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Positive Solutions for Singular Initial Value Problems with Sign Changing Nonlinearities Depending on y'

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Abstract. Using the theory of fixed point index, this paper presents the existence of positive solutions for the singular second-order initial value problems, where $f(t, y, y')$ may be singular at $y = 0$ and $y' = 0$, and f may change sign.

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1. Introduction

In this paper, we consider the singular initial value problems

$$\begin{cases} y''(t) = \Phi(t)f(t, y, y'), & t \in (0, T] \\ y(0) = y'(0) = 0, \end{cases} \tag{1.1}$$

where $f(t, y, y')$ may change sign and may be singular at $y = 0$ and $y' = 0$.

When $f(t, y, y') > 0$ may be singular at $t = 0$, $y = 0$ or $y' = 0$ and suplinear at $y = +\infty$, R.P. Agarwal and D. O'Regan considered the existence of positive solutions to (1.1) in [1]. Also, in [4], H.Wang and W.Ge presented the existence of positive solutions to (1.1) by improving the work in [1] when $f(t, y, y')$ is nonnegative. In [5, 6], G.Yang considered the existence of positive solutions to (1.1) ($T = 1$) when $f(t, y, y') > 0$ is singular at $y = 0$ and $y' = 0$, but the boundedness of $f(t, y, y')$ at $+\infty$ is necessary. In this paper, $f(t, y, y')$ changes sign and may be singular at $y = 0$ and $y' = 0$ and $f(t, y, y')$ may be superlinear at $y = +\infty$.

There are main two sections in our paper. In section 3, using the theory of fixed point index on a cone (see [3]) we discuss the existence of positive solutions to (1.1) when $f(t, y, y')$ is singular at $y' = 0$ but not $y = 0$ and when f may change sign. In section 4, we discuss the existence of positive solutions to (1.1) when $f(t, y, y')$ is singular at $y' = 0$ and $y = 0$ and when f may change sign. Some ideas come from [2] and [7].

2. Preliminaries

Let

$$C^1[0, T] = \{y : [0, T] \to R \mid y(t) \text{ is continuously differentiable on } [0, T]\}$$

with norm $\| y \| = \max\{\max_{t\in[0,T]} |y(t)|, \max_{t\in[0,T]} |y'(t)|\}$ and

$$P = \{y \in C^1[0, T] : y(t) \geq 0 \text{ and } y'(t) \geq 0,\ \forall t \in [0, T]\}.$$

Obviously, $C^1[0, T]$ is a Banach space and P is a cone in $C^1[0, T]$.

The following lemma is needed later.

Lemma 2.1. *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta \in \Omega$, $\Omega \cap P$ is a relatively open set in P and $A : \bar{\Omega} \bigcap P \longrightarrow P$ be continuous and compact. Suppose*

$$\lambda Ax \neq x,\ \forall x \in \partial\Omega \cap P,\ \lambda \in (0, 1]. \tag{2.1}$$

Then

$$i(A, \Omega \cap P, P) = 1.$$

Suppose the following condition holds:

$$\Phi \in C[0, T] \cap L^1[0, T] \text{ for } t \in (0, T], \quad \text{and } f \in C([0, T] \times [0, \infty) \times [0, \infty), R). \tag{2.2}$$

For $y \in P$, define an operator by

$$(Ay)(t) = \int_0^t \max\left\{0, \int_0^s \Phi(\tau) f(\tau, y(\tau), y'(\tau)) d\tau\right\} ds,\ \forall t \in [0, T]. \tag{2.3}$$

A standard argument in the literature [1, 4] yields:

Lemma 2.2 *Suppose that (2.2) holds. Then $A : P \to P$ is continuous and completely continuous.*

3. Singularities at $y' = 0$ but not $y = 0$

In this section our nonlinearity f may be singular at $y' = 0$, but not at $y = 0$. Throughout this section we will assume that the following conditions hold:

(H_1) $\Phi \in C[0,T]$ with $\Phi(t) > 0$ on $(0,T]$;

(H_2) $f : [0,T] \times [0,+\infty) \times (0,+\infty) \to R$ is continuous with $|f(t,x,y)| \le h(x)[g(y) + r(y)]$ on $[0,T] \times [0,+\infty) \times (0,+\infty)$ with $g(y) > 0$ continuous and nonincreasing on $(0,+\infty)$, and $h(x) \ge 0$, $r(y) \ge 0$ continuous and nondecreasing on $[0,\infty)$;

(H_3)

$$\sup_{c\in(0,+\infty)} \frac{c}{\max\{1,T\} I^{-1}(|\Phi|_0 \int_0^c h(x)dx)} > 1,$$

where $I(z) = \int_0^z \frac{udu}{g(u)+r(u)}$, $z \in (0,+\infty)$, and $|\Phi|_0 = \max_{t\in[0,T]} |\Phi(t)|$;

(H_4) there is a $\beta \in C((0,T),(0,+\infty))$ and constants $\delta > 0$ and $1 > \gamma \ge 0$ such that

$$f(t,x,y) \ge \beta(t)x^\gamma, \ \ \forall(t,x,y) \in (0,T) \times [0,+\infty) \times (0,\delta].$$

For $y \in P$ and each $n \in \{1,2,\cdots\}$, define operators by

$$(A_n y)(t) = \int_0^t \max\left\{0, \int_0^s \Phi(\tau) f(\tau, y(\tau) + \frac{\tau}{n}, y'(\tau) + \frac{1}{n}) d\tau \right\} ds, \ \ \forall\, t \in [0,T]. \tag{3.1$_n$}$$

Theorem 3.1 *Suppose that $(H_1) - (H_4)$ hold. Then (1.1) has at least one nonnegative solution $y_0 \in C^1[0,T] \cap C^2(0,T)$ with $y_0(t) > 0$ on $(0,T]$.*

Proof. From (H_3), choose $R_1 > 0$ with

$$\frac{R_1}{\max\{1,T\} I^{-1}(|\Phi|_0 \int_0^{R_1} h(x)dx)} > 1.$$

From the continuity of I^{-1}, I and $\int_0^z h(u)du$, we can choose $\varepsilon > 0$ and $\varepsilon < R_1$ such that

$$\frac{R_1}{\max\{1,T\} I^{-1}(|\Phi|_0 \int_0^{R_1+\varepsilon} h(x)dx + I(\varepsilon))} > 1. \tag{3.2}$$

Let $n_0 \in \{1,2,\cdots\}$ be chosen so that $\frac{1}{n_0} < \delta/2$, $\frac{T}{n_0} < \varepsilon$ and let $N_0 = \{n_0, n_0+1, \cdots\}$. Now (H_1), (H_2) and Lemma 2.2 guarantee that for each $n \in N_0$, $A_n : P \to P$ is continuous and completely continuous. Now let

$$\Omega_1 = \{y \in C^1[0,T] : \| y \| < R_1\}.$$

We now show that

$$y \ne \mu A_n y, \forall y \in P \cap \partial\Omega_1, \ \ \mu \in (0,1], \ \ n \in N_0. \tag{3.3}$$

Suppose there exist a $y_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0,1]$ such that $y_0 = \mu_0 A_n y_0$, i.e.,

$$y_0(t) = \mu_0 \int_0^t \max\left\{0, \int_0^s \Phi(\tau) f\left(\tau, y_0(\tau) + \frac{\tau}{n}, y_0'(\tau) + \frac{1}{n}\right) d\tau\right\} ds, \ \ t \in [0,T],$$

which yields

$$y_0'(t) = \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\}, \ \ t \in [0,T].$$

Obviously, $y_0'(t) \geq 0$, $t \in (0,T]$ and $\lim_{t\to 0^+} y_0'(t) = 0$. Then, from $\frac{1}{n} < \delta/2$, there is a $t_0 > 0$ such that $0 \leq y_0'(t) + \frac{1}{n} \leq \delta$ for all $t \in (0, t_0]$. (H_4) implies $f(t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n}) \geq \beta(t)(y_0(t) + \frac{t}{n})^\gamma > 0$ for all $t \in (0, t_0]$, which means that

$$\begin{aligned}
&\max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\
&\quad \geq \max\left\{0, \int_0^t \Phi(s)\beta(s)\left(y_0(s) + \frac{s}{n}\right)^\gamma ds\right\} \\
&\quad = \int_0^t \Phi(s)\beta(s)\left(y_0(s) + \frac{s}{n}\right)^\gamma ds > 0, \ \ t \in (0, t_0].
\end{aligned}$$

and

$$\begin{aligned}
y_0'(t) &= \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\
&\geq \mu_0 \max\left\{0, \int_0^t \Phi(s)\beta(s)\left(y_0(s) + \frac{s}{n}\right)^\gamma ds\right\} \\
&= \mu_0 \int_0^t \Phi(s)\beta(s)\left(y_0(s) + \frac{s}{n}\right)^\gamma ds > 0, \ \ t \in (0, t_0].
\end{aligned}$$

Let $t^* = \sup\{t \in (0,T] \mid y_0'(s) > 0 \text{ for all } s \in (0,t]\}$. Then, we claim that $t^* = T$, which means that $y_0'(t) > 0$ for all $t \in (0,T)$, and so

$$y_0'(t) = \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} > 0, \ \ \forall t \in (0,T).$$

Hence

$$y_0'(t) = \mu_0 \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds, \ \ t \in (0,T). \tag{3.4}$$

Suppose that $t^* < T$, which means $y_0'(t^*) = 0$ and $y_0'(t) > 0$ for all $t \in (0, t^*)$ and

$$\begin{aligned} 0 < y_0'(t) &= \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\ &= \mu_0 \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds, \quad t \in (0, t^*). \end{aligned} \tag{3.5}$$

The continuity of $y_0'(t)$ at t^* and $\frac{1}{n} < \delta/2$ guarantee that there is a $t_0^* \in (0, t^*)$ such that $0 < y_0'(t) + \frac{1}{n} \le \delta$ for all $t \in [t_0^*, t^*]$, which implies that $f(t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n}) \ge \beta(t)(y_0(t) + \frac{t}{n})^\gamma$ for all $t \in [t_0^*, t^*]$. Thus, from (3.5), we have

$$y_0'(t_0^*) = \mu_0 \int_0^{t_0^*} \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds$$

and

$$\begin{aligned} 0 &= y_0'(t^*) \\ &= \mu_0 \max\left\{0, \int_0^{t^*} \Phi(s) f(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}) ds\right\} \\ &= \mu_0 \max\left\{0, \int_{t_0^*}^{t^*} \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds \right. \\ &\quad \left. + \int_0^{t_0^*} \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\ &= \mu_0 \max\left\{0, \int_{t_0^*}^{t^*} \Phi(s) f\left(s, y_0(s) + \frac{s}{n}, y_0'(s) + \frac{1}{n}\right) ds + \frac{1}{\mu_0} y_0'(t_0^*)\right\} \\ &\ge \mu_0 \max\left\{0, \int_{t_0^*}^{t^*} \Phi(s)\beta(s) \left(y_0(s) + \frac{s}{n}\right)^\gamma ds + \frac{1}{\mu_0} y_0'(t_0^*)\right\} \\ &= \mu_0 \int_{t_0^*}^{t^*} \Phi(s)\beta(s) \left(y_0(s) + \frac{s}{n}\right)^\gamma ds + y_0'(t_0^*) > 0. \end{aligned}$$

This is a contradiction.

Consequently, $t^* = T$ and (3.4) is true. Since $y_0(0) = 0$, one has $y_0(t) > 0$ for all $t \in (0, T]$. And by direct differentiation, (3.4) yields

$$\begin{cases} y_0''(t) = \mu_0 \Phi(t) f(t, y_0(t) + \frac{t}{n}, y_0'(t) + \frac{1}{n}), \ \ t \in (0, T), \\ y_0(0) = 0, \ \ y_0'(0) = 0. \end{cases} \tag{3.6}$$

Therefore,

$$\begin{aligned} y_0''(t) &= \mu_0 \Phi(t) f\left(t, y_0(t)+\frac{t}{n}, y_0'(t)+\frac{1}{n}\right) \\ &\le \Phi(t)\left|f\left(t, y_0(t)+\frac{t}{n}, y_0'(t)+\frac{1}{n}\right)\right| \\ &\le \Phi(t) h\left(y_0(t)+\frac{t}{n}\right)\left(g\left(y_0'(t)+\frac{1}{n}\right)+r\left(y_0'(t)+\frac{1}{n}\right)\right), \quad \forall t\in(0,T), \end{aligned}$$

which means that

$$\frac{y_0''(t)(y_0'(t)+\frac{1}{n})}{g(y_0'(t)+\frac{1}{n})+r(y_0'(t)+\frac{1}{n})} \le \Phi(t) h\left(y_0(t)+\frac{t}{n}\right)\left(y_0'(t)+\frac{1}{n}\right), \quad \forall t\in(0,T).$$

Integration from 0 to t yields

$$\begin{aligned} I\left(y_0'(t)+\frac{1}{n}\right) &\le I(\varepsilon)+\int_0^t \Phi(s) h\left(y_0(s)+\frac{s}{n}\right) d\left(y_0(s)+\frac{s}{n}\right) \\ &\le |\Phi|_0 \int_0^{y_0(t)+\frac{t}{n}} h(x)dx + I(\varepsilon). \end{aligned}$$

Thus

$$\begin{aligned} y_0'(t) &\le I^{-1}\left(|\Phi|_0 \int_0^{y_0(t)+\frac{t}{n}} h(x)dx + I(\varepsilon)\right) \\ &\le I^{-1}\left(|\Phi|_0 \int_0^{R_1+\varepsilon} h(x)dx + I(\varepsilon)\right), \quad t\in(0,T). \end{aligned}$$

Integration from 0 to T yields

$$\begin{aligned} y_0(T)-y_0(0)=y_0(T) &\le \int_0^T I^{-1}\left(|\Phi|_0 \int_0^{R_1+\varepsilon} h(x)dx + I(\varepsilon)\right) dt \\ &= T I^{-1}\left(|\Phi|_0 \int_0^{R_1+\varepsilon} h(x)dx + I(\varepsilon)\right) \end{aligned}$$

Then we have

$$R_1 = \|y_0\| \le \max\{1,T\} I^{-1}\left(|\Phi|_0 \int_0^{R_1+\varepsilon} h(x)dx + I(\varepsilon)\right),$$

which means that

$$\frac{R_1}{\max\{1,T\} I^{-1}(|\Phi|_0 \int_0^{R_1+\varepsilon} h(x)dx + I(\varepsilon))} \le 1.$$

This is a contradiction to (3.2). Thus (3.3) is true.

From Lemma 2.1, for each $n \in N_0$, we have

$$i(A_n, P \cap \Omega_1, P) = 1. \tag{3.7}$$

As a result, for each $n \in N_0$, there exists an $y_n \in P \cap \Omega_1$ such that $y_n = A_n y_n$, i.e. ,

$$\begin{aligned} y_n(t) &= (A_n y_n)(t) \\ &= \int_0^t \max\left\{0, \int_0^s \Phi(\tau) f\left(\tau, y_n(\tau) + \frac{\tau}{n}, y'(\tau) + \frac{1}{n}\right) d\tau\right\} ds, \ \ t \in [0, T]. \end{aligned}$$

A similar argument to show (3.4) yileds

$$y_n'(t) > 0, \ \ \text{and } y_n'(t) = \int_0^t \Phi(s) f\left(s, y_n(s) + \frac{s}{n}, y_n'(s) + \frac{1}{n}\right) ds, \ \ t \in (0, T), \ \ n \in N_0.$$

Now we consider $\{y_n\}_{n \in N_0}$. Since $\|y_n\| \leq R_1$, obviously

$$\text{the functions belonging to } \{y_n(t)\} \text{ are uniformly bounded on } [0, T] \tag{3.8}$$

and

$$\text{the functions belonging to } \{y_n'(t)\} \text{ are uniformly bounded on } [0, T]. \tag{3.9}$$

And moreover, (3.9) guarantees that

$$\text{the functions belonging to } \{y_n(t)\} \text{ are equicontnuous on } [0, T]. \tag{3.10}$$

A similar argument to show (3.6) yields that

$$\begin{cases} y_n''(t) = \Phi(t) f(t, y_n(t) + \frac{t}{n}, y_n'(t) + \frac{1}{n}), & t \in (0, T), \\ y_n(0) = 0, \ \ y_n'(0) = 0. \end{cases}$$

And then,

$$\begin{aligned} y_n''(t) &= \Phi(t) f\left(t, y_n(t) + \frac{t}{n}, y_n'(t) + \frac{1}{n}\right) \\ &\leq \Phi(t) \left| f\left(t, y_n(t) + \frac{t}{n}, y_n'(t) + \frac{1}{n}\right)\right| \\ &\leq \Phi(t) h\left(y_n(t) + \frac{t}{n}\right)\left(g\left(y_n'(t) + \frac{1}{n}\right) + r\left(y_n'(t) + \frac{1}{n}\right)\right), \ \ \forall t \in (0, T), \end{aligned} \tag{3.11}$$

which means that

$$\frac{(y_n'(t)+\frac{1}{n})'(y_n'(t)+\frac{1}{n})}{g(y_n'(t)+\frac{1}{n})+r(y_n'(t)+\frac{1}{n})} \le \Phi(t)h\left(y_n(t)+\frac{t}{n}\right)\left(y_n'(t)+\frac{1}{n}\right), \quad \forall t\in(0,T). \tag{3.12}$$

Therefore, for any $t_1,\ t_2\in[0,T]$, $t_1<t_2$, one has

$$\begin{aligned} I\left(y_n'(t_2)+\frac{1}{n}\right)-I\left(y_n'(t_1)+\frac{1}{n}\right) &\le \int_{t_1}^{t_2}\Phi(s)h\left(y_n(s)+\frac{s}{n}\right)d\left(y_n(s)+\frac{s}{n}\right)\\ &\le |\Phi|_0\int_{y_n(t_1)+\frac{t_1}{n}}^{y_n(t_2)+\frac{t_2}{n}} h(x)dx. \end{aligned} \tag{3.13}$$

On the other hand

$$\begin{aligned} -y_n''(t) &= -\Phi(t)f\left(t,y_n(t)+\frac{t}{n},y_n'(t)+\frac{1}{n}\right)\\ &\le \Phi(t)\left|f\left(t,y_n(t)+\frac{t}{n},y_n'(t)+\frac{1}{n}\right)\right|\\ &\le \Phi(t)h\left(y_n(t)+\frac{t}{n}\right)\left(g\left(y_n'(t)+\frac{1}{n}\right)+r\left(y_n'(t)+\frac{1}{n}\right)\right), \quad \forall t\in(0,T). \end{aligned} \tag{3.14}$$

Therefore, for any $t_1,\ t_2\in[0,T]$, $t_1<t_2$, one has

$$I\left(y_n'(t_1)+\frac{1}{n}\right)-I\left(y_n'(t_2)+\frac{1}{n}\right) \le |\Phi|_0\int_{y_n(t_1)+\frac{t_1}{n}}^{y_n(t_2)+\frac{t_2}{n}} h(x)dx. \tag{3.15}$$

(3.13) and (3.15) imply that

$$\left|I\left(y_n'(t_1)+\frac{1}{n}\right)-I\left(y_n'(t_2)+\frac{1}{n}\right)\right| \le |\Phi|_0\left|\int_{y_n(t_1)+\frac{t_1}{n}}^{y_n(t_2)+\frac{t_2}{n}} h(x)dx\right|,$$

which together with (3.10) implies that

$$\text{the functions belonging to } \left\{I\left(y_n'(t)+\frac{1}{n}\right)\right\} \text{ are equicontinuous on } [0,T]. \tag{3.16}$$

Since I^{-1} are uniformly continuous on $[0,I(R_1+\varepsilon)]$, for any $\tilde{\varepsilon}>0$, there is a $\varepsilon'>0$ such that

$$|I^{-1}(s_1)-I^{-1}(s_2)|<\tilde{\varepsilon}, \quad \forall |s_1-s_1|<\varepsilon', s_1,s_2\in[0,I(R_1+\varepsilon)]. \tag{3.17}$$

From (3.16), for $\varepsilon' > 0$, there is a $\delta' > 0$ such that

$$\left| I\left(y_n'(t_1) + \frac{1}{n}\right) - I\left(y_n'(t_2) + \frac{1}{n}\right)\right| < \varepsilon', \quad \forall |t_1 - t_2| < \delta', \ t_1, t_1 \in [0, T]. \tag{3.18}$$

(3.17) and (3.18) yield that

$$\begin{aligned} |y_n'(t_1) - y_n'(t_2)| &= \left| y_n'(t_1) + \frac{1}{n} - \left(y_n'(t_2) + \frac{1}{n}\right)\right| \\ &= \left| I^{-1}\left(I\left(y_n'(t_1) + \frac{1}{n}\right)\right) - I^{-1}\left(I\left(y_n'(t_2) + \frac{1}{n}\right)\right)\right| \\ &< \tilde{\varepsilon}, \ \ \forall |t_1 - t_2| < \delta', \ t_1, t_2 \in [0, T], n \in N_0, \end{aligned}$$

which means that

$$\text{the functions belonging to } \{y_n'(t)\} \text{ are equicontinuous on } [0, T]. \tag{3.19}$$

Consequently, from (3.8), (3.9), (3.10) and (3.19), the Arzela-Ascoli Theorem guarantees that $\{y_n(t)\}$ and $\{y_n'(t)\}$ are relatively compact in $C[0, T]$, i.e., there is a function $y_0 \in C^1[0, T]$, and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to+\infty} \max_{t\in[0,T]} |y_{n_j}(t) - y_0(t)| = 0, \quad \lim_{j\to+\infty} \max_{t\in[0,T]} |y_{n_j}'(t) - y_0'(t)| = 0.$$

Since $y_{n_j}(0) = 0$ and $y_{n_j}'(0) = 0$, $y_{n_j}(t) > 0$, $y_{n_j}'(t) > 0$, $t \in (0, T)$, $j \in \{1, 2, \cdots\}$, one has

$$y_0(0) = 0, \quad y_0'(0) = 0, \quad y_0(t) \geq 0, \quad y_0'(t) \geq 0, \quad \forall\, t \in (0, T). \tag{3.20}$$

Following we show that $y_0'(t) > 0$, $t \in (0, T)$. By the continuity of $y_0'(t)$ at $t = 0$, there is a $t_0 < T$ such that $y_0'(t) \leq \frac{1}{2}\delta$ for all $t \in [0, t_0]$. By

$$\lim_{j\to+\infty} \max_{t\in[0,T]} |y_{n_j}'(t) - y_0'(t)| = 0,$$

there is a $j_0 > 0$ such that $0 < y_{n_j}'(t) + \frac{1}{n_j} \leq \delta$ for all $t \in [0, t_0]$, $j \geq j_0$. Thus, (H_4) implies

$$f\left(t, y_{n_j}(t) + \frac{t}{n_j}, y_{n_j}'(t) + \frac{1}{n_j}\right) \geq \beta(t)\left(y_0(t) + \frac{t}{n}\right)^{\gamma}, \quad t \in [0, t_0],$$

which yields

$$\begin{aligned} y_{n_j}'(t) &= \int_0^t \Phi(s) f\left(s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j}\right) ds \\ &\geq \int_0^t \Phi(s)\beta(s)\left(y_0(s) + \frac{s}{n}\right)^{\gamma} ds > 0, \ \ t \in [0, t_0], \ j \in \{j_0, j_0 + 1, \cdots\}. \end{aligned}$$

Therefore, $y_0'(t) \ge \int_0^t \Phi(s)\beta(s)ds > 0$, $t \in [0, t_0]$. Let $t^* = \sup\{t \in (0, T] | y_0'(s) > 0 \text{ for all } s \in (0, t]\}$. We claim that $t^* = T$.

Suppose that $t^* < T$, which means $y_0'(t^*) = 0$ and $y_0'(t) > 0$ for all $t \in (0, t^*)$. The continuity of $y_0'(t)$ at t^* guarantees that there is a $0 < t_0^* < t^*$ such that $0 < y_0'(t) \le \frac{1}{2}\delta$ for all $t \in [t_0^*, t^*]$. And the uniform convergence of $\{y_{n_j}'(t)\}$ on $[0, T]$ guarantees that there is a $j_0 > 0$ such that $0 < y_{n_j}'(t) + \frac{1}{n_j} \le \delta$, $t \in [t_0^*, t^*], j \ge j_0$. Therefore, for $t \in [t_0^*, t^*]$,

$$f\left(t, y_{n_j}(t) + \frac{t}{n_j}, y_{n_j}'(t) + \frac{1}{n_j}\right) \ge \beta(t), \quad t \in [t_0^*, t^*],$$

which implies that

$$\begin{aligned}
0 = y_{n_j}'(t^*) &= \int_0^{t^*} \Phi(s) f\left(s, y_{n_j}(s) + \frac{s}{n_j}(t), y_{n_j}'(s) + \frac{1}{n_j}\right) ds \\
&= \int_{t_0^*}^{t^*} \Phi(s) f\left(s, y_{n_j}(s) + \frac{s}{n_j}(t), y_{n_j}'(s) + \frac{1}{n_j}\right) ds \\
&\quad + \int_0^{t_0^*} \Phi(s) f\left(s, y_{n_j}(s) + \frac{s}{n_j}(t), y_{n_j}'(s) + \frac{1}{n_j}\right) ds \\
&\ge \int_{t_0^*}^{t^*} \Phi(s)\beta(s)ds + \int_0^{t_0^*} \Phi(s) f\left(s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j}\right) ds \\
&= y_{n_j}'(t_0^*) + \int_{t_0^*}^{t^*} \Phi(s)\beta(s)ds > 0, \ \ j \in \{j_0, j_0 + 1, \cdots\}.
\end{aligned}$$

Letting $j \to +\infty$, one has $0 = y_0'(t^*) \ge y_0'(t_0^*) + \int_{t_0^*}^{t^*} \Phi(s)\beta(s)ds$, a contradiction.

Consequently, $t^* = T$ and $y_0'(t) > 0$ for all $t \in (0, T)$. In addition to $y_0(0) = 0$, one has $y_0(t) > 0$ for all $t \in (0, T]$. Therefore,

$$\min_{s \in [\frac{T}{2}, t]} y_0'(s) > 0, \quad \text{for all } t \in \left[\frac{T}{2}, T\right] \text{ and } \min_{s \in [t, \frac{T}{2}]} y_0'(s) > 0, \quad \text{for all } t \in \left(0, \frac{T}{2}\right].$$

Since

$$y_{n_j}'(t) - y_{n_j}'\left(\frac{T}{2}\right) = \int_{\frac{T}{2}}^t \Phi(s) f\left(s, y_{n_j}(s) + \frac{s}{n_j}, y_{n_j}'(s) + \frac{1}{n_j}\right) ds, \ t \in (0, T),$$

letting $j \to +\infty$, the Lebesgue Dominated Convergence Theorem guarantees that

$$y_0'(t) - y_0'\left(\frac{T}{2}\right) = \int_{\frac{T}{2}}^t \Phi(s) f(s, y_0(s), y_0'(s)) ds, \ \ t \in (0, T).$$

By direct differentiation, we have

$$y_0''(t) = \Phi(t) f(t, y_0(t), y_0'(t)), \quad t \in (0, T).$$

In addition to (3.20), $y_0 \in C^1[0,T] \cap C^2(0,T)$ is a nonnegative solution to (1.1) with $y_0(t) > 0$ for all $t \in (0,1]$.

Example 3.1 Consider the initial value problem

$$\begin{cases} y''(t) = \mu[\cos\frac{1}{t} - (y')^e + (y')^{-a}][1+y^b], t \in (0,T), \\ y'(0) = 0, \ \ y(0) = 0 \end{cases} \tag{3.21}$$

with $a > 0$, $e \geq 0$, $b \geq 0$ and $\mu > 0$. Then there is a $\mu_0 > 0$ such that (3.21) has at least one nonnegative solution $y_0 \in C^1[0,T] \cap C^2(0,T)$ with $y_0(t) > 0$ on (0,T] for all $0 < \mu < \mu_0$.

To see that (3.21) has at least one nonnegative solution, we will apply Theorem 3.1 with $\Phi(t) \equiv \mu$, $g(y') = (y')^{-a}$, $r(y') = 1 + (y')^e$, $h(y) = 1 + y^b$. Clearly, (H_1), (H_2) and (H_4) ($\beta(t) \equiv 1$ and $\delta = (\frac{1}{3})^{1/a}$) hold. Since $\lim\limits_{z\to 0+} I^{-1}(z) = 0$, there is a $\mu_0 > 0$ such that

$$\frac{1}{\max\{1,T\}I^{-1}(2\mu_0)} \geq 1,$$

and so

$$\sup_{c\in(0,+\infty)} \frac{c}{\max\{1,T\}I^{-1}(|\Phi|_0 \int_0^c h(x)dx)}$$
$$\geq \sup_{c\in(0,+\infty)} \frac{c}{\max\{1,T\}I_1^{-1}(\mu(1+c^b))} > 1, \ \forall 0 < \mu < \mu_0,$$

which guarantees that (H_3) holds.

4. Singularities at $y' = 0$ and $y = 0$

In this section our nonlinearity f may be singular at $y' = 0$ and $y = 0$. Throughout this section we will assume that the following conditions hold:

(P_1) $\Phi \in C[0,T]$, with $\Phi(t) > 0$ on $(0,T)$;

(P_2) $f : [0,T] \times (0,+\infty) \times (0,+\infty) \to R$ is continuous with $|f(t,x,y)| \leq [h(x)+w(x)][g(y)+r(y)]$ on $[0,T] \times (0,+\infty) \times (0,+\infty)$ with $w(x) > 0, g(y) > 0$ continuous and nonincreasing on $(0,+\infty)$ and $w \in L^1[0,T]$, $h(x) \geq 0$, $r(y) \geq 0$ continuous and nondecreasing on $(0,\infty)$;

(P_3) there is a $\beta \in C((0,T),(0,+\infty))$ and a constant $\delta > 0$ such that $f(t,x,y) \geq \beta(t)$, $\forall (t,x,y) \in (0,T) \times (0,+\infty) \times (0,\delta]$;

(P_4)

$$\sup_{c\in(0,+\infty)} \frac{c}{\max\{1,T\}I^{-1}(ch(c)|\Phi|_0 + |\Phi|_0 \int_0^c w(s)ds)} > 1,$$

where $I(z) = \displaystyle\int_0^z \frac{u}{g(u)+r(u)}du, \ z \in (0,+\infty), |\Phi|_0 = \max_{t\in[0,T]} |\Phi(t)|.$

For $y \in P$ and each $n \in \{1, 2, \cdots\}$, define operators by

$$(A_n y)(t) = \int_0^t \max\{0, \int_0^s \Phi(\tau) f(\tau, y(\tau) + \frac{1}{n}\tau + \frac{1}{n}, y'(\tau) + \frac{1}{n}) d\tau\} ds, \quad \forall t \in [0, T]. \tag{4.1$_n$}$$

Theorem 4.1 *Suppose* $(P_1) - (P_4)$ *hold. Then equation (1.1) has at least one nonnegative solution* $y_0 \in C^1[0, T] \cap C^2(0, T)$ *with* $y_0(t) > 0$ *on* $(0, T]$.

Proof. From (P_4), choose $R_1 > 0$ such that

$$\frac{R_1}{\max\{1, T\} I^{-1}(R_1 h(R_1)|\Phi|_0 + |\Phi|_0 \int_0^{R_1} w(s) ds)} > 1.$$

Since I^{-1}, I, h and $\int_0^z h(u) du$ are continuous, we choose a $\frac{R_1}{2} > \varepsilon > 0$ small enough such that

$$\frac{R_1}{\max\{1, T\} I^{-1}((R_1 + \varepsilon) h(R + \varepsilon)|\Phi|_0 + |\Phi|_0 \int_0^{R+\varepsilon} w(s) ds + I(\varepsilon))} > 1. \tag{4.2}$$

Let $n_0 \in \{1, 2, \cdots\}$ be chosen so that $\frac{1}{n_0} < \delta/2$ and $\frac{1+T}{n_0} < \varepsilon$. Let $N_0 = \{n_0, n_0 + 1, \cdots\}$.

From (P_1) and (P_2), Lemma 2.2 guarantees that for each $n \in N_0$, $A_n : P \to P$ is continuous and completely continuous .

Now let

$$\Omega_1 = \{y \in C^1[0, T] : \|y\| < R_1\}.$$

We show that

$$y \neq \mu A_n y, \quad \forall y \in P \cap \partial\Omega_1, \ \mu \in (0, 1], \ n \in N_0. \tag{4.3}$$

Suppose there exist a $y_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0, 1]$ such that $y_0 = \mu_0 A_n y_0$, i.e.,
$y_0(t) = \mu_0 \int_0^t \max\{0, \int_0^s \Phi(\tau) f(\tau, y_0(\tau) + \frac{1}{n}\tau + \frac{1}{n}, y_0'(\tau) + \frac{1}{n}) d\tau\} ds$, $t \in [0, T]$, which yields

$$y_0'(t) = \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\}, \quad t \in [0, T].$$

Obviously, $y_0'(t) \geq 0$, $t \in (0, T)$ and $\lim_{t \to 0+} y_0'(t) = 0$. Then, since $\frac{1}{n} < \delta/2$, there is a $t_0 > 0$ such that $0 \leq y_0'(t) + \frac{1}{n} \leq \delta$ for all $t \in (0, t_0]$. From (P_3), one has $f(t, y_0(t) + \frac{1}{n}t + \frac{1}{n}, y_0'(t) + \frac{1}{n}) \geq \beta(t) > 0$ for all $t \in (0, t_0]$, which implies that

$$\max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\}$$
$$\geq \int_0^t \Phi(s)\beta(s) ds > 0, \quad t \in (0, t_0]$$

and

$$\begin{aligned} y_0'(t) &= \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\ &\geq \mu_0 \max\left\{0, \int_0^t \Phi(s)\beta(s) ds\right\} \\ &= \mu_0 \int_0^t \Phi(s)\beta(s) ds > 0, \ \ t \in (0, t_0]. \end{aligned}$$

Let $t^* = \sup\{t \in (0,T] | y_0'(s) > 0 \text{ for all } s \in (0,t]\}$. We claim that $t^* = T$, which means that $y_0'(t) > 0$ for all $t \in (0,T)$, and so

$$y_0'(t) = \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} > 0, \ \ \forall t \in (0,T).$$

Hence

$$\begin{aligned} y_0'(t) &= \mu_0 \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds \\ &= \mu_0 \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds, \ \ t \in (0,T). \end{aligned} \tag{4.4}$$

Suppose that $t^* < T$, then $y_0'(t^*) = 0$ and $y_0'(t) > 0$ for all $t \in (0, t^*)$ and

$$\begin{aligned} 0 < y_0'(t) &= \mu_0 \max\left\{0, \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\ &= \mu_0 \int_0^t \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds, \ \ t \in (0, t_0^*). \end{aligned} \tag{4.5}$$

Hence, since $\frac{1}{n} < \delta/2$, there is a $0 < t_0^* < t^*$ such that $0 < y_0'(t) + \frac{1}{n} \leq \delta$ for all $t \in [t_0^*, t^*]$, which implies that $f(t, y_0(t) + \frac{1}{n}t + \frac{1}{n}, y_0'(t) + \frac{1}{n}) \geq \beta(t)$ for all $t \in [t_0^*, t^*]$. Thus, from (4.5), we have

$$y_0'(t_0^*) = \mu_0 \int_0^{t_0^*} \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds$$

and

$$\begin{aligned}
0 &= y_0'(t^*) \\
&= \mu_0 \max\left\{0, \int_0^{t^*} \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\
&= \mu_0 \max\left\{0, \int_{t_0^*}^{t^*} \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right. \\
&\quad \left. + \int_0^{t_0^*} \Phi(s) f\left(s, y_0(s) + \frac{1}{n}s + \frac{1}{n}, y_0'(s) + \frac{1}{n}\right) ds\right\} \\
&\geq \mu_0 \max\left\{0, \int_{t_0^*}^{t^*} \Phi(s)\beta(s) ds + \frac{1}{\mu_0} y_0'(t_0^*)\right\} \\
&= \mu_0 \int_{t_0^*}^{t^*} \Phi(s)\beta(s) ds + y_0'(t_0^*) > 0.
\end{aligned}$$

This is a contradiction. Consequently, $t^* = T$ and (4.4) is true.

Since $y_0'(0) = 0$, one has $y_0(t) > 0$ for all $t \in (0, T)$. By direct differentiation, we have

$$\begin{cases} y_0''(t) = \mu_0 \Phi(t) f(t, y_0(t) + \frac{1}{n}t + \frac{1}{n}, y_0'(t) + \frac{1}{n}), \quad t \in (0, T), \\ y_0(0) = 0, \ \ y_0'(0) = 0. \end{cases} \tag{4.6}$$

Therefore,

$$\begin{aligned}
& y_0''(t) \\
&= \mu_0 \Phi(t) f\left(t, y_0(t) + \frac{1}{n}t + \frac{1}{n}, y_0'(t) + \frac{1}{n}\right) \\
&\leq \Phi(t) \left| f\left(t, y_0(t) + \frac{1}{n}t + \frac{1}{n}, y_0'(t) + \frac{1}{n}\right) \right| \\
&\leq \Phi(t) \left[h\left(y_0(t) + \frac{1}{n}t + \frac{1}{n}\right) + w\left(y_0(t) + \frac{1}{n}t + \frac{1}{n}\right) \right] \\
&\quad \cdot \left[g\left(y_0'(t) + \frac{1}{n}\right) + r\left(y_0'(t) + \frac{1}{n}\right) \right], \ \ \forall t \in (0, T),
\end{aligned}$$

which means that

$$\begin{aligned}
& \frac{y_0''(t)}{g(y_0'(t) + \frac{1}{n}) + r(y_0'(t) + \frac{1}{n})} \\
&\leq \Phi(t) \left[h\left(y_0(t) + \frac{1}{n}t + \frac{1}{n}\right) + w\left(y_0(t) + \frac{1}{n}t + \frac{1}{n}\right) \right], \ \ \forall t \in (0, T)
\end{aligned}$$

and

$$\frac{(y_0'(t)+\frac{1}{n})y_0''(t)}{g(y_0'(t)+\frac{1}{n})+r(y_0'(t)+\frac{1}{n})}$$

$$\leq \Phi(t)\left[h\left(y_0(t)+\frac{1}{n}t+\frac{1}{n}\right)+w\left(y_0(t)+\frac{1}{n}t+\frac{1}{n}\right)\right]\left(y_0'(t)+\frac{1}{n}\right), \quad \forall t\in(0,T).$$

Integration from 0 to t yields

$$\begin{aligned} &I(y_0'(t)+\frac{1}{n})-I(\frac{1}{n})\\ &\leq |\Phi|_0[h(R_1+\varepsilon)(R_1+\varepsilon)+\int_0^T w(y_0(s)+\frac{1}{n}s+\frac{1}{n})]d(y_0(s)+\frac{1}{n}s+\frac{1}{n})]\\ &\leq |\Phi|_0[h(R_1+\varepsilon)(R_1+\varepsilon)+\int_0^{R_1+\varepsilon} w(s)ds], \end{aligned}$$

and so

$$I\left(y_0'(t)+\frac{1}{n}\right)\leq I(\varepsilon)+|\Phi|_0\left[h(R_1+\varepsilon)(R_1+\varepsilon)+\int_0^{R_1+\varepsilon} w(s)ds\right].$$

Thus

$$y_0'(t)\leq I^{-1}\left(I(\varepsilon)+|\Phi|_0\left[h(R_1+\varepsilon)(R_1+\varepsilon)+\int_0^{R_1+\varepsilon} w(s)ds\right]\right), \quad \forall t\in[0,T]. \tag{4.7}$$

Integrate from 0 to T to obtain

$$y_0(T)=y_0(T)-y_0(0)\leq I^{-1}\left(I(\varepsilon)+|\Phi|_0\left[h(R_1+\varepsilon)(R_1+\varepsilon)+\int_0^{R_1+\varepsilon} w(s)ds\right]\right)T. \tag{4.8}$$

(4.7) and (4.8) guarantee that

$$R_1=\|y_0\|\leq \max\{1,T\}I^{-1}\left(I(\varepsilon)+|\Phi|_0\left[h(R_1+\varepsilon)(R_1+\varepsilon)+\int_0^{R_1+\varepsilon} w(s)ds\right]\right),$$

which means

$$\frac{R_1}{\max\{1,T\}I^{-1}(I(\varepsilon)+|\Phi|_0[h(R_1+\varepsilon)(R_1+\varepsilon)+\int_0^{R_1+\varepsilon} w(s)ds])}\leq 1.$$

This is a contradiction to (4.2). Thus (4.3) is true.

From Lemma 2.1, for each $n\in N_0$, we have

$$i(A_n, P\cap\Omega_1, P)=1. \tag{4.9}$$

As a result, for each $n \in N_0$, there exists a $y_n \in P \cap \Omega_1$ such that $y_n = A_n y_n$, i.e.,

$$y_n(t) = (A_n y_n)(t)$$
$$= \int_0^t \max\left\{0, \int_0^s \Phi(\tau) f\left(\tau, y_n(\tau) + \frac{1}{n}\tau + \frac{1}{n}, y_n'(\tau) + \frac{1}{n}\right) d\tau\right\} ds,\ \ t \in [0,T].$$

A similar to show (4.4) yields that

$$y_n'(t) > 0,\ \ y_n'(t) = \int_0^t \Phi(s) f(s, y_n(s) + \frac{1}{n}s + \frac{1}{n}, y_n'(s) + \frac{1}{n}) ds,\ \ t \in (0,T),\ \ n \in N_0.$$

Now we consider $\{y_n\}_{n\in N_0}$. Since $\|y_n\| \le R_1$, obviously

$$\text{the functions belonging to } \{y_n(t)\} \text{ are uniformly bounded on } [0,T] \tag{4.10}$$

and

$$\text{the functions belonging to } \{y_n'(t)\} \text{ are uniformly bounded on } [0,T]. \tag{4.11}$$

And moreover, (4.11) yields that

$$\text{the functions belonging to } \{y_n(t)\} \text{ are equicontinuous on } [0,T]. \tag{4.12}$$

Similarly as (4.6), one has

$$\begin{cases} y_n''(t) = \Phi(t) f(t, y_n(t) + \frac{1}{n}t + \frac{1}{n}, y_n'(t) + \frac{1}{n}),\ \ t \in (0,T) \\ y_n(0) = 0,\ \ y_n'(0) = 0. \end{cases}$$

Then,

$$\begin{aligned} \pm y_n''(t) &= \pm\Phi(t) f\left(t, y_n(t) + \frac{1}{n}t + \frac{1}{n}, y_n'(t) + \frac{1}{n}\right) \\ &\le \Phi(t)\left|f\left(t, y_n(t) + \frac{1}{n}t + \frac{1}{n}, y_n'(t) + \frac{1}{n}\right)\right| \\ &\le \Phi(t)\left[h\left(y_n(t) + \frac{1}{n}t + \frac{1}{n}\right) + w\left(y_n(t) + \frac{1}{n}t + \frac{1}{n}\right)\right] \\ &\quad \cdot \left[g\left(y_n'(t) + \frac{1}{n}\right) + r\left(y_n'(t) + \frac{1}{n}\right)\right],\ \ \forall t \in (0,T), \end{aligned}$$

which means that

$$\frac{\pm(y_0'(t) + \frac{1}{n}) y_n''(t)}{g(y_n'(t) + \frac{1}{n}) + r(y_n'(t) + \frac{1}{n})}$$
$$\le \Phi(t)[h(y_n(t) + \frac{1}{n}t + \frac{1}{n}) + w(y_n(t) + \frac{1}{n}t + \frac{1}{n})](y_0'(t) + \frac{1}{n}),\ \ \forall t \in (0,T). \tag{4.13}$$

For any t_1, $t_2 \in [0,T]$, integration from t_1 to t_2 yields that

$$\begin{aligned}&\left|I\left(y_n'(t_1)+\frac{1}{n}\right)-I\left(y_n'(t_2)+\frac{1}{n}\right)\right|\\ &\le |\Phi|_0\left|\int_{t_1}^{t_2}\left[h(y_n(s)+\frac{1}{n}s+\frac{1}{n})+w(y_n(s)+\frac{1}{n}s+\frac{1}{n})\right]d(y_n(s)+\frac{1}{n}s+\frac{1}{n})\right|\\ &=|\Phi|_0\left|\int_{y_n(t_1)+\frac{1}{n}t_1+\frac{1}{n}}^{y_n(t_2)+\frac{1}{n}t_2+\frac{1}{n}}[h(s)+w(s)]ds\right|.\end{aligned}$$

Since I^{-1} is uniformly continuous on $[0,I(R_1+\varepsilon)]$, for any $\varepsilon'>0$, there is a $\delta'>0$ such that

$$|I^{-1}(s_1)-I^{-1}(s_2)|<\varepsilon',\ \ \forall|s_1-s_2|<\delta',\ \ s_1,s_2\in[0,I(R_1+\varepsilon)]. \tag{4.14}$$

Since $\int_0^z(h(s)+w(s))ds$ is uniformly continuous on $[0,R_1+\varepsilon]$, there is a $\delta''>0$ such that

$$\left|\int_{u_1}^{u_2}(h(s)+w(s))ds\right|<\frac{\delta'}{|\Phi|_0},\ \ \forall|u_1-u_2|<\delta'',u_1,u_2\in[0,R_1+\varepsilon]. \tag{4.15}$$

From (4.12), there is a $\tilde{\delta}>0$ such that

$$|(y_n(t_1)+t_1)-(y_n(t_2)+t_2)|<\delta'',\ \ \forall|t_1-t_2|<\tilde{\delta},\ \ t_1,t_2\in[0,T]. \tag{4.16}$$

(4.15) and (4.16) yield that

$$\begin{aligned}&\left|I\left(y_n'(t_1)+\frac{1}{n}\right)-I\left(y_n'(t_2)+\frac{1}{n}\right)\right|\\ &\le|\Phi|_0\left|\int_{y_n(t_1)+\frac{1}{n}t_1+\frac{1}{n}}^{y_n(t_2)+\frac{1}{n}t_2+\frac{1}{n}}[h(s)+w(s)]ds\right|\\ &<|\Phi|_0\frac{\delta'}{|\Phi|_0}=\delta',\ \ |t_1-t_2|<\tilde{\delta},\ \ t_1,t_2\in[0,T],\ \ n\in N_0.\end{aligned} \tag{4.17}$$

From (4.14) and (4.17), we have

$$\begin{aligned}|y_n'(t_1)-y_n'(t_2)|&=\left|y_n'(t_1)+\frac{1}{n}-\left(y_n'(t_2)+\frac{1}{n}\right)\right|\\ &=\left|I^{-1}\left(I\left(y_n'(t_1)+\frac{1}{n}\right)\right)-I^{-1}\left(I\left(y_n'(t_2)+\frac{1}{n}\right)\right)\right|\\ &<\varepsilon',\ \ \forall|t_1-t_2|<\tilde{\delta},\ \ t_1,t_2\in[0,T],\ \ n\in N_0,\end{aligned}$$

which means that

$$\text{the functions belonging to } \{y_n'(t)\} \text{ are equicontinuous on } [0,T]. \tag{4.18}$$

Consequently, from (4.10), (4.11), (4.12) and (4.18), the Arzela-Ascoli Theorem guarantees that $\{y_n(t)\}$ and $\{y_n'(t)\}$ are relatively compact in $C[0,T]$, i.e., there is a $y_0 \in C^1[0,T]$ and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to+\infty} \max_{t\in[0,T]} |y_{n_j}(t) - y_0(t)| = 0, \quad \lim_{j\to+\infty} \max_{t\in[0,T]} |y_{n_j}'(t) - y_0'(t)| = 0.$$

Since $y_{n_j}(0) = 0$, $y_{n_j}'(0) = 0$, $y_{n_j}(t) > 0$, $y_{n_j}'(t) > 0$, $t \in (0,T)$,

$$y_0(0) = 0,\ y_0'(0) = 0,\ y_0(t) \ge 0,\ y_0'(t) \ge 0,\ t \in (0,T). \tag{4.19}$$

Following we show that $y_0'(t) > 0$, $t \in (0,T)$. Since $y_0'(t)$ is right continuous at $t = 0$ and $y_0'(0) = 0$, there is a $0 < t_0 < 1$ such that $y_0'(t) \le \frac{1}{2}\delta$ for all $t \in [0,t_0]$. By

$$\lim_{j\to+\infty} \max_{t\in[0,T]} |y_{n_j}'(t) - y_0'(t)| = 0,$$

there is a $j_0 > 0$ such that $0 < y_{n_j}'(t) + \frac{1}{n_j} \le \delta$ for all $t \in [0,t_0]$, $j \ge j_0$. Thus, (H_3) implies

$$f\left(t, y_{n_j}(t) + \frac{1}{n_j}t + \frac{1}{n_j}, y_{n_j}'(t) + \frac{1}{n_j}\right) \ge \beta(t),\ \ t \in [0,t_0],$$

which yields

$$\begin{aligned} y_{n_j}'(t) &= \int_0^t \Phi(s) f\left(s, y_{n_j}(s) + \frac{1}{n_j}s + \frac{1}{n_j}, y_{n_j}'(s) + \frac{1}{n_j}\right) ds \\ &\ge \int_0^t \Phi(s)\beta(s)ds > 0,\ \ t \in [0,t_0],\ j \in \{j_0, j_0+1, \cdots\}. \end{aligned}$$

Therefore, $y_0'(t) \ge \int_0^t \Phi(s)\beta(s)ds > 0$, $t \in [0,t_0]$.

Let $t^* = \sup\{t \in (0,T) | y_0'(s) > 0 \text{ for all } s \in (0,t]\}$. Then, we claim that $t^* = T$. Suppose that $t^* < T$, which means $y_0'(t^*) = 0$ and $y_0'(t) > 0$ for all $t \in (0,t^*)$. The continuity of $y_0'(t)$ at t^* guarantees that there is a $0 < t_0^* < t^*$ such that $0 < y_0'(t) \le \frac{1}{2}\delta$ for all $t \in [t_0^*, t^*]$. And the uniform convergence of $\{y_{n_j}'(t)\}$ on $[0,T]$ guarantees that there is a $j_0 > 0$ such that $0 < y_{n_j}'(t) + \frac{1}{n_j} \le \delta$, $t \in [t_0^*, t^*]$, $j \ge j_0$. Then, for $t \in [t_0^*, t^*]$,

$$f\left(t, y_{n_j}(t) + \frac{1}{n_j}t + \frac{1}{n_j}, y_{n_j}'(t) + \frac{1}{n_j}\right) \ge \beta(t),$$

which implies that

$$\begin{aligned}
y'_{n_j}(t^*) &= \int_0^{t^*} \Phi(s) f\left(s, y_{n_j}(s) + \frac{1}{n_j}s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j}\right) ds \\
&= \int_{t_0^*}^{t^*} \Phi(s) f\left(s, y_{n_j}(s) + \frac{1}{n_j}s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j}\right) ds \\
&\quad + \int_0^{t_0^*} \Phi(s) f\left(s, y_{n_j}(s) + \frac{1}{n_j}s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j}\right) ds \\
&\geq \int_{t_0^*}^{t^*} \Phi(s)\beta(s) ds + y'_{n_j}(t_0^*) \\
&> \int_{t_0^*}^{t^*} \Phi(s)\beta(s) ds, \;\; j \in \{j_0, j_0+1, \cdots\}.
\end{aligned}$$

Hence, letting $j \to +\infty$, we have $y'_0(t^*) \geq \int_{t_0^*}^{t^*} \Phi(s)\beta(s)ds > 0$, a contradiction.

Consequently, $t^* = T$ and $y'_0(t) > 0$ for all $t \in (0, T)$. In addition to $y_0(0) = 0$, one has $y_0(t) > 0$ for all $t \in (0, T]$. Therefore,

$$\min\{\min_{s\in[\frac{T}{2},t]} y_0(s), \min_{s\in[\frac{T}{2},t]} y'_0(s)\} > 0, \quad \text{for all } t \in \left[\frac{T}{2}, T\right],$$

and

$$\min\{\min_{s\in[t,\frac{T}{2}]} y_0(s), \min_{s\in[t,\frac{T}{2}]} y'_0(s)\} > 0, \quad \text{for all } t \in \left(0, \frac{T}{2}\right].$$

Since

$$y'_{n_j}(t) - y'_{n_j}\left(\frac{T}{2}\right) = \int_{\frac{T}{2}}^{t} \Phi(s) f\left(s, y_{n_j}(s) + \frac{1}{n_j}s + \frac{1}{n_j}, y'_{n_j}(s) + \frac{1}{n_j}\right) ds, \; t \in (0, T),$$

letting $j \to +\infty$, the Lebesgue Dominated Convergence Theorem guarantees that

$$y'_0(t) - y'_0\left(\frac{T}{2}\right) = \int_{\frac{T}{2}}^{t} \Phi(s) f(s, y_0(s), y'_0(s)) ds, \;\; t \in (0, T).$$

By direct differentiation, we have

$$y''_0(t) = \Phi(t) f(t, y_0(t), y'_0(t)), \quad t \in (0, T).$$

In addition to (4.19), $y_0 \in C^1[0, T] \cap C^2(0, T)$ is a nonnegative solution to equation (1.1) with $y_0(t) > 0$ for all $t \in (0, 1]$.

Example 4.1 Consider the initial value problems

$$\left\{ \begin{array}{l} y''(t) = \mu[\sin\frac{1}{t} - (y')^e + (y')^{-a}][1 + y^b + y^{-d}], \;\; t \in (0, T), \\ y'(0) = 0, \;\; y(0) = 0 \end{array} \right. \tag{4.20}$$

with $a > 0$, $e \geq 0$, $b \geq 0$, $0 < d < 1$ and $\mu > 0$. Then there is a $\mu_0 > 0$ such that (4.20) has at least one nonnegative solution $y_0 \in C^1[0,T] \cap C^2(0,T)$ with $y_0(t) > 0$ on $(0,T]$ for all $0 < \mu < \mu_0$.

To see that (4.20) has at least one nonnegative solution, we will apply Theorem 4.1 with $\Phi(t) \equiv \mu$, $g(y') = (y')^{-a}$, $r(y') = 1 + (y')^e$, $h(y) = 1 + y^b$, $w(y) = y^{-d}$. Clearly, (P_1), (P_2) and (P_3) ($\beta(t) \equiv 1$ and $\delta = (\frac{1}{3})^{1/a}$) hold. Since $\lim\limits_{z \to 0+} I^{-1}(z) = 0$ there is a $\mu_0 > 0$ such that

$$\frac{1}{\max\{1,T\} I^{-1}[\mu(2 + 1/(1+d))]} \geq 1$$

and so

$$\sup_{c \in (0,+\infty)} \frac{c}{\max\{1,T\} I^{-1}(ch(c)|\Phi|_0 + |\Phi|_0 \int_0^c w(s)ds)} > 1,$$

which guarantees that (P_4) holds.

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References

[1] R.P.Agarwal, D.O'Regan, Second-order initial value problems of singular type, *Journal of Math. Anal. Appl.,* **229**(1999), pp414–430.

[2] R.P.Agarwal, D.O'Regan, Second order boundary value problems of singular type, *Journal of Mathematical Analysis and Applications,* **226**(1998), pp414–430.

[3] D. Guo and V.Lakshmikantham, ″Nonlinear problems in abstract cones″, Academic Press. Inc. , New York, 1988.

[4] H. Wang and W.Ge, Positive solutions of second-order singular initial value problems, *Acta Mathematica Sinica,* **44**(2001), pp227–232.

[5] G.Yang, Minimal positive solutions to some singular second order differential equations, *J. math. Anal. Appl.* **266**(2002), pp479–491.

[6] G.Yang, Positive solutions of some second-order nonlinear singular differential equations, *Computer and Mathematics with Applications* ,**45**(2003), pp605–614.

[7] G.Yang, Second order singular boundary value problems with sign-changing nonlinearities on infinity intervals. *Nonlinear Analysis Forum,* **9**(2004), pp169–174.

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