Positive Solutions for Singular Initial Value Problems with Sign Changing Nonlinearities Depending on $y'$

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Abstract. Using the theory of fixed point index, this paper presents the existence of positive solutions for the singular second-order initial value problems, where $f(t, y, y')$ may be singular at $y = 0$ and $y' = 0$, and $f$ may change sign.

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1. Introduction

In this paper, we consider the singular initial value problems

\[
\begin{aligned}
y''(t) &= \Phi(t)f(t, y, y'), \\ y(0) &= y'(0) = 0,
\end{aligned}
\]

(1.1)

where $f(t, y, y')$ may change sign and may be singular at $y = 0$ and $y' = 0$.

When $f(t, y, y') > 0$ may be singular at $t = 0$, $y = 0$ or $y' = 0$ and superlinear at $y = +\infty$, R.P. Agarwal and D. O’Regan considered the existence of positive solutions to (1.1) in [1]. Also, in [4], H.Wang and W.Ge presented the existence of positive solutions to (1.1) by improving the work in [1] when $f(t, y, y')$ is nonnegative. In [5, 6], G.Yang considered the existence of positive solutions to (1.1) ($T = 1$) when $f(t, y, y') > 0$ is singular at $y = 0$ and $y' = 0$, but the boundedness of $f(t, y, y')$ at $+\infty$ is necessary. In this paper, $f(t, y, y')$ changes sign and may be singular at $y = 0$ and $y' = 0$ and $f(t, y, y')$ may be superlinear at $y = +\infty$.

There are main two sections in our paper. In section 3, using the theory of fixed point index on a cone (see [3]) we discuss the existence of positive solutions...
solutions to (1.1) when \( f(t, y, y') \) is singular at \( y' = 0 \) but not \( y = 0 \) and when \( f \) may change sign. In section 4, we discuss the existence of positive solutions to (1.1) when \( f(t, y, y') \) is singular at \( y' = 0 \) and \( y = 0 \) and when \( f \) may change sign. Some ideas come from [2] and [7].

2. Preliminaries

Let

\[ C^1[0, T] = \{ y : [0, T] \to \mathbb{R} \mid y(t) \text{ is continuously differentiable on } [0, T] \} \]

with norm \( \| y \| = \max \{ \max_{t \in [0, T]} |y(t)|, \max_{t \in [0, T]} |y'(t)| \} \) and

\[ P = \{ y \in C^1[0, T] : y(t) \geq 0 \text{ and } y'(t) \geq 0, \forall t \in [0, T] \}. \]

Obviously, \( C^1[0, T] \) is a Banach space and \( P \) is a cone in \( C^1[0, T] \).

The following lemma is needed later.

**Lemma 2.1.** Let \( \Omega \) be a bounded open set in real Banach space \( E \), \( P \) be a cone of \( E \), \( \theta \in \Omega \), \( \Omega \cap P \) is a relatively open set in \( P \) and \( A : \bar{\Omega} \cap P \to P \) be continuous and compact. Suppose

\[ \lambda Ax \neq x, \forall x \in \partial \Omega \cap P, \lambda \in (0, 1]. \quad (2.1) \]

Then

\[ i(A, \Omega \cap P, P) = 1. \]

Suppose the following condition holds:

\[ \Phi \in C[0, T] \cap L^1[0, T] \text{ for } t \in (0, T], \quad \text{and } f \in C([0, T] \times [0, \infty) \times [0, \infty), \mathbb{R}). \quad (2.2) \]

For \( y \in P \), define an operator by

\[ (Ay)(t) = \int_0^t \max \left\{ 0, \int_0^s \Phi(\tau)f(\tau, y(\tau), y'(\tau))d\tau \right\} ds, \forall t \in [0, T]. \quad (2.3) \]

A standard argument in the literature [1, 4] yields:

**Lemma 2.2** Suppose that (2.2) holds. Then \( A : P \to P \) is continuous and completely continuous.
3. Singularities at \( y' = 0 \) but not \( y = 0 \)

In this section our nonlinearity \( f \) may be singular at \( y' = 0 \), but not at \( y = 0 \). Throughout this section we will assume that the following conditions hold:

\[
(H_1) \quad \Phi \in C[0, T] \text{ with } \Phi(t) > 0 \text{ on } (0, T);
\]

\[
(H_2) \quad f : [0, T] \times [0, +\infty) \times (0, +\infty) \to R \text{ is continuous with } |f(t, x, y)| \leq h(x)[g(y) + r(y)] \text{ on } [0, T] \times [0, +\infty) \times (0, +\infty) \text{ with } g(y) > 0 \text{ continuous and nonincreasing on } (0, +\infty), \text{ and } h(x) \geq 0, r(y) \geq 0 \text{ continuous and nondecreasing on } [0, \infty);
\]

\[
(H_3) \quad \sup_{c \in (0, +\infty)} \max \{1, T\} I^{-1}(\Phi_0) \int_0^c h(x) dx > 1,
\]

where \( I(z) = \int_0^z \frac{udu}{g(u)+r(u)} \), \( z \in (0, +\infty) \), and \( |\Phi_0| = \max_{t \in [0, T]} |\Phi(t)|; \)

\[
(H_4) \quad \text{there is a } \beta \in C((0, T), (0, +\infty)) \text{ and constants } \delta > 0 \text{ and } 1 > \gamma \geq 0 \text{ such that }
\]

\[
f(t, x, y) \geq \beta(t)x^\gamma, \quad \forall(t, x, y) \in (0, T) \times [0, +\infty) \times (0, \delta].
\]

For \( y \in P \) and each \( n \in \{1, 2, \cdots \} \), define operators by

\[
(A_n y)(t) = \int_0^t \max \left\{ 0, \int_0^s \Phi(\tau) f(\tau, y(\tau) + \frac{\tau}{n}, y'(\tau) + \frac{1}{n} d\tau \right\} ds, \quad \forall t \in [0, T].
\]

(3.1)

**Theorem 3.1** Suppose that \((H_1) - (H_4)\) hold. Then (1.1) has at least one nonnegative solution \( Y_0 \in C^1[0, T] \cap C^2(0, T) \) with \( y_0(t) > 0 \) on \((0, T]\).

**Proof.** From \((H_3)\), choose \( R_1 > 0 \) with

\[
\frac{R_1}{\max \{1, T\} I^{-1}(\Phi_0) \int_0^{R_1} h(x) dx} > 1.
\]

From the continuity of \( I^{-1}, I \) and \( \int_0^z h(u) du \), we can choose \( \varepsilon > 0 \) and \( \varepsilon < R_1 \) such that

\[
\frac{R_1}{\max \{1, T\} I^{-1}(\Phi_0) \int_0^{R_1+\varepsilon} h(x) dx + I(\varepsilon)} > 1. \quad (3.2)
\]

Let \( n_0 \in \{1, 2, \cdots \} \) be chosen so that \( \frac{1}{n_0} < \delta/2 \), \( \frac{T}{n_0} < \varepsilon \) and let \( N_0 = \{n_0, n_0 + 1, \cdots \} \). Now \((H_1), (H_2)\) and Lemma 2.2 guarantee that for each \( n \in N_0, A_n : P \to P \) is continuous and completely continuous. Now let

\[
\Omega_1 = \{y \in C^1[0, T] : \|y\| < R_1\}.
\]

We now show that

\[
y \neq \mu A_n y, \forall y \in P \cap \partial \Omega_1, \quad \mu \in (0, 1], \quad n \in N_0. \quad (3.3)
\]
Suppose there exist a \( y_0 \in P \cap \partial \Omega_1 \) and a \( \mu_0 \in (0, 1] \) such that \( y_0 = \mu_0 A_n y_0 \), i.e.,

\[
y_0(t) = \mu_0 \int_0^t \max \left\{ 0, \int_0^s \Phi(\tau) f \left( \tau, y_0(\tau) + \frac{\tau}{n}, y_0(\tau) + \frac{1}{n} \right) d\tau \right\} ds, \quad t \in [0, T],
\]

which yields

\[
y_0'(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0(s) + \frac{1}{n} \right) ds \right\}, \quad t \in [0, T].
\]

Obviously, \( y_0'(t) \geq 0, t \in (0, T] \) and \( \lim_{t \to 0} y_0'(t) = 0 \). Then, from \( \frac{1}{n} < \delta/2 \), there is a \( t_0 > 0 \) such that \( 0 \leq y_0(t) + \frac{1}{n} \leq \delta \) for all \( t \in (0, t_0] \). (H4) implies

\[
f(t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n}) \geq \beta(t)(y_0(t) + \frac{1}{n})^{\gamma} > 0 \quad \text{for all } t \in (0, t_0],
\]

which means that

\[
\max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0(s) + \frac{1}{n} \right) ds \right\} \\
\geq \max \left\{ 0, \int_0^t \Phi(s) \beta(s) \left( y_0(s) + \frac{s}{n} \right)^{\gamma} ds \right\} \\
= \int_0^t \Phi(s) \beta(s) \left( y_0(s) + \frac{s}{n} \right)^{\gamma} ds > 0, \quad t \in (0, t_0].
\]

and

\[
y_0'(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0(s) + \frac{1}{n} \right) ds \right\} \\
\geq \mu_0 \max \left\{ 0, \int_0^t \Phi(s) \beta(s) \left( y_0(s) + \frac{s}{n} \right)^{\gamma} ds \right\} \\
= \mu_0 \int_0^t \Phi(s) \beta(s) \left( y_0(s) + \frac{s}{n} \right)^{\gamma} ds > 0, \quad t \in (0, t_0].
\]

Let \( t^* = \sup \{t \in (0, T] \mid y_0'(s) > 0 \text{ for all } s \in (0, t)\} \). Then, we claim that \( t^* = T \), which means that \( y_0'(t) > 0 \) for all \( t \in (0, T) \), and so

\[
y_0'(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds \right\} > 0, \quad \forall t \in (0, T).
\]

Hence

\[
y_0'(t) = \mu_0 \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds, \quad t \in (0, T).
\]

(3.4)
Suppose that \( t^* < T \), which means \( y_0'(t^*) = 0 \) and \( y_0'(t) > 0 \) for all \( t \in (0, t^*) \) and

\[
0 < y_0'(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds \right\}
\]

\[
= \mu_0 \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds, \quad t \in (0, t^*). \tag{3.5}
\]

The continuity of \( y_0'(t) \) at \( t^* \) and \( \frac{1}{n} < \delta/2 \) guarantee that there is a \( t^*_0 \in (0, t^*) \) such that \( 0 < y_0'(t) + \frac{1}{n} \leq \delta \) for all \( t \in [t^*_0, t^*] \), which implies that \( f(t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n}) \geq \beta(t)(y_0(t) + \frac{1}{n})^{\gamma} \) for all \( t \in [t^*_0, t^*] \). Thus, from (3.5), we have

\[
y_0'(t^*_0) = \mu_0 \int_0^{t^*_0} \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds
\]

and

\[
0 = y_0'(t^*)
\]

\[
= \mu_0 \max \left\{ 0, \int_0^{t^*} \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds \right\}
\]

\[
= \mu_0 \max \left\{ 0, \int_{t^*_0}^{t^*} \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds \right\}
\]

\[
+ \int_0^{t^*_0} \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds
\]

\[
= \mu_0 \max \left\{ 0, \int_{t^*_0}^{t^*} \Phi(s) f \left( s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n} \right) ds + \frac{1}{\mu_0} y_0'(t^*_0) \right\}
\]

\[
\geq \mu_0 \max \left\{ 0, \int_{t^*_0}^{t^*} \Phi(s) \beta(s) \left( y_0(s) + \frac{s}{n} \right)^{\gamma} ds + \frac{1}{\mu_0} y_0'(t^*_0) \right\}
\]

\[
= \mu_0 \int_{t^*_0}^{t^*} \Phi(s) \beta(s) \left( y_0(s) + \frac{s}{n} \right)^{\gamma} ds + y_0'(t^*_0) > 0.
\]

This is a contradiction.

Consequently, \( t^* = T \) and (3.4) is true. Since \( y_0(0) = 0 \), one has \( y_0(t) > 0 \) for all \( t \in (0, T] \). And by direct differentiation, (3.4) yields

\[
\begin{align*}
\left\{ \begin{array}{ll}
y_0''(t) = \mu_0 \Phi(t) f(t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n}), & t \in (0, T), \\
y_0(0) = 0, & y_0'(0) = 0.
\end{array} \right.
\end{align*}
\]

(3.6)
Therefore,

\[
y_0''(t) = \mu_0 \Phi(t) f \left( t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n} \right) \\
\leq \Phi(t) \left| f \left( t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n} \right) \right| \\
\leq \Phi(t) h \left( y_0(t) + \frac{t}{n} \right) \left( g \left( y_0'(t) + \frac{1}{n} \right) + r \left( y_0'(t) + \frac{1}{n} \right) \right), \quad \forall t \in (0, T),
\]

which means that

\[
\frac{y_0''(t)(y_0'(t) + \frac{1}{n})}{g(y_0'(t) + \frac{1}{n}) + r(y_0'(t) + \frac{1}{n})} \leq \Phi(t) h \left( y_0(t) + \frac{t}{n} \right) \left( y_0'(t) + \frac{1}{n} \right), \quad \forall t \in (0, T).
\]

Integration from 0 to \( t \) yields

\[
I \left( y_0'(t) + \frac{1}{n} \right) \leq I(\varepsilon) + \int_0^t \Phi(s) h \left( y_0(s) + \frac{s}{n} \right) d \left( y_0(s) + \frac{s}{n} \right) \\
\quad \leq |\Phi|_0 \int_0^{y_0(t) + \frac{1}{n}} h(x) dx + I(\varepsilon).
\]

Thus

\[
y_0'(t) \leq I^{-1} \left( |\Phi|_0 \int_0^{y_0(t) + \frac{1}{n}} h(x) dx + I(\varepsilon) \right) \\
\leq I^{-1} \left( |\Phi|_0 \int_0^{R_1 + \varepsilon} h(x) dx + I(\varepsilon) \right), \quad t \in (0, T).
\]

Integration from 0 to \( T \) yields

\[
y_0(T) - y_0(0) = y_0(T) \leq \int_0^T I^{-1} \left( |\Phi|_0 \int_0^{R_1 + \varepsilon} h(x) dx + I(\varepsilon) \right) dt \\
= TI^{-1} \left( |\Phi|_0 \int_0^{R_1 + \varepsilon} h(x) dx + I(\varepsilon) \right)
\]

Then we have

\[
R_1 = \|y_0\| \leq \max\{1, T\} I^{-1} \left( |\Phi|_0 \int_0^{R_1 + \varepsilon} h(x) dx + I(\varepsilon) \right),
\]

which means that

\[
\frac{R_1}{\max\{1, T\} I^{-1} (|\Phi|_0 \int_0^{R_1 + \varepsilon} h(x) dx + I(\varepsilon))} \leq 1.
\]
This is a contradiction to (3.2). Thus (3.3) is true.

From Lemma 2.1, for each \( n \in N_0 \), we have

\[
i(A_n, P \cap \Omega_1, P) = 1.
\]

(3.7)

As a result, for each \( n \in N_0 \), there exists an \( y_n \in P \cap \Omega_1 \) such that

\[
y_n = A_n y_n, \quad i.e.,
\]

\[
y_n(t) = (A_n y_n)(t)
\]

\[
= \int_0^t \max \left\{ 0, \int_0^s \Phi(\tau) f \left( \tau, y_n(\tau) + \frac{\tau}{n}, y_n'(\tau) + \frac{1}{n} \right) d\tau \right\} d\tau, \quad t \in [0, T].
\]

A similar argument to show (3.4) yields

\[
y_n'(t) > 0, \quad \text{and} \quad y_n'(t) = \int_0^t \Phi(s) f \left( s, y_n(s) + \frac{s}{n}, y_n'(s) + \frac{1}{n} \right) ds, \quad t \in (0, T), \quad n \in N_0.
\]

Now we consider \( \{y_n\}_{n \in N_0} \). Since \( \|y_n\| \leq R_1 \), obviously

the functions belonging to \( \{y_n(t)\} \) are uniformly bounded on \([0, T]\)  

(3.8)

and

the functions belonging to \( \{y_n'(t)\} \) are uniformly bounded on \([0, T]\).  

(3.9)

And moreover, (3.9) guarantees that

the functions belonging to \( \{y_n(t)\} \) are equicontinuous on \([0, T]\).  

(3.10)

A similar argument to show (3.6) yields that

\[
\begin{cases}
y''_n(t) = \Phi(t) f \left( t, y_n(t) + \frac{t}{n}, y'_n(t) + \frac{1}{n} \right), & t \in (0, T), \\
y_n(0) = 0, \quad y'_n(0) = 0.
\end{cases}
\]

And then,

\[
y_n''(t) = \Phi(t) f \left( t, y_n(t) + \frac{t}{n}, y'_n(t) + \frac{1}{n} \right)
\]

\[
\leq \Phi(t) \left| f \left( t, y_n(t) + \frac{t}{n}, y'_n(t) + \frac{1}{n} \right) \right|
\]

\[
\leq \Phi(t) h \left( y_n(t) + \frac{t}{n} \right) \left( g \left( y_n(t) + \frac{1}{n} \right) + r \left( y'_n(t) + \frac{1}{n} \right) \right), \quad \forall t \in (0, T),
\]

(3.11)
which means that
\[
\frac{(y_n'(t) + \frac{1}{n})(y_n'(t) + \frac{1}{n})}{g(y_n'(t) + \frac{1}{n}) + r(y_n'(t) + \frac{1}{n})} \leq \Phi(t)h \left( y_n(t) + \frac{1}{n} \right) \left( y_n'(t) + \frac{1}{n} \right), \quad \forall t \in (0, T).
\]
(3.12)

Therefore, for any \( t_1, t_2 \in [0, T], t_1 < t_2, \) one has
\[
I \left( y_n'(t_2) + \frac{1}{n} \right) - I \left( y_n'(t_1) + \frac{1}{n} \right) \leq \Phi[0, \int_{y_n(t)}^{y_n(t) + \frac{1}{n}} h(x)dx].
\]
(3.13)

On the other hand
\[
-y_n''(t) = -\Phi(t)f \left( t, y_n(t) + \frac{1}{n}, y_n'(t) + \frac{1}{n} \right) \\
\leq \Phi(t) \left| f \left( t, y_n(t) + \frac{1}{n}, y_n'(t) + \frac{1}{n} \right) \right| \\
\leq \Phi(t)h \left( y_n(t) + \frac{1}{n} \right) \left( g \left( y_n'(t) + \frac{1}{n} \right) + r \left( y_n'(t) + \frac{1}{n} \right) \right), \quad \forall t \in (0, T).
\]
(3.14)

Therefore, for any \( t_1, t_2 \in [0, T], t_1 < t_2, \) one has
\[
I \left( y_n'(t_1) + \frac{1}{n} \right) - I \left( y_n'(t_2) + \frac{1}{n} \right) \leq |\Phi[0, \int_{y_n(t)}^{y_n(t) + \frac{1}{n}} h(x)dx].
\]
(3.15)

(3.13) and (3.15) imply that
\[
I \left( y_n'(t_1) + \frac{1}{n} \right) - I \left( y_n'(t_2) + \frac{1}{n} \right) \leq |\Phi[0, \int_{y_n(t)}^{y_n(t) + \frac{1}{n}} h(x)dx],
\]
which together with (3.10) implies that

the functions belonging to \( \left\{ I \left( y_n'(t) + \frac{1}{n} \right) \right\} \) are equicontinuous on \([0, T]\.\)
(3.16)

Since \( I^{-1} \) are uniformly continuous on \([0, I(R_1 + \varepsilon)], \) for any \( \varepsilon > 0, \) there is a \( \varepsilon' > 0 \) such that
\[
|I^{-1}(s_1) - I^{-1}(s_2)| < \varepsilon, \quad \forall |s_1 - s_2| < \varepsilon', s_1, s_2 \in [0, I(R_1 + \varepsilon)].
\]
(3.17)
From (3.16), for \( \varepsilon' > 0 \), there is a \( \delta' > 0 \) such that
\[
|I\left(y'_n(t_1) + \frac{1}{n}\right) - I\left(y'_n(t_2) + \frac{1}{n}\right)| < \varepsilon', \quad \forall|t_1 - t_2| < \delta', \ t_1, t_2 \in [0, T].
\]
(3.17) and (3.18) yield that
\[
|y'_n(t_1) - y'_n(t_2)| = \left|\frac{1}{n} - \left(y'_n(t_2) + \frac{1}{n}\right)\right|
= \left|I^{-1}\left(I\left(y'_n(t_1) + \frac{1}{n}\right)\right) - I^{-1}\left(I\left(y'_n(t_2) + \frac{1}{n}\right)\right)\right|
< \bar{\varepsilon}, \quad \forall|t_1 - t_2| < \delta', \ t_1, t_2 \in [0, T], n \in N_0,
\]
which means that
\[
\text{the functions belonging to } \{y'_n(t)\} \text{ are equicontinuous on } [0, T]. \quad (3.19)
\]
Consequently, from (3.8), (3.9), (3.10) and (3.19), the Arzela-Ascoli Theorem guarantees that \( \{y_n(t)\} \) and \( \{y'_n(t)\} \) are relatively compact in \( C[0, T] \), i.e., there is a function \( y_0 \in C^1[0, T] \), and a subsequence \( \{y_{n_j}\} \) of \( \{y_n\} \) such that
\[
\lim_{j \to +\infty} \max_{t \in [0, T]} |y_{n_j}(t) - y_0(t)| = 0, \quad \lim_{j \to +\infty} \max_{t \in [0, T]} |y'_{n_j}(t) - y'_0(t)| = 0.
\]
Since \( y_{n_j}(0) = 0 \) and \( y'_{n_j}(0) = 0 \), \( y_{n_j}(t) > 0, \ y'_{n_j}(t) > 0 \), \( t \in (0, T), j \in \{1, 2, \cdots\} \), one has
\[
y_0(0) = 0, \ y'_0(0) = 0, \ y_0(t) \geq 0, \ y'_0(t) \geq 0, \ \forall \ t \in (0, T). \quad (3.20)
\]
Following we show that \( y'_0(t) > 0, \ t \in (0, T) \). By the continuity of \( y'_0(t) \) at \( t = 0 \), there is a \( t_0 < T \) such that \( y'_0(t) \leq \frac{1}{2}\delta \) for all \( t \in [0, t_0] \). By
\[
\lim_{j \to +\infty} \max_{t \in [0, T]} |y'_{n_j}(t) - y'_0(t)| = 0,
\]
there is a \( j_0 > 0 \) such that \( 0 < y'_{n_j}(t) + \frac{1}{n_j} \leq \delta \) for all \( t \in [0, t_0], j \geq j_0 \). Thus, \((H_4)\) implies
\[
f\left(t, y_{n_j}(t) + \frac{t}{n_j}, y'_{n_j}(t) + \frac{1}{n_j}\right) \geq \beta(t) \left(y_0(t) + \frac{t}{n}\right) ^\gamma, \quad t \in [0, t_0],
\]
which yields
\[
y'_{n_j}(t) = \int_0^t \Phi(s) f\left(s, y_{n_j}(s) + \frac{s}{n_j}, y'_{n_j}(s) + \frac{1}{n_j}\right) ds
\geq \int_0^t \Phi(s) \beta(s) \left(y_0(s) + \frac{s}{n}\right) ^\gamma ds > 0, \quad t \in [0, t_0], j \in \{j_0, j_0 + 1, \cdots\}.\]
Therefore, $y_0'(t) \geq \int_0^t \Phi(s)\beta(s)ds > 0, \; t \in [0, t_0]$. Let $t^* = \sup \{t \in (0, T) | y_0'(s) > 0 \text{ for all } s \in (0, t]\}$. We claim that $t^* = T$.

Suppose that $t^* < T$, which means $y_0'(t^*) = 0$ and $y_0'(t) > 0$ for all $t \in (0, t^*)$. The continuity of $y_0'(t)$ at $t^*$ implies that there is a $0 < t_0^* < t^*$ such that $0 < y_0'(t) \leq \frac{1}{2}\delta$ for all $t \in [t_0^*, t^*]$. And the uniform convergence of \{\{y_{n_j}'(t)\}\} on $[0, T]$ guarantees that there is a $j_0 > 0$ such that $0 < y_{n_j}'(t) + \frac{1}{n_j} \leq \delta, \; t \in [t_0^*, t^*], \; j \geq j_0$. Therefore, for $t \in [t_0^*, t^*],$

$$f \left( t, y_{n_j}(t) + \frac{t}{n_j}, y_{n_j}'(t) + \frac{1}{n_j} \right) \geq \beta(t), \; t \in [t_0^*, t^*],$$

which implies that

$$0 = y_{n_j}'(t^*) = \int_0^{t^*} \Phi(s) f \left( s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j} \right) ds$$

$$= \int_0^{t_0^*} \Phi(s) f \left( s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j} \right) ds$$

$$+ \int_0^{t_0^*} \Phi(s) f \left( s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j} \right) ds$$

$$\geq \int_0^{t_0^*} \Phi(s)\beta(s)ds + \int_0^{t_0^*} \Phi(s) f \left( s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j} \right) ds$$

$$= y_{n_j}'(t_0^*) + \int_0^{t^*} \Phi(s)\beta(s)ds > 0, \; j \in \{j_0, j_0 + 1, \cdots \}.$$

Letting $j \to +\infty$, one has $0 = y_0'(t^*) \geq y_0'(t_0^*) + \int_0^{t_0^*} \Phi(s)\beta(s)ds$, a contradiction.

Consequently, $t^* = T$ and $y_0'(t) > 0$ for all $t \in (0, T)$. In addition to $y_0(0) = 0$, one has $y_0(t) > 0$ for all $t \in (0, T]$. Therefore,

$$\min_{s \in [\frac{T}{2}, T]} y_0'(s) > 0, \; \text{for all } t \in \left[ \frac{T}{2}, T \right] \quad \text{and} \quad \min_{s \in [t, \frac{T}{2}]} y_0'(s) > 0, \; \text{for all } t \in \left( 0, \frac{T}{2} \right].$$

Since

$$y_{n_j}'(t) - y_{n_j}' \left( \frac{T}{2} \right) = \int_{\frac{T}{2}}^t \Phi(s) f \left( s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j} \right) ds, \; t \in (0, T),$$

letting $j \to +\infty$, the Lebesgue Dominated Convergence Theorem guarantees that

$$y_0(t) - y_0 \left( \frac{T}{2} \right) = \int_{\frac{T}{2}}^t \Phi(s) f(s, y_0(s), y_0'(s))ds, \; t \in (0, T).$$

By direct differentiation, we have

$$y_0''(t) = \Phi(t)f(t, y_0(t), y_0'(t)), \; t \in (0, T).$$
Consider the initial value problem

Example 3.1 Consider the initial value problem

\[
\begin{align*}
    y''(t) &= \mu \cos \frac{t}{a} - (y'(t))^{e} + (y'(t))^{-a}[1 + y^b], t \in (0, T), \\
    y(0) &= a, y(0) = 0
\end{align*}
\]

with \(a > 0, e \geq 0, b \geq 0\) and \(\mu > 0\). Then there is a \(\mu_0 > 0\) such that (3.21) has at least one nonnegative solution \(y_0 \in C^1[0, T] \cap C^2(0, T)\) with \(y_0(t) > 0\) on \((0, T)\) for all \(0 < \mu < \mu_0\).

To see that (3.21) has at least one nonnegative solution, we will apply Theorem 3.1 with \(\Phi(t) \equiv \mu, g(y') = (y')^{-a}, r(y') = 1 + (y')^c, h(y) = 1 + y^h\). Clearly, \((H_1), (H_2)\) and \((H_3)\) \((\beta(t) \equiv 1)\) and \(\delta = (1/3)^{1/a}\) hold. Since \(\lim_{z \to 0^+} I^{-1}(z) = 0\), there is a \(\mu_0 > 0\) such that

\[
\frac{1}{\max\{1, T\} I^{-1}(2\mu_0)} \geq 1,
\]

and so

\[
\sup_{c \in (0, +\infty)} \frac{c}{\max\{1, T\} I^{-1}(\mu |\Phi|_0 \int_0^c h(x)dx)} \geq \sup_{c \in (0, +\infty)} \frac{c}{\max\{1, T\} I^{-1}(\mu(1 + c^h))} > 1, \forall 0 < \mu < \mu_0,
\]

which guarantees that \((H_3)\) holds.

4. Singularities at \(y' = 0\) and \(y = 0\)

In this section our nonlinearity \(f\) may be singular at \(y' = 0\) and \(y = 0\). Throughout this section we will assume that the following conditions hold:

\((P_1)\) \(\Phi \in C[0, T]\), with \(\Phi(t) > 0\) on \((0, T)\):

\((P_2)\) \(f : [0, T] \times (0, +\infty) \times (0, +\infty) \to R\) is continuous with \(|f(t, x, y)| \leq |h(x)w(x)|(|g(y)|r(y))\) on \([0, T] \times (0, +\infty) \times (0, +\infty)\) with \(w(x) > 0, g(y) > 0\) continuous and nonincreasing on \((0, +\infty)\) and \(w \in L^1[0, T], h(x) \geq 0, r(y) \geq 0\) continuous and nondecreasing on \((0, +\infty)\);

\((P_3)\) there is a \(\beta \in C((0, T), (0, +\infty))\) and a constant \(\delta > 0\) such that \(f(t, x, y) \geq \beta(t), \forall (t, x, y) \in (0, T) \times (0, +\infty) \times (0, \delta)\):

\((P_4)\)

\[
\sup_{c \in (0, +\infty)} \frac{c}{\max\{1, T\} I^{-1}(ch(c)|\Phi|_0 + |\Phi|_0 \int_0^c w(s)ds)} > 1,
\]

where \(I(z) = \int_0^z \frac{u}{g(u) + r(u)}du, \ z \in (0, +\infty), |\Phi|_0 = \max_{t \in [0, T]} |\Phi(t)|.\)
For \( y \in P \) and each \( n \in \{1, 2, \cdots \} \), define operators by
\[
(A_n y)(t) = \int_0^t \max \left\{ 0, \int_0^s \Phi(\tau) f(\tau, y(\tau), y(\tau), y(\tau)) + \frac{1}{n} \tau + \frac{1}{n} y(\tau) + \frac{1}{n} d\tau \right\} ds, \quad \forall t \in [0, T].
\]

(4.1)

**Theorem 4.1** Suppose \((P_1) - (P_4)\) hold. Then equation (1.1) has at least one nonnegative solution \( y_0 \in C^1[0, T] \cap C^2(0, T) \) with \( y_0(t) > 0 \) on \((0, T)\).

**Proof.** From \((P_4)\), choose \( R_1 > 0 \) such that
\[
\frac{R_1}{\max\{1, T\} I^{-1}(R_1 h(R_1) + \Phi_0 + \int_0^{R_1} w(s) ds)} > 1.
\]

Since \( I^{-1}, I, h \) and \( \int_0^s h(u) du \) are continuous, we choose a \( \frac{R_1}{2} > \varepsilon > 0 \) small enough such that
\[
\frac{R_1}{\max\{1, T\} I^{-1}((R_1 + \varepsilon) h(R + \varepsilon) + \Phi_0 + \int_0^{R_1 + \varepsilon} w(s) ds + I(\varepsilon))} > 1.
\]

(4.2)

Let \( n_0 \in \{1, 2, \cdots \} \) be chosen so that \( \frac{1}{n_0} < \delta/2 \) and \( \frac{1+T}{n_0} < \varepsilon \). Let \( N_0 = \{n_0, n_0 + 1, \cdots \} \).

From \((P_1)\) and \((P_2)\), Lemma 2.2 guarantees that for each \( n \in N_0 \), \( A_n : P \to P \) is continuous and completely continuous.

Now let
\[
\Omega_1 = \{y \in C^1[0, T] : \|y\| < R_1\}.
\]

We show that
\[
y \neq \mu A_n y, \quad \forall y \in P \cap \partial \Omega_1, \quad \mu \in (0, 1], \quad n \in N_0.
\]

(4.3)

Suppose there exist a \( y_0 \in P \cap \partial \Omega_1 \) and a \( \mu_0 \in (0, 1] \) such that \( y_0 = \mu_0 A_n y_0 \), i.e.,
\[
y_0(t) = \mu_0 \int_0^t \max \left\{ 0, \int_0^s \Phi(\tau) f(\tau, y_0(\tau), y_0(\tau), y_0(\tau)) + \frac{1}{n} \tau + \frac{1}{n} y_0(\tau) + \frac{1}{n} d\tau \right\} ds, \quad t \in [0, T],
\]

which yields
\[
y_0'(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n} y_0(s) + \frac{1}{n} \right) ds \right\}, \quad t \in [0, T].
\]

Obviously, \( y_0'(t) \geq 0, \quad t \in (0, T) \) and \( \lim_{t \to 0^+} y_0'(t) = 0 \). Then, since \( \frac{1}{n} < \delta/2 \), there is a \( t_0 > 0 \) such that \( 0 \leq y_0'(t) + \frac{1}{n} \leq \delta \) for all \( t \in (0, t_0] \). From \((P_3)\), one has \( f(t, y_0(t) + \frac{1}{n} t + \frac{1}{n}, y_0'(t) + \frac{1}{n}) \geq \beta(t) > 0 \) for all \( t \in (0, t_0] \), which implies that
\[
\max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n} y_0(s) + \frac{1}{n} \right) ds \right\} \geq \int_0^t \Phi(s) \beta(s) ds > 0, \quad t \in (0, t_0]
\]
and

\[ y'(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y'(s) + \frac{1}{n} \right) ds \right\} \]

\[ \geq \mu_0 \max \left\{ 0, \int_0^t \Phi(s) \beta(s) ds \right\} \]

\[ = \mu_0 \int_0^t \Phi(s) \beta(s) ds > 0, \quad t \in (0, t_0]. \]

Let \( t^* = \sup \{ t \in (0, T] | y'_0(s) > 0 \text{ for all } s \in (0, t] \} \). We claim that \( t^* = T \), which means that \( y'_0(t) > 0 \) for all \( t \in (0, T) \), and so

\[ y'_0(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y'_0(s) + \frac{1}{n} \right) ds \right\} > 0, \quad \forall t \in (0, T). \]

Hence

\[ y'(t) = \mu_0 \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y'(s) + \frac{1}{n} \right) ds \]

\[ = \mu_0 \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y'_0(s) + \frac{1}{n} \right) ds, \quad t \in (0, T). \tag{4.4} \]

Suppose that \( t^* < T \), then \( y'_0(t^*) = 0 \) and \( y'_0(t) > 0 \) for all \( t \in (0, t^*) \) and

\[ 0 < y'_0(t) = \mu_0 \max \left\{ 0, \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y'_0(s) + \frac{1}{n} \right) ds \right\} \]

\[ = \mu_0 \int_0^t \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y'_0(s) + \frac{1}{n} \right) ds, \quad t \in (0, t^*_0). \tag{4.5} \]

Hence, since \( \frac{1}{n} < \delta/2 \), there is a \( 0 < t^*_0 < t^* \) such that \( 0 < y'_0(t) + \frac{1}{n} \leq \delta \) for all \( t \in [t^*_0, t^*] \), which implies that \( f(t, y_0(t) + \frac{1}{n} t + \frac{1}{n}, y'_0(t) + \frac{1}{n}) \geq \beta(t) \) for all \( t \in [t^*_0, t^*] \). Thus, from (4.5), we have

\[ y'_0(t^*_0) = \mu_0 \int_0^{t^*_0} \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y'_0(s) + \frac{1}{n} \right) ds \]
\[0 = y_0'(t^*)\]
\[= \mu_0 \max \left\{ 0, \int_0^{t^*} \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y_0'(s) + \frac{1}{n} \right) ds \right\}\]
\[= \mu_0 \max \left\{ 0, \int_0^{t^*} \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y_0'(s) + \frac{1}{n} \right) ds \right\}
+ \int_0^{t^*} \Phi(s) f \left( s, y_0(s) + \frac{1}{n} s + \frac{1}{n}, y_0'(s) + \frac{1}{n} \right) ds\]
\[\geq \mu_0 \max \left\{ 0, \int_0^{t^*} \Phi(s) \beta(s) ds + \frac{1}{\mu_0} y_0'(t^*_0) \right\}\]
\[= \mu_0 \int_{t^*_0}^{t^*} \Phi(s) \beta(s) ds + y_0'(t^*_0) > 0.\]

This is a contradiction. Consequently, \(t^* = T\) and (4.4) is true.

Since \(y_0'(0) = 0\), one has \(y_0(t) > 0\) for all \(t \in (0, T)\). By direct differentiation, we have
\[
\begin{cases}
y_0''(t) = \mu_0 \Phi(t) f(t, y_0(t) + \frac{1}{n} t + \frac{1}{n}, y_0'(t) + \frac{1}{n}), & t \in (0, T), \\
y_0(0) = 0, & y_0'(0) = 0.
\end{cases}
\tag{4.6}
\]

Therefore,
\[
y_0''(t) = \mu_0 \Phi(t) f \left( t, y_0(t) + \frac{1}{n} t + \frac{1}{n}, y_0'(t) + \frac{1}{n} \right)
\leq \Phi(t) \left| f \left( t, y_0(t) + \frac{1}{n} t + \frac{1}{n}, y_0'(t) + \frac{1}{n} \right) \right|
\leq \Phi(t) \left[ h \left( y_0(t) + \frac{1}{n} t + \frac{1}{n} \right) + w \left( y_0(t) + \frac{1}{n} t + \frac{1}{n} \right) \right]
\cdot \left[ g \left( y_0'(t) + \frac{1}{n} \right) + r \left( y_0'(t) + \frac{1}{n} \right) \right], \quad \forall t \in (0, T),
\]
which means that
\[
\frac{y_0''(t)}{g(y_0'(t) + \frac{1}{n}) + r(y_0'(t) + \frac{1}{n})}
\leq \Phi(t) \left[ h \left( y_0(t) + \frac{1}{n} t + \frac{1}{n} \right) + w \left( y_0(t) + \frac{1}{n} t + \frac{1}{n} \right) \right], \quad \forall t \in (0, T)\]
and

\[
\frac{(y'_0(t) + \frac{1}{n})y''_0(t)}{g(y'_0(t) + \frac{1}{n}) + r(y'_0(t) + \frac{1}{n})} \leq \Phi(t) \left[ h \left( y_0(t) + \frac{1}{n} t + \frac{1}{n} \right) + w \left( y_0(t) + \frac{1}{n} t + \frac{1}{n} \right) \right] \left( y'_0(t) + \frac{1}{n} \right), \quad \forall t \in (0, T).
\]

Integration from 0 to \( t \) yields

\[
I \left( y'_0(t) + \frac{1}{n} \right) - I(\frac{1}{n}) \leq |\Phi|_0 \left[ h(R_1 + \varepsilon)(R_1 + \varepsilon) + \int_0^T w(y_0(s) + \frac{1}{n} s + \frac{1}{n}) \right]
\]

\[
\leq |\Phi|_0 [h(R_1 + \varepsilon)(R_1 + \varepsilon) + \int_0^{R_1 + \varepsilon} w(s) ds],
\]

and so

\[
I \left( y'_0(t) + \frac{1}{n} \right) \leq I(\varepsilon) + |\Phi|_0 \left[ h(R_1 + \varepsilon)(R_1 + \varepsilon) + \int_0^{R_1 + \varepsilon} w(s) ds \right].
\]

Thus

\[
y'_0(t) \leq I^{-1} \left( I(\varepsilon) + |\Phi|_0 \left[ h(R_1 + \varepsilon)(R_1 + \varepsilon) + \int_0^{R_1 + \varepsilon} w(s) ds \right] \right), \quad \forall t \in [0, T].
\]

Integrate from 0 to \( T \) to obtain

\[
y_0(T) = y_0(T) - y_0(0) \leq I^{-1} \left( I(\varepsilon) + |\Phi|_0 \left[ h(R_1 + \varepsilon)(R_1 + \varepsilon) + \int_0^{R_1 + \varepsilon} w(s) ds \right] \right) T.
\]

(4.7) and (4.8) guarantee that

\[
R_1 = \|y_0\| \leq \max\{1, T\} I^{-1} \left( I(\varepsilon) + |\Phi|_0 \left[ h(R_1 + \varepsilon)(R_1 + \varepsilon) + \int_0^{R_1 + \varepsilon} w(s) ds \right] \right),
\]

which means

\[
\frac{R_1}{\max\{1, T\} I^{-1} (I(\varepsilon) + |\Phi|_0 [h(R_1 + \varepsilon)(R_1 + \varepsilon) + \int_0^{R_1 + \varepsilon} w(s) ds])} \leq 1.
\]

This is a contradiction to (4.2). Thus (4.3) is true.

From Lemma 2.1, for each \( n \in N_0 \), we have

\[
i(A_n, P \cap \Omega_1, P) = 1.
\]

(4.9)
As a result, for each \( n \in N_0 \), there exists a \( y_n \in P \cap \Omega_1 \) such that \( y_n = A_n y_n \), i.e.,

\[
y_n(t) = (A_n y_n)(t)
\]

\[
= \int_0^t \max \left\{ 0, \int_0^s \Phi(\tau)f \left( \tau, y_n(\tau) + \frac{1}{n}\tau + \frac{1}{n}, y_n'(\tau) + \frac{1}{n} \right) d\tau \right\} ds, \quad t \in [0, T].
\]

A similar to show (4.4) yields that

\[
y_n'(t) > 0, \quad y_n'(t) = \int_0^t \Phi(s)f(s, y_n(s) + \frac{1}{n}s + \frac{1}{n}, y_n'(s) + \frac{1}{n}) ds, \quad t \in (0, T), \quad n \in N_0.
\]

Now we consider \( \{y_n\}_{n \in N_0} \). Since \( \|y_n\| \leq R_1 \), obviously the functions belonging to \( \{y_n(t)\} \) are uniformly bounded on \([0, T]\) (4.10) and the functions belonging to \( \{y_n'(t)\} \) are uniformly bounded on \([0, T]\). (4.11)

And moreover, (4.11) yields that the functions belonging to \( \{y_n(t)\} \) are equicontinuous on \([0, T]\). (4.12)

Similarly as (4.6), one has

\[
\left\{ \begin{array}{l}
y_n''(t) = \Phi(t)f(t, y_n(t) + \frac{1}{n}t + \frac{1}{n}, y_n'(t) + \frac{1}{n}), \quad t \in (0, T) \\
y_n(0) = 0, \quad y_n'(0) = 0.
\end{array} \right.
\]

Then,

\[
\pm y_n''(t) = \pm \Phi(t)f \left( t, y_n(t) + \frac{1}{n}t + \frac{1}{n}, y_n'(t) + \frac{1}{n} \right)
\]

\[
\leq \Phi(t) \left| f \left( t, y_n(t) + \frac{1}{n}t + \frac{1}{n}, y_n'(t) + \frac{1}{n} \right) \right|
\]

\[
\leq \Phi(t) \left[ h \left( y_n(t) + \frac{1}{n}t + \frac{1}{n} \right) + w \left( y_n(t) + \frac{1}{n}t + \frac{1}{n} \right) \right]
\]

\[
\cdot \left[ g \left( y_n'(t) + \frac{1}{n} \right) + r \left( y_n'(t) + \frac{1}{n} \right) \right], \quad \forall t \in (0, T),
\]

which means that

\[
\frac{\pm (y_n'(t) + \frac{1}{n}) y_n''(t)}{g(y_n'(t) + \frac{1}{n}) + r(y_n'(t) + \frac{1}{n})}
\]

\[
\leq \Phi(t)[h(y_n(t) + \frac{1}{n}t + \frac{1}{n}) + w(y_n(t) + \frac{1}{n}t + \frac{1}{n})](y_n'(t) + \frac{1}{n}), \quad \forall t \in (0, T). \tag{4.13}
\]
For any \( t_1, t_2 \in [0, T] \), integration from \( t_1 \) to \( t_2 \) yields that
\[
\begin{align*}
\left| I \left( y_n'(t_1) + \frac{1}{n} \right) - I \left( y_n'(t_2) + \frac{1}{n} \right) \right| \\
\leq |\Phi_0| \int_{t_1}^{t_2} \left[ h(y_n(s) + \frac{1}{n} s + \frac{1}{n}) + w(y_n(s) + \frac{1}{n} s + \frac{1}{n}) \right] d(y_n(s) + \frac{1}{n} s + \frac{1}{n}) \\
= |\Phi_0| \int_{y_n(t_1)}^{y_n(t_2) + \frac{1}{n} t_2 + \frac{1}{n}} [h(s) + w(s)] ds.
\end{align*}
\]

Since \( I^{-1} \) is uniformly continuous on \([0, I(R_1 + \varepsilon)]\), for any \( \varepsilon' > 0 \), there is a \( \delta' > 0 \) such that
\[
\left| I^{-1}(s_1) - I^{-1}(s_2) \right| < \varepsilon', \quad \forall|s_1 - s_2| < \delta', \quad s_1, s_2 \in [0, I(R_1 + \varepsilon)]. \tag{4.14}
\]

Since \( \int_0^\varepsilon (h(s) + w(s))ds \) is uniformly continuous on \([0, R_1 + \varepsilon]\), there is a \( \delta'' > 0 \) such that
\[
\left| \int_{u_1}^{u_2} (h(s) + w(s))ds \right| < \frac{\delta'}{|\Phi_0|}, \quad \forall|u_1 - u_2| < \delta'', \quad u_1, u_2 \in [0, R_1 + \varepsilon]. \tag{4.15}
\]

From (4.12), there is a \( \tilde{\delta} > 0 \) such that
\[
\left| (y_n(t_1) + t_1) - (y_n(t_2) + t_2) \right| < \delta'', \quad \forall|t_1 - t_2| < \tilde{\delta}, \quad t_1, t_2 \in [0, T]. \tag{4.16}
\]

(4.15) and (4.16) yield that
\[
\begin{align*}
\left| I \left( y_n'(t_1) + \frac{1}{n} \right) - I \left( y_n'(t_2) + \frac{1}{n} \right) \right| \\
\leq |\Phi_0| \int_{y_n(t_1)}^{y_n(t_2) + \frac{1}{n} t_2 + \frac{1}{n}} [h(s) + w(s)] ds \\
< |\Phi_0| \frac{\delta'}{|\Phi_0|} = \delta', \quad |t_1 - t_2| < \tilde{\delta}, \quad t_1, t_2 \in [0, T], \quad n \in N_0. \tag{4.17}
\end{align*}
\]

From (4.14) and (4.17), we have
\[
\left| y_n'(t_1) - y_n'(t_2) \right| = \left| y_n'(t_1) + \frac{1}{n} - \left( y_n'(t_2) + \frac{1}{n} \right) \right| \\
= \left| I^{-1} \left( I \left( y_n'(t_1) + \frac{1}{n} \right) \right) - I^{-1} \left( I \left( y_n'(t_2) + \frac{1}{n} \right) \right) \right| \\
< \varepsilon', \quad \forall|t_1 - t_2| < \tilde{\delta}, \quad t_1, t_2 \in [0, T], \quad n \in N_0,
\]

which means that
\[
\text{the functions belonging to \{y_n'(t)\} are equicontinuous on [0, T].} \tag{4.18}
\]

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Consequently, from (4.10), (4.11), (4.12) and (4.18), the Arzela-Ascoli Theorem guarantees that \( \{y_n(t)\} \) and \( \{y'_n(t)\} \) are relatively compact in \( C[0,T] \), i.e., there is a \( y_0 \in C^1[0,T] \) and a subsequence \( \{y_{n_j}\} \) of \( \{y_n\} \) such that

\[
\lim_{j \to +\infty} \max_{t \in [0,T]} |y_{n_j}(t) - y_0(t)| = 0, \quad \lim_{j \to +\infty} \max_{t \in [0,T]} |y'_{n_j}(t) - y'_0(t)| = 0.
\]

Since \( y_{n_j}(0) = 0, y'_{n_j}(0) = 0, y_{n_j}(t) > 0, y'_{n_j}(t) > 0, t \in (0,T) \),

\[
y_0(0) = 0, \quad y'_0(0) = 0, \quad y_0(t) \geq 0, \quad y'_0(t) \geq 0, \quad t \in (0,T).
\]

Following we show that \( y'_0(t) > 0, \ t \in (0,T) \). Since \( y'_0(t) \) is right continuous at \( t = 0 \) and \( y'_0(0) = 0 \), there is a \( 0 < t_0 < 1 \) such that \( y'_0(t) \leq \frac{1}{2} \delta \) for all \( t \in [0,t_0] \). By

\[
\lim_{j \to +\infty} \max_{t \in [0,T]} |y'_{n_j}(t) - y'_0(t)| = 0,
\]

there is a \( j_0 > 0 \) such that \( 0 < y'_{n_j}(t) + \frac{1}{n_j} \leq \delta \) for all \( t \in [0,t_0], \ j \geq j_0 \). Thus, \( (H_3) \) implies

\[
f \left( t, y_{n_j}(t) + \frac{1}{n_j} t + \frac{1}{n_j} y'_{n_j}(t) + \frac{1}{n_j} \right) \geq \beta(t), \quad t \in [0,t_0],
\]

which yields

\[
y'_{n_j}(t) = \int_0^t \Phi(s) f \left( s, y_{n_j}(s) + \frac{1}{n_j} s + \frac{1}{n_j} y'_{n_j}(s) + \frac{1}{n_j} \right) ds
\]

\[
\geq \int_0^t \Phi(s) \beta(s) ds > 0, \quad t \in [0,t_0], \ j \in \{j_0,j_0 + 1, \cdots\}.
\]

Therefore, \( y'_0(t) \geq \int_0^t \Phi(s) \beta(s) ds > 0, \ t \in [0,t_0] \).

Let \( t^* = \sup\{t \in (0,T)|y'_0(s) > 0 \ \text{for all} \ s \in (0,t)\} \). Then, we claim that \( t^* = T \). Suppose that \( t^* < T \), which means \( y'_0(t^*) = 0 \) and \( y'_0(t) > 0 \) for all \( t \in (0,t^*) \). The continuity of \( y'_0(t) \) at \( t^* \) guarantees that there is a \( 0 < t_0^* < t^* \) such that \( 0 < y'_0(t) \leq \frac{1}{2} \delta \) for all \( t \in [t_0^*, t^*) \). And the uniform convergence of \( \{y'_{n_j}(t)\} \) on \( [0,T] \) guarantees that there is a \( j_0 > 0 \) such that

\[
0 < y'_{n_j}(t) + \frac{1}{n_j} \leq \delta, \ t \in [t_0^*, t^*], \ j \geq j_0. \quad \text{Then, for} \ t \in [t_0^*, t^*],
\]

\[
f \left( t, y_{n_j}(t) + \frac{1}{n_j} t + \frac{1}{n_j} y'_{n_j}(t) + \frac{1}{n_j} \right) \geq \beta(t),
\]
which implies that
\[ y'_{n_j}(t^*) = \int_0^{t^*} \Phi(s) f \left( s, y_{n_j}(s) + \frac{1}{n_j} s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j} \right) ds \]
\[ = \int_0^{t^*} \Phi(s) f \left( s, y_{n_j}(s) + \frac{1}{n_j} s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j} \right) ds \]
\[ + \int_0^{t^*} \Phi(s) f \left( s, y_{n_j}(s) + \frac{1}{n_j} s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j} \right) ds \]
\[ \geq \int_0^{t^*} \Phi(s) \beta(s) ds + y'_{n_j}(t^*) \]
\[ > \int_0^{t^*} \Phi(s) \beta(s) ds, \ j \in \{ j_0, j_0 + 1, \ldots \}. \]

Hence, letting \( j \to +\infty \), we have \( y'_0(t^*) \geq \int_0^{t^*} \Phi(s) \beta(s) ds > 0 \), a contradiction.

Consequently, \( t^* = T \) and \( y'_0(t) > 0 \) for all \( t \in (0, T) \). In addition to \( y_0(0) = 0 \), one has \( y_0(t) > 0 \) for all \( t \in (0, T] \). Therefore,
\[ \min\{ \min_{s \in [t, T]} y_0(s), \min_{s \in [\frac{T}{2}, t]} y'_0(s) \} > 0, \text{ for all } t \in \left[ \frac{T}{2}, T \right], \]
and
\[ \min\{ \min_{s \in [t, \frac{T}{2}]} y_0(s), \min_{s \in [\frac{T}{2}, t]} y'_0(s) \} > 0, \text{ for all } t \in \left( 0, \frac{T}{2} \right]. \]

Since
\[ y'_{n_j}(t) - y'_{n_j} \left( \frac{T}{2} \right) = \int_{\frac{T}{2}}^t \Phi(s) f \left( s, y_{n_j}(s) + \frac{1}{n_j} s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j} \right) ds, \ t \in (0, T), \]
letting \( j \to +\infty \), the Lebesgue Dominated Convergence Theorem guarantees that
\[ y'_0(t) - y_0 \left( \frac{T}{2} \right) = \int_{\frac{T}{2}}^t \Phi(s) f(s, y_0(s), y'_0(s)) ds, \ t \in (0, T). \]

By direct differentiation, we have
\[ y''_0(t) = \Phi(t)f(t, y_0(t), y'_0(t)), \ t \in (0, T). \]

In addition to (4.19), \( y_0 \in C^1[0, T] \cap C^2(0, T) \) is a nonnegative solution to equation (1.1) with \( y_0(t) > 0 \) for all \( t \in (0, 1] \).

**Example 4.1** Consider the initial value problems
\[
\begin{align*}
\begin{cases}
  y''(t) = \mu[\sin \frac{1}{2} - (y')^c + (y')^{-a}][1 + y^b + y^{-d}], & t \in (0, T), \\
  y'(0) = 0, \ y(0) = 0
\end{cases}
\end{align*}
\]
with $a > 0$, $e \geq 0$, $b \geq 0$, $0 < d < 1$ and $\mu > 0$. Then there is a $\mu_0 > 0$ such that (4.20) has at least one nonnegative solution $y_0 \in C^1[0, T] \cap C^2(0, T)$ with $y_0(t) > 0$ on $(0, T]$ for all $0 < \mu < \mu_0$.

To see that (4.20) has at least one nonnegative solution, we will apply Theorem 4.1 with $\Phi(t) \equiv \mu$, $g(y) = (y')^{-a}$, $r(y) = 1 + (y')^e$, $h(y) = 1 + y^b$, $w(y) = y^{-d}$. Clearly, $(P_1)$, $(P_2)$ and $(P_3)$ $(\beta(t) \equiv 1$ and $\delta = (\frac{1}{3})^{1/a})$ hold. Since $\lim_{z \to 0^+} I^{-1}(z) = 0$ there is a $\mu_0 > 0$ such that

$$\frac{1}{\max\{1, T\} I^{-1}[\mu(2 + 1/(1 + d))] \geq 1}$$

and so

$$\sup_{c \in (0, +\infty)} \frac{c}{\max\{1, T\} I^{-1}(ch(c)|\Phi|_0 + |\Phi|_0 \int_0^c w(s) ds)} > 1,$$

which guarantees that $(P_4)$ holds.

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**References**


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