = \tau_k, \quad k = 1, 2, \dots \tag{13}$$

are valid for each $t \ge t_0$, each function $x \in \Omega_1$, $V \in \mathcal{V}_0$ and $c \in \boldsymbol{R}^+$.

Then the zero solution of the problem (1), (2), (3) *is globally equiasymptotically stable.*

Proof. Let $\varepsilon > 0$. From the condition $V(t_0, 0) = 0$ and the properties of the function $V(t_0, x)$ at the point $x = 0$ it follows that there exists a constant $\delta = \delta(t_0, \varepsilon) > 0$ such that if $|x| < \delta$, then

$$\sup_{|x|<\delta} V(t_0 + 0, x) < a(\varepsilon).$$

Let $\varphi_0 \in B_\delta(t_0, C_0)$ and let $x(t) = x(t; t_0, \varphi_0)$ be the solution of the problem (1), (2), (3). From Lemma 1 it follows that $J^+(t_0, \varphi_0) = [t_0, \infty)$. Since the conditions of Corollary 1 are fulfilled, we have

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0)), \qquad t \in [t_0, \infty). \tag{14}$$

Since $\varphi_0 \in B_\delta(t_0, C_0)$ implies $|\varphi_0(t_0)| < \delta$, we have $V(t_0 + 0, \varphi_0(t_0)) < a(\varepsilon)$.

From (11) and (14) and the above inequality we get the inequalities

$$a(|x(t; t_0, \varphi_0)|) \leq V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0)) < a(\varepsilon),$$

hence $|x(t; t_0, \varphi_0)| < \varepsilon$ for $t \in [t_0, \infty)$. This implies the stability of the solution $x(t) \equiv 0$ of the problem (1), (2), (3).

We shall show that it is also globally equiattractive. Let $\alpha = \text{const} > 0$ and $\varphi_0 \in B_\alpha(t_0, C_0)$. From (12) and (13) follows the inequality

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0, \varphi_0(t_0)) \exp[-c(t - t_0)]. \tag{15}$$

Let $M(t_0, \alpha) = \sup\{V(t_0, x) \colon |x| < \alpha\}$ and $\sigma > \dfrac{1}{c} \ln \dfrac{M(t_0, \alpha)}{a(\varepsilon)}$. Then from (15) for $t \geq t_0 + \sigma$ follows the inequality

$$V(t, x(t; t_0, \varphi_0)) \leq a(\varepsilon).$$

From the above inequality and (11) we obtain

$$|x(t; t_0, \varphi_0)| < \varepsilon \quad \text{for} \quad \mathrm{t} \geq \mathrm{t}_0 + \sigma.$$

Hence the zero solution of the problem (1), (2), (3) is globally equiattractive.
□

Theorem 2. *Let the following conditions hold:*

1. *Conditions* H1–H7 *are met.*
2. *The functions* $V \in \mathcal{V}_0$, $a,\ b \in \mathcal{K}$ *and* $\gamma \colon [t_0, \infty) \to [1, \infty)$ *are such that*

$$a(|x(t)|) \leq V(t, x(t)) \leq \gamma(t) b(|x(t)|), \qquad t \in [t_0, \infty), \tag{16}$$
$$x \in PC[[t_0, \infty), \boldsymbol{R}^n].$$

3. *The inequalities*

$$D_{-}V(t, x(t)) \leq -g(t)c(|x(t)|), \quad t \neq \tau_k, \ k = 1, 2, \ldots, \tag{17}$$

$$V(t+0, x(t) + I_k(x(t))) \leq V(t, x(t)), \quad t = \tau_k, \ k = 1, 2, \ldots \tag{18}$$

are valid for any $t \geq t_0$, *any function* $x \in \Omega_1$ *with* $V \in \mathcal{V}_0$, $g\colon [t_0, \infty) \to (0, \infty)$, *and* $c \in \mathcal{K}$.

4. *The equality*

$$\int_{t_0}^{\infty} g(s)c\left[b^{-1}\left(\frac{\eta}{\gamma(s)}\right)\right] ds = \infty$$

is valid for sufficiently small values of $\eta > 0$.

Then the zero solution of the problem (1), (2), (3) *is globally equiasymptotically stable.*

Proof. As in the proof of Theorem 1 it is shown that the zero solution of the problem (1), (2), (3) is stable.

Let $\alpha = \text{const} > 0$, $\varepsilon > 0$ and $\eta = \frac{a(\varepsilon)}{2}$. From the condition 4 of Theorem 2 it follows that we can choose a positive number $\sigma = \sigma(t_0, \alpha, \varepsilon)$ so that

$$\int_{t_0}^{t_0+\sigma} g(s)c\left[b^{-1}\left(\frac{\eta}{\gamma(s)}\right)\right] ds > \gamma(t_0)b(\alpha). \tag{19}$$

Let $\varphi_0 \in B_\alpha(t_0, C_0)$. Suppose that for each $t \in [t_0, t_0 + \sigma]$ the inequality

$$|x(t; t_0, \varphi_0)| \geq b^{-1}\left(\frac{\eta}{\gamma(t)}\right) \tag{20}$$

holds. Then from (17) and (18) we deduce the inequalities

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0)) - \int_{t_0}^{t} g(s)c(|x(s; t_0, \varphi_0)|)ds$$

$$\leq V(t_0 + 0, \varphi_0(t_0)) - \int_{t_0}^{t} g(s)c\left[b^{-1}\left(\frac{\eta}{\gamma(s)}\right)\right] ds, \quad t \in [t_0, t_0 + \sigma].$$

From the above inequalities, (16) and (19) for $t = t_0 + \sigma$ we obtain the inequalities

$$V(t, x(t; t_0, \varphi_0)) \leq \gamma(t_0)b(\alpha) - \int_{t_0}^{t} g(s)c\left[b^{-1}\left(\frac{\eta}{\gamma(s)}\right)\right] ds < 0,$$

which contradicts (16).

Hence inequality (20) is not valid for each $t \in [t_0, t_0 + \sigma]$, i.e., there exists $t' \in [t_0, t_0 + \sigma]$ such that

$$|x(t'; t_0, \varphi_0)| < b^{-1}\left(\frac{\eta}{\gamma(t')}\right).$$

Then from (16), (17) and (18) we obtain that for $t \geq t'$ (and therefore for $t \geq t_0 + \sigma$ too) the following inequalities are valid:

$$\begin{aligned} a(|x(t; t_0, \varphi_0)|) &\leq V(t, x(t; t_0, \varphi_0)) \leq V(t', x(t'; t_0, \varphi_0)) \\ &\leq \gamma(t')b(|x(t'; t_0, \varphi_0)|) \leq \eta < a(\varepsilon). \end{aligned}$$

Consequently, $|x(t; t_0, \varphi_0)| < \varepsilon$ for $t \geq t_0 + \sigma$, i.e., the zero solution of the problem (1), (2), (3) is globally equiattractive. □

Theorem 3. *Let the following conditions hold:*

1. *Conditions* H1–H7 *are met.*
2. *The functions* $V \in \mathcal{V}_0$ *and* $a,\ b \in \mathcal{K}$ *are such that*

$$a(|x(t)|) \leq V(t, x(t)) \leq b(|x(t)|), \qquad t \in [t_0, \infty), \quad x \in PC[[t_0, \infty), \boldsymbol{R}^n], \tag{21}$$

and $a(r) \to \infty$ *as* $r \to \infty$.

3. *The inequalities*

$$D_- V(t, x(t)) \leq -c(|x(t)|), \quad t \neq \tau_k,\ k = 1, 2, \ldots, \tag{22}$$

$$V(t + 0, x(t) + I_k(x(t))) \leq V(t, x(t)), \quad t = \tau_k,\ k = 1, 2, \ldots \tag{23}$$

are valid for any $t \geq t_0$, *each function* $x \in \Omega_1$ *with* $V \in \mathcal{V}_0$, $c \in \mathcal{K}$.

Then the zero solution of the problem (1), (2), (3) *is uniformly globally asymptotically stable.*

Proof. We shall first prove that the zero solution of the problem (1), (2), (3) is uniformly stable.

Let $\varepsilon > 0$. Choose $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$. Let $\varphi_0 \in B_\delta(t_0, C_0)$ and let $x(t) = x(t; t_0, \varphi_0)$ be the solution of the problem (1), (2), (3).

From (21) and (23) we deduce the inequalities

$$\begin{aligned} a(|x(t; t_0, \varphi_0)|) &\leq V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0)) \\ &\leq b(|\varphi_0(t_0)|) \leq b(\|\varphi_0\|) < b(\delta) < a(\varepsilon), \end{aligned}$$

from which it follows that $|x(t;t_0,\varphi_0)| < \varepsilon$ for $t \in [t_0,\infty)$. Hence the zero solution of the problem (1), (2), (3) is uniformly stable.

Let $\alpha > 0$. Choose a constant $\beta = \beta(\alpha) > 0$ so that $a(\beta) > b(\alpha)$. This is possible in view of the condition $a(r) \to \infty$ as $r \to \infty$. Let $\varphi_0 \in B_\alpha(t_0, C_0)$. Since the conditions of Corollary 1 are met, we have

$$V(t, x(t;t_0,\varphi_0)) \le V(t_0+0, \varphi_0(t_0)), \qquad t \in [t_0,\infty).$$

From the above inequality, (21) and (23), we obtain the inequalities

$$\begin{aligned} a(|x(t;t_0,\varphi_0)|) &\le V(t,x(t;t_0,\varphi_0)) \le V(t_0+0,\varphi_0(t_0)) \\ &\le b(|\varphi_0(t_0)|) \le b(\|\varphi_0\|) < b(\alpha) < a(\beta), \quad t \in [t_0,\infty). \end{aligned}$$

Hence $|x(t;t_0,\varphi_0)| < \beta, \qquad t \in [t_0,\infty)$.

Finally we shall prove that the zero solution of the problem (1), (2), (3) is uniformly globally attractive.

Let $\varepsilon > 0$ and $\alpha > 0$ be chosen arbitrarily. Choose $\eta = \eta(\varepsilon) > 0$ so that $b(\eta) < a(\varepsilon)$ and let $\sigma = \sigma(\varepsilon,\alpha) > 0$ be such that $\sigma > \frac{b(\alpha)}{c(\eta)}$. Let $\varphi_0 \in B_\alpha(t_0, C_0)$. Suppose that for each $t \in [t_0, t_0+\sigma]$ the inequality $|x(t;t_0,\varphi_0)| \ge \eta$ holds. Then from (22) and (23) we deduce the inequalities

$$\begin{aligned} V(t,x(t;t_0,\varphi_0)) &\le V(t_0+0,\varphi_0(t_0)) - \int_{t_0}^{t} c(|x(\tau;t_0,\varphi_0)|)d\tau \\ &\le b(\alpha) - c(\eta)\sigma < 0, \end{aligned}$$

which contradicts (21). Hence there exists $t^* \in [t_0, t_0+\sigma]$ such that $|x(t^*;t_0,\varphi_0)| < \eta$. Then from (21), (22) and (23) we obtain that for $t \ge t^*$ (hence for $t \ge t_0+\sigma$ too) the following inequalities are valid

$$\begin{aligned} a(|x(t;t_0,\varphi_0)|) &\le V(t,x(t;t_0,\varphi_0)) \le V(t^*,x(t^*;t_0,\varphi_0)) \\ &\le b(|x(t^*;t_0,\varphi_0)|) \le b(\eta) < a(\varepsilon). \end{aligned}$$

This shows that the zero solution of the problem (1), (2), (3) is uniformly globally attractive. □

Corollary 2. *If in Theorem* 3 *the condition* (22) *is replaced with the condition*

$$D_-V(t,x(t)) \le -cV(t,x(t)), \quad t \ne \tau_k, \quad k = 1,2,\dots, \tag{24}$$

$$\text{for } t \ge t_0,\ x \in \Omega_1, V \in \mathcal{V}_0,\ c = \text{const} > 0,$$

then the zero solution of the problem (1), (2), (3) *is uniformly globally asymptotically stable.*

The proof of Corollary 2 is analogous to the proof of Theorem 3. But we shall use the inequality instead

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0)) \exp[-c(t - t_0)], \quad t \geq t_0$$

which follows from (23) and (24).

Theorem 4. *Let the following conditions hold:*

1. *Conditions* H1–H7 *are met.*
2. *The function* $V \in \mathcal{V}_0$ *is such that for any* $\alpha > 0$, *there exists* $k(\alpha) > 0$ *such that*

$$|x(t)| \leq V(t, x(t)) \leq k(\alpha)|x(t)|, \qquad t \in [t_0, \infty), \quad x \in PC[[t_0, \infty), \boldsymbol{R}^n]. \tag{25}$$

3. *The inequalities*

$$D_- V(t, x(t)) \leq -cV(t, x(t)), \quad t \neq \tau_k, \ k = 1, 2, \ldots, \tag{26}$$

$$V(t + 0, x(t) + I_k(x(t))) \leq V(t, x(t)), \quad t = \tau_k, \ k = 1, 2, \ldots \tag{27}$$

are valid for each $t \geq t_0$, *each function* $x \in \Omega_1$ *with* $V \in \mathcal{V}_0$, $c = \text{const} > 0$.

Then the zero solution of the problem (1), (2), (3) *is exponentially globally asymptotically stable.*

Proof. Let $\varphi_0 \in B_\alpha(t_0, C_0)$. From (26) and (27) we obtain

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0)) \exp[-c(t - t_0)], \quad t \geq t_0.$$

From the above estimate and (25) we deduce the inequalities

$$\begin{aligned} |x(t; t_0, \varphi_0)| &\leq k(\alpha)|\varphi_0(t_0)| \exp[-c(t - t_0)] \\ &\leq k(\alpha)\|\varphi_0\| \exp[-c(t - t_0)], \quad t \geq t_0. \end{aligned}$$

This shows that the zero solution of the problem (1), (2), (3) is exponentially globally asymptotically stable. □

§4. Examples

Example 1. Consider the problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)x(t-h), \quad t > 0, \quad t \neq \tau_k, \\ x(t) = \varphi_1(t), \quad t \in [-h, 0], \\ \Delta x(\tau_k) = C_k x(\tau_k), \quad k = 1, 2, \dots, \end{cases} \tag{28}$$

where $x \in \boldsymbol{R}^n$; $A(t)$ and $B(t)$ are $n \times n$ matrix-valued functions which are continuous in $\boldsymbol{R}^+$; $B(t)$ is diagonal, and $A(t)$ is skewsymmetric; $C_k = \operatorname{diag}(c_{1k}, \dots, c_{nk})$, $-1 < c_{ik} \leq 0$; $h > 0$; $\varphi_1 \in C[[-h, 0], \boldsymbol{R}^n]$, $0 < \tau_1 < \tau_2 < \dots < \tau_k < \tau_{k+1} < \dots$ and $\lim_{k\to\infty} \tau_k = \infty$.

Let $V(t, x) = \langle x, x \rangle$, where $\langle x, y \rangle$ is the scalar product of the vectors $x,\ y \in \boldsymbol{R}^n$. Then the set

$$\Omega_1 = \Big\{ x \in PC[\boldsymbol{R}^+, \boldsymbol{R}^n] \colon \langle x(s), x(s) \rangle \leq \langle x(t), x(t) \rangle,\ t - h \leq s \leq t,\ t \geq 0 \Big\}.$$

For $t \geq 0$ and $x \in \Omega_1$ we shall have

$$D_- V(t, x(t)) = 2\langle x(t), B(t)x(t-h) \rangle \leq 2\langle x(t), B(t)x(t) \rangle, \quad t \neq \tau_k, \quad k = 1, 2, \dots$$

and

$$V(\tau_k + 0, x(\tau_k) + C_k x(\tau_k)) = \sum_{i=1}^{n} (1 + c_{ki})^2 x_i^2 \leq V(\tau_k, x(\tau_k)), \quad k = 1, 2, \dots.$$

If $B(t) = \operatorname{diag}(b_1(t), \dots, b_n(t))$ and $b_k(t) \leq -\gamma_k < 0$ for $k = 1, 2, \dots, n$ and $t \in \boldsymbol{R}^+$, then by Theorem 3 the zero solution of the problem (28) is uniformly globally asymptotically stable.

In the case when $n = 1$, $A(t) = -a < 0$ and $B(t) = b$, $1 + bh > 0$, the solution of the problem (28) satisfies the inequality

$$|x(t; t_0, \varphi_0)| < (1 + bh)\|\varphi_0\| \exp[-a(t - t_0)] \prod_{t_0 < \tau_k < t} (1 + C_k).$$

In this case the zero solution of (28) is uniformly globally asymptotically stable.

Example 2. Consider the problem

$$\begin{cases} \dot{x}(t) = a(t)y(t) + b(t)x(t)[x^2(t-h) + y^2(t-h)], \quad t \neq \tau_k, \quad t > 0 \\ \dot{y}(t) = -a(t)x(t) + b(t)y(t)[x^2(t-h) + y^2(t-h)], \quad t \neq \tau_k, \quad t > 0 \\ X(t) = \operatorname{col}(x(t), y(t)) = \varphi_2(t), \quad t \in [-h, 0], \\ \Delta x(\tau_k) = c_k x(\tau_k), \qquad \Delta y(\tau_k) = d_k y(\tau_k), \quad k = 1, 2, \dots, \end{cases} \tag{29}$$

where $t \in \boldsymbol{R}^+$; the functions $a(t)$ and $b(t)$ are continuous for $t \in \boldsymbol{R}^+$; $b(t) \leq -\gamma < 0$ for $t \in \boldsymbol{R}^+$; $-1 < c_k \leq 0$, $-1 < d_k \leq 0$, $0 < \tau_1 < \tau_2 < \ldots < \tau_k < \tau_{k+1} < \ldots$, $\lim_{k\to\infty} \tau_k = \infty$; $h > 0$; $\varphi_2 \in C[[-h, 0], \boldsymbol{R}^2]$.

Let $V(t, x, y) = x^2 + y^2$. Then the set

$$\Omega_1 = \Big\{X \in PC[\boldsymbol{R}^+, \boldsymbol{R}^2]\colon x^2(s) + y^2(s) \leq x^2(t) + y^2(t),\ t - h \leq s \leq t,\ t \geq 0\Big\}.$$

For $t \geq 0$ and $\mathrm{col(x, y)} \in \Omega_1$ we obtain

$$D_-V(t, x(t), y(t)) = 2b(t)\Big[x^2(t) + y^2(t)\Big]\Big[x^2(t-h) + y^2(t-h)\Big]$$

$$\leq 2b(t)\Big[x^2(t) + y^2(t)\Big]^2 \leq -2\gamma\Big[x^2(t) + y^2(t)\Big]^2, \quad t \neq \tau_k,$$

$$V(\tau_k + 0, x(\tau_k) + c_k x(\tau_k), y(\tau_k) + d_k y(\tau_k)) =$$

$$(1 + c_k)^2 x^2(\tau_k) + (1 + d_k)^2 y^2(\tau_k) \leq V(\tau_k, x(\tau_k), y(\tau_k)).$$

Since the conditions of Theorem 3 are met, the zero solution of the problem (29) is uniformly globally asymptotically stable.

Example 3. Consider the problem

$$\begin{cases} \dot{x}(t) = -g(t)x(t-h), \quad t \neq \tau_k, \quad t > 0, \\ x(t) = \varphi_3(t), \quad t \in [-h, 0], \\ \Delta x(\tau_k) = c_k x(\tau_k), \end{cases} \tag{30}$$

where the function $g(t)$ is continuous for $t \in \boldsymbol{R}^+$, $g(t) \to 0$ as $t \to \infty$ and $\int_0^\infty g(t)dt = \infty$, $g(t) > 0$; $-1 < c_k \leq 0$; $h > 0$; $\varphi_3 \in C[[-h, 0], \boldsymbol{R}]$.

Consider the function $V(t, x) = \frac{1}{2}x^2$. Then

$$\Omega_1 = \Big\{x \in PC[\boldsymbol{R}^+, \boldsymbol{R}]\colon \frac{1}{2}x^2(s) \leq \frac{1}{2}x^2(t),\ t - h \leq s \leq t,\ t \geq 0\Big\}.$$

For $t \geq 0$ and $x \in \Omega_1$

$$D_-V(t, x(t)) = -g(t)x(t)x(t-h) \leq -g(t)x^2(t), \quad t \neq \tau_k$$

and

$$V(\tau_k + 0, x(\tau_k) + c_k x(\tau_k)) = \frac{1}{2}(1 + c_k)^2 x^2(\tau_k) \leq V(\tau_k, x(\tau_k)).$$

Hence the conditions of Theorem 2 are satisfied and the zero solution of the problem (30) is globally equiasymptotically stable.

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