SUT Journal of Mathematics
Vol. 43, No. 2 (2007), 251–261

Some expressions of double and triple sine functions

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(Received March 2, 2007)

Abstract. We show some expressions of double and triple sine functions. Then we apply the results to special values of Dirichlet L-functions and $\zeta(3)$.

AMS 2000 *Mathematics Subject Classification.* Primary 11M06.

Key words and phrases. multiple sine function, multiple gamma function, multiple Hurwitz zeta function.

§1. Introduction

The double sine function $S_2(x)$ and the triple sine function $S_3(x)$ (see [2]) are defined as

$$S_2(x) := \Gamma_2(x)^{-1}\Gamma_2(2-x)$$

and

$$S_3(x) := \Gamma_3(x)^{-1}\Gamma_3(3-x)^{-1}$$

respectively, where

$$\Gamma_2(x) := \exp(\zeta_2'(0,x)) = \exp\left(\frac{\partial}{\partial s}\zeta_2(s,x)\,\bigg|_{s=0}\right)$$

and

$$\Gamma_3(x) := \exp(\zeta_3'(0,x)) = \exp\left(\frac{\partial}{\partial s}\zeta_3(s,x)\,\bigg|_{s=0}\right)$$

are the double gamma function and the triple gamma function. We notice that the double Hurwitz zeta function and the triple Hurwitz zeta function are constructed by

$$\zeta_2(s,x) := \sum_{n_1,n_2\geq 0}(x+n_1+n_2)^{-s}$$

and

$$\zeta_3(s,x) := \sum_{n_1,n_2,n_3\geq 0} (x+n_1+n_2+n_3)^{-s}.$$

We recall the classical objects:

$$S_1(x) := \Gamma_1(x)^{-1}\Gamma_1(1-x)^{-1},$$
$$\Gamma_1(x) := \exp(\zeta_1'(0,x)) = \exp\left(\frac{\partial}{\partial s}\zeta_1(s,x)\Big|_{s=0}\right)$$

and

$$\zeta_1(s,x) = \zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}.$$

Then we have

$$S_1(x) = \frac{2\pi}{\Gamma(x)\Gamma(1-x)}$$

where we used

$$\Gamma_1(x) = \frac{\Gamma(x)}{\sqrt{2\pi}}$$

which was obtained by Lerch [10]. First we describe the double gamma function $\Gamma_2(x)$ and the double sine function $S_2(x)$ using the logarithm of the usual gamma function $\Gamma(x)$ as follows:

Theorem 1.1. *We have*

$$\Gamma_2(x) = \frac{\Gamma(x)^{1-x}}{\sqrt{2\pi}}\exp\left(\frac{x^2-x}{2}+\zeta'(-1)+\int_0^x \log\Gamma(t)\,dt\right),$$
$$S_2(x) = \left(\frac{(1-x)\pi}{e\sin(\pi x)}\right)^{x-1}\exp\left(\int_x^{2-x}\log\Gamma(t)\,dt\right).$$

We notice that this result is considered to be an analogue of Raabe's formula

$$\int_x^{x+1}\log\Gamma(t)\,dt = x\log x - x + \frac{1}{2}\log(2\pi)$$

proved in [12] (1844). We refer to [6, 7, 9] for another kind of generalization of Raabe's formula using the generalized gamma function

$$\mathbf{\Gamma}_r(x) := \exp\left(\frac{\partial}{\partial s}\zeta(s,x)\Big|_{s=1-r}\right)$$

suggested by Milnor [11] and the generalized sine function

$$\mathbf{S}_r(x) := \mathbf{\Gamma}_r(x)^{-1}\mathbf{\Gamma}_r(1-x)^{(-1)^r}$$

introduced in [7]. Here we remark that this result is considered as a kind of "Raabe's formula" from the view point of the generalized gamma function and the generalized sine function (see [6, 7, 9] for details). From Theorem 1.1 we get expressions for special values of some Dirichlet L-functions:

Theorem 1.2. *We obtain*

$$L(2,\chi_{-3}) = \frac{8\sqrt{3}\pi}{9}\log\Big(\frac{4\pi}{3e}\Big) - \frac{4\sqrt{3}\pi}{3}\int_{\frac{1}{3}}^{\frac{5}{3}} \log\Gamma(t)\,dt,$$
$$L(2,\chi_{-4}) = \frac{3\pi}{2}\log\Big(\frac{3\pi}{2e}\Big) - 2\pi\int_{\frac{1}{4}}^{\frac{7}{4}} \log\Gamma(t)\,dt,$$

where χ_{-3} and χ_{-4} are non-trivial Dirichlet characters of modulo 3 and 4 respectively.

Remark 1.1. We can rewrite Theorem 1.2 as

$$L(2,\chi_{-3}) = \frac{4\sqrt{3}\pi}{3}\Big(\zeta'(-1,\frac{1}{3}) - \zeta'(-1,\frac{5}{3})\Big) + \frac{8\sqrt{3}\pi}{9}\log\Big(\frac{2}{3}\Big),$$
$$L(2,\chi_{-4}) = 2\pi\Big(\zeta'(-1,\frac{1}{4}) - \zeta'(-1,\frac{7}{4})\Big) + \frac{3\pi}{2}\log\Big(\frac{3}{4}\Big).$$

Next we consider the triple gamma function $\Gamma_3(x)$ and the triple sine function $S_3(x)$.

Theorem 1.3. *We have*

$$\Gamma_3(x) = \frac{\Gamma(x)^{\frac{(x-1)(x-2)}{2}}}{\sqrt{2\pi}}\exp\Big(-\frac{x^3}{4} + \frac{7}{8}x^2 - \frac{17}{24}x + \frac{3}{2}\zeta'(-1) + \frac{\zeta'(-2)}{2}$$
$$+ \int_0^x\int_0^t \log\Gamma(u)\,dudt + \frac{3-2x}{2}\int_0^x \log\Gamma(t)\,dt\Big),$$
$$S_3(x) = 2\pi\Big(\frac{e\sin(\pi x)}{\pi(x-1)(x-2)}\Big)^{\frac{(x-1)(x-2)}{2}}$$
$$\times\exp\Big(-\Big(\int_0^x + \int_0^{3-x}\Big)\int_0^t \log\Gamma(u)\,dudt + \frac{3-2x}{2}\int_x^{3-x}\log\Gamma(t)\,dt$$
$$-3\zeta'(-1) - \zeta'(-2)\Big).$$

From this we get the following result:

Theorem 1.4. *We obtain*

$$\zeta(3) = \frac{8\pi^2}{7}\Big(-\frac{7}{4}\log 2 - \frac{9}{4}\log\pi + \frac{1}{4} + 6\zeta'(-1) + 4\int_0^{\frac{3}{2}}\int_0^t \log\Gamma(u)\,dudt\Big).$$

Remark 1.2. We show also that

$$\int_0^{\frac{3}{2}} \int_0^t \log\Gamma(u)\,dudt = \int_0^{\frac{3}{2}} \zeta'(-1,t)\,dt + \frac{9}{16}\log(2\pi) - \frac{3}{2}\zeta'(-1).$$

So we can rewrite Theorem 1.4 as

$$\zeta(3) = \frac{32\pi^2}{7}\int_0^{\frac{1}{2}} \zeta'(-1,t)\,dt.$$

§2. Proofs of results

Lemma 2.1.

$$\zeta'(-1,x) = \int_0^x \log\Gamma(t)\,dt + \frac{x^2}{2} - \frac{1+\log(2\pi)}{2}x + \zeta'(-1).$$

Proof of Lemma 2.1. Let

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) = 0.5772156649\ldots$$

be the Euler constant, then by the infinite product expression for the gamma function

$$-\log\Gamma(x) = \log x + \gamma x + \sum_{n=1}^{\infty}\left(\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right).$$

Hence we have

$$\frac{d^3}{dx^3}\left(-\int_0^x \log\Gamma(t)\,dt\right) = -\sum_{n=0}^{\infty}(n+x)^{-2}.$$

Also

$$\frac{\partial^3}{\partial x^3}\zeta(s,x) = -s(s+1)(s+2)\sum_{n=0}^{\infty}(n+x)^{-s-3}$$

converges absolutely for $\Re(s) > -2$. Therefore

$$\frac{\partial}{\partial s}\left(\frac{\partial^3}{\partial x^3}\zeta(s,x)\right)\Bigg|_{s=-1} = \sum_{n=0}^{\infty}(n+x)^{-2}.$$

So we can write

$$\zeta'(-1,x) - \int_0^x \log\Gamma(t)\,dt = ax^2 + bx + c,$$

where a, b, c are some constant numbers. Here under the change of the variable, it is easy to verify that

$$\begin{aligned}\int_0^x \log\Gamma(t+1)\,dt &= \int_1^{x+1} \log\Gamma(t)\,dt \\ &= \int_0^{x+1} \log\Gamma(t)\,dt - \int_0^1 \log\Gamma(t)\,dt.\end{aligned}$$

Using $\log\Gamma(t+1) - \log\Gamma(t) = \log t$, we get

$$\int_0^x \log\Gamma(t)\,dt - \int_0^{x+1} \log\Gamma(t)\,dt = -x\log x + x - \int_0^1 \log\Gamma(t)\,dt.$$

Moreover, from $\zeta(s, x+1) - \zeta(s, x) = -x^{-s}$ we have

$$\zeta'(-1, x+1) - \zeta'(-1, x) = x\log x.$$

Thus we obtain

$$\begin{aligned}\zeta'(-1, x+1) - \zeta'(-1, x) - \int_0^{x+1} \log\Gamma(t)\,dt + \int_0^x \log\Gamma(t)\,dt &= 2ax + a + b, \\ x - \int_0^1 \log\Gamma(t)\,dt &= 2ax + a + b.\end{aligned}$$

To decide b we recall Euler's integral (see [1, 5, 8]):

$$\int_0^{\frac{\pi}{2}} \log(\sin\varphi)\,d\varphi = -\frac{\pi}{2}\log 2.$$

Note that

$$\int_0^1 \log\Gamma(t)\,dt = \int_0^1 \log\Gamma(1-t)\,dt.$$

Denote the integral of the left-hand side by I. Then we know

$$\begin{aligned}2I &= \int_0^1 \log(\Gamma(t)\Gamma(1-t))\,dt \\ &= \int_0^1 \log\Big(\frac{\pi}{\sin(\pi t)}\Big)\,dt \\ &= \log\pi - \int_0^1 \log(\sin(\pi t))\,dt.\end{aligned}$$

Also we can calculate

$$\begin{aligned}\int_0^1 \log(\sin(\pi t))\,dt &= \frac{1}{\pi}\int_0^{\pi} \log(\sin\varphi)\,d\varphi \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log(\sin\varphi)\,d\varphi.\end{aligned}$$

Therefore from Euler's integral we find that

$$\int_0^1 \log\Gamma(t)\,dt = \frac{1}{2}\log(2\pi)$$

and

$$a = \frac{1}{2},\ b = -\frac{1}{2} - \frac{1}{2}\log(2\pi).$$

Putting $x = 1$ we note

$$c = \zeta'(-1),$$

where we used $\zeta'(-1,1) = \zeta'(-1)$. Then we obtain the result. □

Now we show Theorems.

Proof of Theorem 1.1. We put

$$f(x) := \frac{\Gamma(x)^{1-x}}{\sqrt{2\pi}} \exp\left(\frac{x^2 - x}{2} + \zeta'(-1) + \int_0^x \log\Gamma(t)\,dt\right).$$

Hence we have

$$\log f(x) = (1-x)\log\Gamma(x) - \frac{1}{2}\log(2\pi) + \frac{x^2-x}{2} + \zeta'(-1) + \int_0^x \log\Gamma(t)\,dt,$$

$$\frac{f'}{f}(x) = \frac{2x-1}{2} + (1-x)\frac{\Gamma'}{\Gamma}(x).$$

On the other hand, we know

$$\begin{aligned}\frac{\partial}{\partial x}\zeta_2(s,x) &= -s\zeta_2(s+1,x)\\ &= -s\big(\zeta(s,x) + (1-x)\zeta(s+1,x)\big)\\ &= -s\zeta(s,x) + (1-x)\frac{\partial}{\partial x}\zeta(s,x),\end{aligned}$$

where we used

$$\frac{\partial}{\partial x}\zeta(s,x) = -s\zeta(s+1,x).$$

Since $\zeta(s,x)$ is analytic in a region containing $s = 0$, we can write

$$\begin{aligned}\frac{\partial}{\partial x}\zeta_2(s,x) = -s&\Big(\zeta(0,x) + \zeta'(0,x)s + \cdots\Big)\\ &+ (1-x)\frac{\partial}{\partial x}\Big(\zeta(0,x) + \zeta'(0,x)s + \cdots\Big).\end{aligned}$$

By

$$\zeta(0,x) = \frac{1}{2} - x$$

and the Lerch's formula (see [10])

$$\zeta'(0,x) = \log\Big(\frac{\Gamma(x)}{\sqrt{2\pi}}\Big),$$

we obtain

$$\frac{\Gamma_2'}{\Gamma_2}(x) = \frac{2x-1}{2} + (1-x)\frac{\Gamma'}{\Gamma}(x).$$

Also, we know

$$f(1) = \frac{1}{\sqrt{2\pi}} \exp\Big(\zeta'(-1) + \int_0^1 \log\Gamma(t)\,dt\Big).$$

So we have

$$f(1) = e^{\zeta'(-1)}.$$

Naturally we see

$$\Gamma_2(1) = e^{\zeta'(-1,1)} = e^{\zeta'(-1)}.$$

Moreover by the definition of $S_2(x)$, the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. The following examples were known by Kurokawa-Koyama [2, 4]:

$$S_2(\tfrac{1}{3}) = 3^{\frac{1}{3}} \exp\Big(-\frac{\sqrt{3}}{4\pi}L(2,\chi_{-3})\Big),$$
$$S_2(\tfrac{1}{4}) = 2^{\frac{3}{8}} \exp\Big(-\frac{1}{2\pi}L(2,\chi_{-4})\Big).$$

Then, applying Lemma 2.1 to Theorem 1.2 we obtain the result in Remark 1.1. □

Proof of Theorem 1.3. We define

$$g(x) := \frac{\Gamma(x)^{\frac{(x-1)(x-2)}{2}}}{\sqrt{2\pi}} \exp\Big(-\frac{x^3}{4} + \frac{7}{8}x^2 - \Big(\frac{17}{24} + \int_0^1 \zeta'(-1,t)\,dt\Big)x + \frac{3}{2}\zeta'(-1)$$
$$+ \frac{\zeta'(-2)}{2} + \int_0^x \int_0^t \log\Gamma(u)\,du dt + \frac{3-2x}{2}\int_0^x \log\Gamma(t)\,dt\Big).$$

Then we show Theorem 1.3 similarly as in the proof of Theorem 1.1. Immediately we obtain

$$\frac{d^2}{dx^2}\log g(x) = \frac{2x-3}{2}\frac{\Gamma'}{\Gamma}(x) + \frac{x^2-3x+2}{2}\cdot\frac{\Gamma''(x)\Gamma(x)-\Gamma'^2(x)}{\Gamma^2(x)} - \frac{3}{2}x + \frac{7}{4}.$$

On the other hand, we note

$$\begin{aligned}\log\Gamma_3(x) &= \zeta_3'(0,x)\\ &= \frac{1}{2}\zeta'(-2,x)+\frac{3-2x}{2}\zeta'(-1,x)+\frac{(x-1)(x-2)}{2}\zeta'(0,x),\end{aligned}$$

where we used

$$\begin{aligned}\zeta_3(s,x) &= \sum_{n_1,n_2,n_3\geq 0}^{\infty}(n_1+n_2+n_3+x)^{-s}\\ &= \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)(n+x)^{-s}\\ &= \frac{1}{2}\zeta(s-2,x)+\frac{3-2x}{2}\zeta(s-1,x)+\frac{(x-1)(x-2)}{2}\zeta(s,x).\end{aligned}$$

Here we have

$$\frac{\partial^2}{\partial x^2}\zeta(s,x) = s(s+1)\zeta(s+2,x).$$

Then we get

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\zeta_3(s,x) &= s(s+1)\zeta_3(s+2,x)\\ &= s(s+1)\Big(\frac{1}{2}\zeta(s,x)+\frac{3-2x}{2}\zeta(s+1,x)\\ &\qquad +\frac{(x-1)(x-2)}{2}\zeta(s+2,x)\Big)\\ &= \frac{s(s+1)}{2}\zeta(s,x)+\frac{(2x-3)(s+1)}{2}\frac{\partial}{\partial x}\zeta(s,x)\\ &\qquad +\frac{(x-1)(x-2)}{2}\frac{\partial^2}{\partial x^2}\zeta(s,x).\end{aligned}$$

So we can write

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\zeta_3(s,x) &= \frac{s(s+1)}{2}\Big(\zeta(0,x)+\zeta'(0,x)s+\cdots\Big)\\ &\quad +\frac{(2x-3)(s+1)}{2}\frac{\partial}{\partial x}\Big(\zeta(0,x)+\zeta'(0,x)s+\cdots\Big)\\ &\quad +\frac{(x-1)(x-2)}{2}\frac{\partial^2}{\partial x^2}\Big(\zeta(0,x)+\zeta'(0,x)s+\cdots\Big).\end{aligned}$$

Hence we obtain

$$\frac{d^2}{dx^2}\log\Gamma_3(x) = \frac{2x-3}{2}\frac{\Gamma'}{\Gamma}(x)+\frac{x^2-3x+2}{2}\cdot\frac{\Gamma''(x)\Gamma(x)-\Gamma'^2(x)}{\Gamma^2(x)}-\frac{3}{2}x+\frac{7}{4}.$$

Therefore for some constants a, b

$$\frac{g(x)}{\Gamma_3(x)} = e^{ax+b}.$$

Moreover by Lemma 2.1 we have

$$\int_0^1 \int_0^t \log \Gamma(u)\, du dt = \int_0^1 \zeta'(-1,t)\, dt + \frac{1}{12} + \frac{\log(2\pi)}{4} - \zeta'(-1),$$

$$\int_0^2 \log \Gamma(t)\, dt = \log(2\pi) - 1$$

and

$$\int_0^2 \int_0^t \log \Gamma(u)\, du dt = -\frac{7}{12} + \log(2\pi) - 2\zeta'(-1) + 2\int_0^1 \zeta'(-1,t)\, dt.$$

Hence we can calculate

$$g(1) = \exp\Big(\frac{\zeta'(-2)}{2} + \frac{\zeta'(-1)}{2}\Big)$$

and

$$g(2) = \exp\Big(\frac{\zeta'(-2)}{2} - \frac{\zeta'(-1)}{2}\Big).$$

On the other hand, treating

$$\zeta_3(s,1) = \frac{1}{2}(\zeta(s-2) + \zeta(s-1))$$

and

$$\zeta_3(s,2) = \frac{1}{2}(\zeta(s-2) - \zeta(s-1)),$$

we have

$$\Gamma_3(1) = \exp\Big(\frac{\zeta'(-2)}{2} + \frac{\zeta'(-1)}{2}\Big)$$

and

$$\Gamma_3(2) = \exp\Big(\frac{\zeta'(-2)}{2} - \frac{\zeta'(-1)}{2}\Big).$$

Finally we show

$$\int_0^1 \zeta'(-k,t)\, dt = 0. \tag{2.1}$$

where $k \geq 1$ be an integer. To prove (2.1) we use the following formula (see [3, 7]).

Lemma 2.2 (Generalized Kummer's Formula)**.** *Let $k \geq 1$ be an integer and $0 < x < 1$.*

(1) *When k is odd,*

$$\zeta'(-k,x) = \frac{2(-1)^{\frac{k+1}{2}} k!}{(2\pi)^{k+1}} \Big\{ \sum_{n=1}^{\infty} \frac{(\log n)\cos(2\pi nx)}{n^{k+1}} + \Big(\log(2\pi) + \gamma - \Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)\Big) \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{k+1}} - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{k+1}} \Big\}.$$

(2) *When k is even,*

$$\zeta'(-k,x) = \frac{2(-1)^{\frac{k}{2}} k!}{(2\pi)^{k+1}} \Big\{ \sum_{n=1}^{\infty} \frac{(\log n)\sin(2\pi nx)}{n^{k+1}} + \Big(\log(2\pi) + \gamma - \Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)\Big) \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n^{k+1}} + \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n^{k+1}} \Big\}.$$

Here we notice that $\int_0^1 \sin(2\pi nt)\,dt = 0$ and $\int_0^1 \cos(2\pi nt)\,dt = 0$. Thus we obtain (2.1) by Lemma 2.2. From the definition of $S_3(x)$, Theorem 1.3 is proved. □

Proof of Theorem 1.4. The following example was shown by Kurokawa-Koyama [2, 5]:

$$S_3(\tfrac{3}{2}) = 2^{-\frac{1}{8}} \exp\Big(-\frac{3}{16\pi^2}\zeta(3)\Big).$$

So by Theorem 1.3 we have the result. Applying Lemma 2.1 and (2.1) to Theorem 1.4, we obtain

$$\zeta(3) = \frac{8\pi^2}{7}\Big(\frac{1}{2}\log 2 + \frac{1}{4} + 4\int_1^{\frac{3}{2}} \zeta'(-1,t)\,dt\Big).$$

Since

$$\int_1^{\frac{3}{2}} \zeta'(-1,t)dt = \int_0^{\frac{1}{2}} \zeta'(-1,t)\,dt + \int_0^{\frac{1}{2}} t\log t\,dt = \int_0^{\frac{1}{2}} \zeta'(-1,t)\,dt - \frac{1}{8}\log 2 - \frac{1}{16},$$

we have Remark 1.2.

□

References

[1] L. Euler, *De summis serierum numeros Bernoullianos involventium*, Novi commentarii academiae scientiarum Petropolitanae **14** (1769), 129–167 [Opera Omnia I-15,91–130].

[2] N. Kurokawa and S. Koyama, *Multiple sine functions*, Forum Math. **15** (2003), 839–876.

[3] N. Kurokawa and S. Koyama, *Kummer's Formula for Multiple Gamma Functions*, J.Ramanujan Math.Soc. **18** No.1 (2003), 87–107.

[4] S. Koyama and N. Kurokawa, *Zeta functions and normalized multiple sine functions*, Kodai Math.J. **28** No.3 (2005), 534–550.

[5] S. Koyama and N. Kurokawa, *Euler's integrals and multiple sine functions*, Proc. Amer. Math.Soc. **13** (2005), 1257–1265.

[6] N. Kurokawa and H. Ochiai, *Generalized Kinkelin's Formulas*, Kodai Math.J. **30** No.2 (2007), 195–212

[7] N. Kurokawa, H. Ochiai and M. Wakayama, *Milnor's multiple gamma functions*, J.Ramanujan Math.Soc. **21** No.2 (2006), 153–167.

[8] N. Kurokawa and M. Wakayama, *Extremal values of double and triple trigonometric functions*, Kyushu J.Math. **58** No.1 (2004), 141–166.

[9] N. Kurokawa and M. Wakayama, *Period deformations and Raabe's formulas for generalized gamma and sine functions*, Kyushu J.Math. to appear.

[10] M. Lerch, *Další studie v oboru Malmsténovských řad*, Rozpravy České Akad. **3** No. 28 (1894), 1–61.

[11] J. Milnor, *On polylogarithms, Hurwitz zeta functions, and the Kubert identities*, Enseig.Math. **29** (1983), 281–322.

[12] J. L. Raabe, *Angenäherte Bestimmung der function* $\Gamma(1+n) = \int_0^\infty x^n e^{-x}\,dx$, *wenn n eine ganze, gebrochene, oder incommensurable sehr grosse positiv Zahl ist*, J.reine angew.Math. (Crelle J.) **28** (1844), 10–18.

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