SUT Journal of Mathematics
Vol. 31, No. 1 (1995), 17–28

OSCILLATION AND NONOSCILLATION OF FIRST ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH MAXIMA

Drumi BAINOV, Vasil PETROV and Valentina PROYCHEVA

(Received September 22, 1994)

Abstract. In the paper first order neutral differential equations with "maxima" are considered. Sufficient conditions for oscillation of all solutions are obtained. The asymptotic behavior of the nonoscillatory solutions is investigated.

AMS 1991 *Mathematics Subject Classification.* Primary 34K40.

Key words and phrases. Oscillation, nonoscillation, neutral differential equations, maxima.

§1. Introduction

Consider the equation

$$\Big[x(t) + p(t)x(t-\tau)\Big]' + q(t)\max_{[t-\sigma,t]} x(s) = 0, \tag{1}$$

where $p(t)$, $q(t)$ are continuous bounded functions and τ, σ are positive constants. The present paper deals with the oscillatory and asymptotic properties of the solutions of (1). We shall note that equations with maxima occur in the problem of automatic regulation of various real systems [3], [4]. The maxima arise when the regulation law corresponds to the maximal deviation of the regulable quantity. The only paper in which oscillatory properties of equations with maxima are considered is [1]. The asymptotic stability of the solutions is investigated in [5].

By a solution of (1) we mean a continuous function $x(t)$ on $[t_0,\infty)$ such that $x(t)+p(t)x(t-\tau)$ is continuously differentiable and $x(t)$ satisfies (1). As is customary, a solution of (1) is said to be *oscillatory* if it has arbitrary large zeros, otherwise it is said to be *nonoscillatory.* In the sequel, for the sake of convenience, we will assume that inequalities concerning values of functions are satisfied *eventually,* that is for all large t.

§2. Preliminary notes

We shall say that conditions (H) are met if the following conditions hold:

H1. $\tau,\ \sigma > 0$;

H2. $p(t) \in C([t_0, \infty), \boldsymbol{R})$;

H3. $q(t) \in C([t_0, \infty), \boldsymbol{R}_+),\ \boldsymbol{R}_+ = [0, \infty)$;

H4. $\int_{t_0}^{\infty} q(t)dt = \infty$.

Define the function $z(t)$ as follows:

$$z(t) = x(t) + p(t)x(t - \tau). \tag{2}$$

Then (1) implies that

$$z'(t) = -q(t) \max_{[t-\sigma,t]} x(s). \tag{3}$$

We shall need the following lemma:

Lemma 1. *Let conditions* (H) *hold. Then the following assertions are valid:*

(i) *If $p \le p(t) \le -1$ and $x(t)$ is a positive solution of* (1), *then $z(t)$ is a nonincreasing function and $z(t) < 0$;*

(ii) *If $p \le p(t) \le -1$ and $x(t)$ is a negative solution of* (1), *then $z(t)$ is a nondecreasing function and $z(t) > 0$;*

(iii) *If $-1 \le p(t) \le 0$ and $x(t)$ is a positive solution of* (1), *then $z(t)$ is a nonincreasing function and $z(t) > 0$;*

(iv) *If $-1 \le p(t) \le 0$ and $x(t)$ is a negative solution of* (1), *then $z(t)$ is a nondecreasing function and $z(t) < 0$.*

Proof. By H4 we see that if $x(t)$ is a nonoscillatory solution of (1), then $z(t)$ is not a constant function. We shall prove only (i) and (iii) since the proofs of (ii) and (iv) are analogous.

(i) Let $x(t)$ be a positive solution of (1). Then from (3) it follows that $z'(t) \le 0$ and $z(t)$ is a nonincreasing function. Suppose that $z(t) > 0$. From (2) there follows the inequalities

$$x(t) > -p(t)x(t - \tau) \ge x(t - \tau).$$

From the inequalities $x(t) > x(t - \tau)$ and $x(t) > 0$ it follows that there exists a constant $m > 0$ such that $x(t) > m$, hence $\max_{[t-\sigma,t]} x(s) > m$. From (3) we obtain

$$z'(t) \le -mq(t).$$

Integrate the last inequality from t_1 to t, where t_1 is a sufficiently large number, and obtain

$$z(t) \le z(t_1) - m \int\limits_{t_1}^{t} q(s)ds.$$

Passing to the limit in the above inequality as $t \to \infty$, from H4 it follows that $\lim\limits_{t\to\infty} z(t) = -\infty$. The contradiction obtained shows that $z(t) < 0$.

(iii) Let $x(t) > 0$. As in the case (i) it is verified that $z(t)$ is a nonincreasing function. Suppose that $z(t) < 0$. From the inequality $-1 \le p(t) \le 0$ and from (2) it follows that

$$x(t) < x(t-\tau). \tag{4}$$

Since $x(t) > 0$, then from (4) we obtain that $x(t)$ is a bounded function, hence $z(t)$ is also bounded. Then there exists the finite limit $\lim\limits_{t\to\infty} z(t) = l$ ($l < 0$). Let $c = \liminf\limits_{t\to\infty} x(t)$. Suppose that $c > 0$. For sufficiently large t the inequality $x(t) \ge \dfrac{c}{2}$ is valid, hence $\max\limits_{[t-\sigma,t]} x(s) \ge \dfrac{c}{2}$. As in (i) it is shown that $\lim\limits_{t\to\infty} z(t) = -\infty$, which contradicts the fact that $z(t)$ is a bounded function. Thus $\liminf\limits_{t\to\infty} x(t) = 0$. There exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and $\lim\limits_{n\to\infty} x(t_n - \tau) = 0$. From (4) we obtain that $\lim\limits_{n\to\infty} x(t_n) = 0$, hence $\lim\limits_{n\to\infty} z(t_n) = 0$, which contradicts the fact that $l = \lim\limits_{t\to\infty} z(t) < 0$. Consequently, $z(t) > 0$. □

Corollary 1. *Let conditions* (H) *hold and* $p(t) \equiv -1$. *Then each solution of* (1) *is oscillatory.*

Remark 1. We shall emphasize that equation (1) is *nonlinear* and in the general case the fact that $x(t)$ is a solution of (1) does not imply that $-x(t)$ is also a solution of (1). Due to this reason in the investigation of nonoscillatory solutions of (1) we have to separately consider the cases $x(t) > 0$ and $x(t) < 0$.

§3. Main results

3.1. Nonoscillatory solutions

In this section we shall consider the asymptotic behaviour of the nonoscillatory solutions of (1).

Theorem 1. *Let conditions* (H) *hold,* $\sigma \geq \tau$ *and*

$$-1 < p_1 \leq p(t) \leq p_2. \tag{5}$$

If $x(t)$ *is a positive solution of* (1)*, then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a positive solution of (1). We shall assume that $-1 < p_1 < 0$ and $p_2 > 0$. (In the case when p_1 and p_2 are both negative the assertion of the theorem is valid even without the condition $\sigma \geq \tau$ (Theorem 3, i)). The case when p_1 and p_2 are both positive can be considered analogously to the case $p_1 < 0$, $p_2 > 0$ but is technically easier). We shall divide the proof of the theorem into three steps: *(i), (ii)* and *(iii).*

(i) $z(t) > 0$.

From (3) it follows that $z(t)$ is a nonincreasing function, and from H4 we obtain that $z(t)$ is not a constant function. Suppose that $z(t) < 0$. There exists $c > 0$ for which $z(t) < -c$. Then

$$-x(t-\tau) < p(t)x(t-\tau) < x(t) + p(t)x(t-\tau) = z(t) < -c.$$

From H4, (3) and from the inequality $x(t-\tau) > c$ we derive $\lim_{t\to\infty} z(t) = -\infty$, hence $\limsup_{t\to\infty} x(t) = \infty$. On the other hand, since $z(t) < 0$, then from (2) and (5) there follows the inequality

$$x(t) < -p(t)x(t-\tau) < x(t-\tau).$$

The inequality $x(t) < x(t-\tau)$, however, contradicts the relation $\limsup_{t\to\infty} x(t) = \infty$, hence $z(t) > 0$.

(ii) $\lim_{t\to\infty} z(t) = 0$.

Since $z(t)$ is a nonincreasing positive function, then there exists the finite limit $l = \lim_{t\to\infty} z(t)$ $(l \geq 0)$. Suppose that $l > 0$. Then $z(t) > l$ and from (2) and (5) we obtain the inequalities

$$l < x(t) + p(t)x(t-\tau) \leq x(t) + p_2 x(t-\tau) \leq (1+p_2)\max\{x(t), x(t-\tau)\}.$$

Thus $\max\{x(t), x(t-\tau)\} > \dfrac{l}{1+p_2}$. Since $\tau \leq \sigma$, then from the last inequality it follows that $\max_{[t-\sigma,t]} x(s) > \dfrac{l}{1+p_2}$. Then from (3) and H4 we obtain that $\lim_{t\to\infty} z(t) = -\infty$. The contradiction obtained shows that $l = 0$, i.e., $\lim_{t\to\infty} z(t) = 0$.

(iii) $\lim_{t\to\infty} x(t) = 0$.

Suppose that $x(t)$ is an unbounded function. There exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \alpha_n = \infty$, $\lim_{n\to\infty} x(\alpha_n) = \infty$ and $\max_{[\alpha_1,\alpha_n]} x(s) = x(\alpha_n)$. Then

$$z(\alpha_n) = x(\alpha_n) + p(\alpha_n)x(\alpha_n - \tau) \geq x(\alpha_n) + p_1 x(\alpha_n - \tau) \geq (1 + p_1)x(\alpha_n).$$

Since $1 + p_1 > 0$, then $\lim_{n\to\infty} z(\alpha_n) = \infty$, which contradicts what was proved in *(ii)*. Hence $x(t)$ is a bounded function. Let $d = \limsup_{t\to\infty} x(t)$. We choose a sequence $\{\beta_n\}_{n=1}^{\infty}$ so that $\lim_{n\to\infty} \beta_n = \infty$ and $\lim_{n\to\infty} x(\beta_n) = d$. Since $\{x(\beta_n - \tau)\}$ and $\{p(\beta_n)\}$ are bounded, we can choose a subsequence $\{\beta_{n_k}\}_{k=1}^{\infty}$ so that $\{p(\beta_{n_k})\}$ and $\{x(\beta_{n_k} - \tau)\}$ be convergent. Then

$$0 = \lim_{k\to\infty} z(\beta_{n_k}) = \limsup_{t\to\infty} x(t) + \lim_{k\to\infty} \Big(p(\beta_{n_k})x(\beta_{n_k} - \tau)\Big).$$

If $\lim_{k\to\infty} p(\beta_{n_k}) \geq 0$, then $0 \geq \limsup_{t\to\infty} x(t) \geq 0$.
If $\lim_{k\to\infty} p(\beta_{n_k}) < 0$, then

$$0 = \limsup_{t\to\infty} x(t) + \lim_{k\to\infty} \Big(p(\beta_{n_k})x(\beta_{n_k} - \tau)\Big) \geq \Big(1 + \lim_{k\to\infty} p(\beta_{n_k})\Big) \limsup_{t\to\infty} x(t) \geq 0.$$

Consequently, $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$. □

Remark 2. The condition $\sigma \geq \tau$ in Theorem 1 is essential. We shall show this by the following example.

Example 1. Consider the equation

$$\Big[x(t) + x(t-2)\Big]' + q(t) \max_{[t-\frac{1}{2},t]} x(s) = 0,$$

where

$$q(t) = \frac{\frac{1}{t^2} + \frac{1}{(t-2)^2}}{\max_{[t-\frac{1}{2},t]} \Big\{\varphi(s) + \frac{1}{s}\Big\}}$$

and $\varphi(t)$ is the 4-periodic function

$$\varphi(t) = \begin{cases} 0, & t \in [0,1], \\ t-1, & t \in (1,2], \\ 1, & t \in (2,3], \\ 4-t, & t \in (3,4]. \end{cases}$$

It is immediately verified that $x(t) = \varphi(t) + \dfrac{1}{t}$ is a solution of the equation and that the limit $\lim\limits_{t\to\infty} x(t)$ does not exist. On the other hand,

$$\int\limits_4^\infty q(t)dt > \sum_{k=1}^\infty \int\limits_{4k+\frac{1}{2}}^{4k+1} \frac{\frac{1}{t^2}+\frac{1}{(t-2)^2}}{\max\limits_{[t-\frac{1}{2},t]}\left\{\varphi(s)+\frac{1}{s}\right\}}dt = \sum_{k=1}^\infty \int\limits_{4k+\frac{1}{2}}^{4k+1} \frac{\frac{1}{t^2}+\frac{1}{(t-2)^2}}{\frac{1}{t-\frac{1}{2}}}dt > \sum_{k=1}^\infty \int\limits_{4k+\frac{1}{2}}^{4k+1} \frac{dt}{t}.$$

The last series, however, is obviously divergent, hence $\int\limits_4^\infty q(t)dt = \infty$, i.e., all conditions of Theorem 1 are met except the condition $\tau \le \sigma$.

Remark 3. The assertion of Theorem 1 is not valid for neutral equations without maxima no matter whether $\tau \le \sigma$, or $\tau > \sigma$. We shall show this in the case $\sigma > \tau$.

Example 2. Consider the equation

$$\Big[x(t) + x(t-2)\Big]' + q(t)x(t-4) = 0,$$

where

$$q(t) = \frac{\frac{1}{t^2}+\frac{1}{(t-2)^2}}{\varphi(t-4)+\frac{1}{t-4}}$$

and $\varphi(t)$ is defined as in Example 1. It is immediately verified that $x(t) = \varphi(t) + \dfrac{1}{t}$ is a positive solution of the equation but the limit $\lim\limits_{t\to\infty} x(t)$ does not exist. As in Example 1 it is shown that condition H4 is met.

In the subsequent theorem we shall show that the assertion of Theorem 1 is still valid without the condition $\sigma \ge \tau$, if condition H4 is replaced by a little stronger condition H5:

H5. $\int\limits_{t_0}^\infty \tilde{q}(t)dt = \infty$, $\tilde{q}(t) = \min\{q(t), q(t+\tau)\}$.

Theorem 2. *Let conditions* H1–H3, H5 *and* (5) *hold. If* $x(t)$ *is a positive solution of* (1), *then* $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. We shall again divide the proof of the theorem into the same three steps *(i), (ii)* and *(iii)* as in Theorem 1. The proofs of *(i)* and *(iii)* are quite analogous to those in Theorem 1. We shall prove *(ii).*

(ii) $\lim_{t\to\infty} z(t) = 0$.

From (3), (5) and the definition of $\tilde{q}(t)$ there follows the estimate

$$\begin{aligned} &z'(t) - \tilde{q}(t-\tau)\max_{[t-\sigma,t]}(p(s)x(s-\tau)) \\ &\quad = -q(t)\max_{[t-\sigma,t]} x(s) - \tilde{q}(t-\tau)\max_{[t-\sigma,t]}(p(s)x(s-\tau)) \\ &\quad \le -\tilde{q}(t-\tau)\Big[\max_{[t-\sigma,t]} x(s) + \max_{[t-\sigma,t]}(p(s)x(s-\tau))\Big] \\ &\quad \le -\tilde{q}(t-\tau)\max_{[t-\sigma,t]}\Big\{x(s) + p(s)x(s-\tau)\Big\} \\ &\quad = -\tilde{q}(t-\tau)\max_{[t-\sigma,t]} z(s) = -\tilde{q}(t-\tau)z(t-\sigma). \end{aligned}$$

Finally

$$z'(t) - \tilde{q}(t-\tau)\max_{[t-\sigma,t]}(p(s)x(s-\tau)) \le -\tilde{q}(t-\tau)z(t-\sigma). \tag{6}$$

Suppose that $\lim_{t\to\infty} z(t) = c > 0$. Then $z(t) \ge c$ and (6) takes on the form

$$z'(t) - \tilde{q}(t-\tau)\max_{[t-\sigma,t]}(p(s)x(s-\tau)) \le -c\tilde{q}(t-\tau).$$

We integrate the last inequality from t_1 to t and obtain

$$z(t) - z(t_1) - \int_{t_1}^{t} \tilde{q}(s-\tau)\max_{[s-\sigma,s]}(p(v)x(v-\tau))ds \le -c\int_{t_1}^{t}\tilde{q}(s-\tau)ds.$$

From H5 it follows that

$$\int_{t_1}^{\infty} \tilde{q}(s-\tau)\max_{[s-\sigma,s]}(p(v)x(v-\tau))ds = \infty.$$

Since $p(t)$ is a bounded function, then

$$\int_{t_1}^{\infty} \tilde{q}(t-\tau)\max_{[t-\sigma,t]} x(s-\tau)dt = \infty$$

or

$$\int\limits_{t_1-\tau}^{\infty} \tilde{q}(t) \max_{[t-\sigma,t]} x(s)dt = \infty.$$

From the definition of $\tilde{q}(t)$ and from the last equality we finally get

$$\int\limits_{t_1-\tau}^{\infty} q(t) \max_{[t-\sigma,t]} x(s)dt = \infty.$$

Then from (3) it follows that $\lim\limits_{t\to\infty} z(t) = -\infty$. The contradiction shows that $\lim\limits_{t\to\infty} z(t) = 0$. □

Remark 4. The assertion of Theorem 2 is still valid for neutral equations without maxima, moreover, both for the positive and negative solutions. We shall note that for the negative solutions of (1) we cannot claim under the conditions of Theorem 2 that $\lim\limits_{t\to\infty} x(t) = 0$. We shall illustrate this fact by the following example.

Example 3. Consider the equation

$$\left[x(t) + x\left(t - \frac{1}{2}\right)\right]' + q(t) \max_{[t-1,t]} x(s) = 0,$$

where

$$q(t) = -\frac{e^{-t}\left(1 + e^{\frac{1}{2}}\right)}{\max\limits_{[t-1,t]} \left(\varphi(s) - e^{-s}\right)}$$

and $\varphi(t)$ is the 1-periodic function

$$\varphi(t) = \begin{cases} -t, & t \in [0, \frac{1}{2}], \\ t - 1, & t \in (\frac{1}{2}, 1]. \end{cases}$$

It is immediately verified that $x(t) = \varphi(t) - e^{-t}$ is a negative solution of the equation and that the limit $\lim\limits_{t\to\infty} x(t)$ does not exist. On the other hand, since

$$\max_{[t-1,t]} \left(\varphi(s) - e^{-s}\right) \geq -e^{-(t-1)} \geq -e^{-(t-2)}$$

then $q(t) \geq \dfrac{1 + e^{\frac{1}{2}}}{e^2}$. Of course, the last inequality implies the validity of condition H5.

We shall end this section by stating the following theorem.

Theorem 3. *Let conditions* (H) *hold as well as one of the conditions:*

i) $-1 < p \le p(t) \le 0$;

ii) $0 \le p(t) \le p < 1$;

iii) $1 < p_1 \le p(t) \le p_2$.

Then each solution of (1) *tends to zero as* $t \to \infty$.

We shall omit the proof of Theorem 3 since it is quite analogous to the proof of the corresponding assertion for neutral equations without maxima. (This problem has been investigated by a number of authors even for considerable more general neutral equations without maxima, here we shall just quote [2] where in Theorem 1*c* an assertion is proved analogous to that of Theorem 3.)

3.2. Sufficient conditions for oscillation

Theorem 4. *Let conditions* (H) *hold,* $\tau > \sigma$,

$$p \le p(t) \le -1 \tag{7}$$

and

$$\limsup_{t\to\infty} \int_t^{t+\tau-\sigma} \frac{q(s)}{\max\limits_{[s-\sigma,s]}(-p(v+\tau))} ds > 1. \tag{8}$$

Then each solution of (1) *oscillates.*

Proof. Suppose that $x(t)$ is a positive solution of (1). From (7) and from Lemma 1 (i) it follows that $z(t)$ is a negative nonincreasing function. Then $z(t) > p(t)x(t-\tau)$ and, consequently,

$$x(t) > \frac{z(t+\tau)}{p(t+\tau)}.$$

From the above inequality and from the fact that $z(t)$ is a nonincreasing negative function there follow the inequalities

$$\max_{[t-\sigma,t]} x(s) > \max_{[t-\sigma,t]} \frac{z(s+\tau)}{p(s+\tau)} \ge z(t+\tau-\sigma) \max_{[t-\sigma,t]} \frac{1}{p(s+\tau)}.$$

From (3) and from the last inequality there follows the estimate

$$z'(t) < -q(t)z(t+\tau-\sigma) \max_{[t-\sigma,t]} \frac{1}{p(s+\tau)} = z(t+\tau-\sigma)\frac{q(t)}{\max\limits_{[t-\sigma,t]}(-p(s+\tau))}.$$

We integrate the above inequality from t to $t+\tau-\sigma$ and obtain

$$z(t+\tau-\sigma)-z(t)\le\int_t^{t+\tau-\sigma}z(s+\tau-\sigma)\frac{q(s)}{\max\limits_{[s-\sigma,s]}(-p(v+\tau))}ds.$$

Hence

$$z(t+\tau-\sigma)<z(s+\tau-\sigma)\int_t^{t+\tau-\sigma}\frac{q(s)}{\max\limits_{[s-\sigma,s]}(-p(v+\tau))}ds$$

and

$$\int_t^{t+\tau-\sigma}\frac{q(s)}{\max\limits_{[s-\sigma,s]}(-p(v+\tau))}ds<1.$$

Obviously the last inequality contradicts (8).

Suppose that $x(t)$ is a negative solution of (1). From (7) and from Lemma 1 (ii) it follows that $z(t)$ is a positive nondecreasing function. As above we get the inequality

$$\max_{[t-\sigma,t]}x(s)<z(t+\tau-\sigma)\max_{[t-\sigma,t]}\frac{1}{p(s+\tau)}.$$

From (3) there follows the estimate

$$z'(t)\ge -q(t)z(t+\tau-\sigma)\max_{[t-\sigma,t]}\frac{1}{p(s+\tau)}=z(t+\tau-\sigma)\frac{q(t)}{\max\limits_{[t-\sigma,t]}(-p(s+\tau))}.$$

Further on we get to a contradiction with (8) just as in the previous case. Thus, since (1) cannot have positive or negative solutions, then each solution of (1) oscillates. □

Theorem 5. *Let conditions* (H) *hold and*

$$-1\le p(t)\le 0, \tag{9}$$

$$\limsup_{t\to\infty}\int_{t-\min(\tau,\sigma)}^{t}q(s)\max_{[s-\sigma,s]}(-p(v))ds>1. \tag{10}$$

Then each solution of (1) *oscillates.*

Proof. Suppose that $x(t)$ is a positive solution of (1). From (9) and from Lemma 1 (iii) it follows that $z(t)$ is a nonincreasing positive function. Then $z(t)\le x(t)$ and, consequently,

$$\max_{[t-\sigma,t]}x(s)\ge\max_{[t-\sigma,t]}z(s)=z(t-\sigma).$$

From (3) there follows the inequality $z'(t) \leq -q(t)z(t-\sigma)$. We integrate this inequality from $t-\sigma$ to t and obtain

$$z(t) - z(t-\sigma) \leq -\int\limits_{t-\sigma}^{t} q(s)z(s-\sigma)ds \leq -z(t-\sigma)\int\limits_{t-\sigma}^{t} q(s)ds.$$

Then

$$-z(t-\sigma) < -z(t-\sigma)\int\limits_{t-\sigma}^{t} q(s)ds$$

and

$$\int\limits_{t-\sigma}^{t} q(s)ds < 1.$$

From (9) it follows that

$$1 > \int\limits_{t-\sigma}^{t} q(s) \max_{[s-\sigma,s]}(-p(v))ds \geq \int\limits_{t-\min(\tau,\sigma)}^{t} q(s) \max_{[s-\sigma,s]}(-p(v))ds.$$

The last inequality, however, contradicts (10).

Suppose that $x(t)$ is a negative solution of (1). From Lemma 1 (iv) it follows that $z(t)$ is a negative nondecreasing function, hence $x(t) \leq z(t)$ and $x(t) < -p(t)x(t-\tau)$. From the last two inequalities there follows the estimate

$$x(t) < -p(t)x(t-\tau) \leq -p(t)z(t-\tau).$$

Consequently,

$$\max_{[t-\sigma,t]} x(s) < \max_{[t-\sigma,t]}(-p(s)z(s-\tau)) \leq z(s-\tau)\max_{[t-\sigma,t]}(-p(s)).$$

From (3) there follows the inequality

$$z'(t) \geq -q(t)z(t-\tau))\max_{[t-\sigma,t]}(-p(s)).$$

Integrate the above inequality from $t-\tau$ to t and obtain

$$z(t) - z(t-\tau) \geq -\int\limits_{t-\tau}^{t} q(s)z(s-\tau)\max_{[s-\sigma,s]}(-p(v))ds.$$

Then

$$-z(t-\tau) \geq -z(t-\tau)\int\limits_{t-\tau}^{t} q(s)\max_{[s-\sigma,s]}(-p(v))ds$$

and

$$\int_{t-\tau}^{t} q(s) \max_{[s-\sigma,s]}(-p(v))ds \le 1.$$

The last inequality obviously contradicts (10). □

Acknowledgement

The present investigation was supported by the Bulgarian Ministry of Education and Science under Grant MM–422.

The authors are grateful to the referee for his valuable suggestions.

References

[1] D. D. Bainov and A. I. Zahariev, *Oscillating and asymptotic properties of a class of functional differential equations with maxima,* Czechoslovak Math. J. **34** (1984), 247–251.

[2] Q. Chuanxi and G. Ladas, *Oscillations of neutral differential equations with variable coefficients,* Applicable Analysis **32** (1989), 215–228.

[3] A. R. Magomedov, *On some problems of differential equations with "maxima",* Izv. Acad. Sci. Azerb. SSR, Ser. Phys.-Techn. and Math. Sci. **108** (1977), 104–108 (in Russian).

[4] E. P. Popov, *Automatic Regulation and Control,* Nauka, Moscow, 1966 (in Russian).

[5] H. D. Voulov and D. D. Bainov, *On the asymptotic stability of differential equations with "maxima",* Rend. Circ. Mat. Palermo, Serie II, Tomo XI (1991), 385–420.

Drumi Bainov
Higher Medical Institute,
P.O. Box 45, Sofia — 1504, Bulgaria

Vasil Petrov
Plovdiv University,
Plovdiv, Bulgaria

Valentina Proycheva
Plovdiv University,
Plovdiv, Bulgaria