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Quarter-symmetric metric connection in a Kenmotsu manifold

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Abstract. We consider a quarter-symmetric metric connection in a Kenmotsu manifold. We investigate the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection. We show that the scalar curvature of an n-dimensional locally symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection is equal to $n(1-n)$. Furthermore, we obtain the non-existence of generalized recurrent, φ-recurrent and pseudosymmetric Kenmotsu manifolds with respect to quarter-symmetric metric connection.

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Introduction

A linear connection $\widetilde{\nabla}$ in a Riemannian manifold M is said to be *quarter-symmetric connection* [7] if the torsion tensor of the connection $\widetilde{\nabla}$

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] \tag{0.1}$$

satisfies

$$T(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \tag{0.2}$$

where η is a 1-form and φ is a $(1,1)$ tensor field. A linear connection $\widetilde{\nabla}$ is called a metric connection with respect to a Riemannian metric g of M, if and only if

$$(\widetilde{\nabla}_X g)(Y,Z) = 0, \tag{0.3}$$

where $X, Y, Z \in \chi(M)$ are arbitrary vector fields on M. A linear connection $\widetilde{\nabla}$ satisfying (0.2) and (0.3) is called a *quarter-symmetric metric connection* [7]. If we change φX by X then the connection is called a *semi-symmetric metric connection* [13]. In [11], M. M. Tripathi and N. Kakkar, in [5] the third named author and G. Pathak studied semi-symmetric metric connection in a Kenmotsu manifold. In [12], M. M. Tripathi studied semi-symmetric non-metric connection in a Kenmotsu manifold.

A non-flat n-dimensional Riemannian manifold M, $n > 3$, is called *generalized recurrent* [4] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{0.4}$$

where ∇ is the Levi-Civita connection and α and β are two 1-forms, $(\beta \neq 0)$. If $\beta = 0$ and $\alpha \neq 0$ then M is called *recurrent*. In [10], the second author studied generalized recurrent Kenmotsu manifolds.

A non-flat n-dimensional Riemannian manifold M, $n > 3$, is called *φ-recurrent* [6] if its curvature tensor R satisfies the condition

$$\varphi^2((\nabla_X R)(Y,Z)W) = \alpha(X)R(Y,Z)W, \tag{0.5}$$

where φ is a $(1,1)$-tensor field and α is a non-zero 1-form. In [1], A. Başarı and C. Murathan studied more general case of φ-recurrent Kenmotsu manifolds as generalized φ-recurrent Kenmotsu manifolds.

A non-flat n-dimensional Riemannian manifold (M, g), $n > 3$, is called *pseudosymmetric* if there exists a 1-form α on M such that

$$\begin{aligned} (\nabla_X R)(Y,Z,W) &= 2\alpha(X)R(Y,Z)W + \alpha(Y)R(X,Z)W \\ &+\alpha(Z)R(Y,X)W + \alpha(W)R(Y,Z)X + g(R(Y,Z)W,X)A, \end{aligned} \tag{0.6}$$

where $X, Y, Z, W \in \chi(M)$ are arbitrary vector fields and α is a non-zero 1-form on M. $A \in \chi(M)$ is the vector field corresponding through g to the 1-form α which is given by $g(X, A) = \alpha(X)$ [3]. If $\nabla R = 0$ then M is called *locally symmetric* [9].

In the present paper, we study quarter-symmetric metric connection in a Kenmotsu manifold. The paper is organized as follows: In Section 1, we give a brief account of Kenmotsu manifolds. In Section 2, we investigate the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection. In Section 3, we investigate the scalar curvature of a locally symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. In Section 4, we consider generalized recurrent, φ-recurrent and pseudosymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. We obtain the non-existence of these type manifolds.

§1. Kenmotsu Manifolds

Let M be an $n = (2m+1)$-dimensional almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g) consisting of a (1,1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g on M satisfying

$$\varphi\xi = 0, \tag{1.1}$$

$$\eta \circ \varphi = 0, \tag{1.2}$$

$$\eta(\xi) = 1,$$

$$\varphi^2 X = -X + \eta(X)\xi, \tag{1.3}$$

$$g(X, \xi) = \eta(X), \tag{1.4}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.5}$$

for all vector fields X, Y on M. If an almost contact metric manifold satisfies

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{1.6}$$

then M is called a *Kenmotsu manifold* [8], where ∇ is the Levi-Civita connection of g. From the above equations it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{1.7}$$

and

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \tag{1.8}$$

Moreover the curvature tensor R and the Ricci tensor S satisfy

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{1.9}$$

and

$$S(X, \xi) = -(n-1)\eta(X), \tag{1.10}$$

(see [8]). A Kenmotsu manifold is normal (that is, the Nijenhuis tensor of φ equals $-2d\eta \otimes \xi$) but not Sasakian. Moreover, it is also not compact since from the equation (1.7) we get $\mathrm{div}\xi = n - 1$. In [8], K. Kenmotsu showed (1) that locally a Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kähler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant; and (2) that a Kenmotsu manifold of constant φ-sectional curvature is a space of constant curvature -1, and so it is locally hyperbolic space.

§2. Curvature Tensor

Let $\widetilde{\nabla}$ be a linear connection and ∇ be a Levi-Civita connection of an almost contact metric manifold M such that

$$\widetilde{\nabla}_X Y = \nabla_X Y + U(X,Y), \tag{2.1}$$

where U is a tensor of type $(1,1)$. For $\widetilde{\nabla}$ to be a quarter-symmetric metric connection in M, we have

$$U(X,Y) = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(Y,X)], \tag{2.2}$$

where

$$g(T^{'}(X,Y),Z) = g(T(Z,X),Y), \tag{2.3}$$

(see [7]). From (0.2) and (2.3) we get

$$T^{'}(X,Y) = g(\varphi Y, X)\xi - \eta(X)\varphi Y \tag{2.4}$$

and by making use of (0.1) and (2.4) in (2.2) we obtain

$$U(X,Y) = -\eta(X)\varphi Y. \tag{2.5}$$

Hence a quarter-symmetric metric connection $\widetilde{\nabla}$ in a Kenmotsu manifold is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi Y. \tag{2.6}$$

Let R and $\widetilde{R}$ be the curvature tensors of ∇ and $\widetilde{\nabla}$ of a Kenmotsu manifold, respectively. In view of (2.6) and (1.7), we obtain

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + \eta(X)(\nabla_Y \varphi)Z - \eta(Y)(\nabla_X \varphi)Z,$$

which in view of (1.6) we get

$$\begin{aligned} \widetilde{R}(X,Y)Z = R(X,Y)Z &+ \eta(X)g(\varphi Y, Z)\xi - \eta(Y)g(\varphi X, Z)\xi \\ &- \eta(X)\eta(Z)\varphi Y + \eta(Y)\eta(Z)\varphi X. \end{aligned} \tag{2.7}$$

A relation between the curvature tensor of M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ and Levi-Civita connection ∇ is given by the equation (2.7). So from (2.7) and (1.9) we have

$$\widetilde{R}(X,\xi)Y = g(X,Y)\xi - \eta(Y)X - g(\varphi X, Y)\xi + \eta(Y)\varphi X, \tag{2.8}$$

$$\widetilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X - \eta(X)\varphi Y + \eta(Y)\varphi X \tag{2.9}$$

and

$$\widetilde{R}(\xi, X)\xi = X - \eta(X)\xi - \varphi X. \tag{2.10}$$

Taking the inner product of (2.7) with W, we have

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) + \eta(X)\eta(W)g(\varphi Y, Z) \\ & - \eta(Y)\eta(W)g(\varphi X, Z) - \eta(X)\eta(Z)g(\varphi Y, W) \\ & + \eta(Y)\eta(Z)g(\varphi X, W).\end{aligned} \tag{2.11}$$

Contracting (2.11) over X and W, we obtain

$$\widetilde{S}(Y,Z) = S(Y,Z) + g(\varphi Y, Z), \tag{2.12}$$

where $\widetilde{S}$ and S are the Ricci tensors of the connections $\widetilde{\nabla}$ and ∇, respectively. So in a Kenmotsu manifold, the Ricci tensor of the quarter-symmetric metric connection is not symmetric. Again, contracting (2.12) over Y and Z, we get

$$\widetilde{r} = r, \tag{2.13}$$

where $\widetilde{r}$ and r are the scalar curvatures of the connections $\widetilde{\nabla}$ and ∇, respectively. So we have the following theorem:

Theorem 1. *For a Kenmotsu manifold M with the quarter-symmetric metric connection $\widetilde{\nabla}$*

(a) *The curvature tensor $\widetilde{R}$ is given by* (2.7)*,*
(b) *The Ricci tensor $\widetilde{S}$ is given by* (2.12)*,*
(c) $\widetilde{R}(X,Y,Z,W) + \widetilde{R}(X,Y,W,Z) = 0,$
(d) $\widetilde{R}(X,Y,Z,W) + \widetilde{R}(Y,X,Z,W) = 0,$
(e) $\widetilde{S}(Y,\xi) = S(Y,\xi) = (1-n)\eta(Y),$
(f) $\widetilde{r} = r,$
(g) *The Ricci tensor $\widetilde{S}$ is not symmetric.*

§3. Locally symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection

In this section, we consider locally symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. We have the following theorem:

Theorem 2. *Let M be a locally symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then the scalar curvature of the Levi-Civita connection of M is equal to $n(1-n)$.*

Proof. Assume that M is a locally symmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then $(\widetilde{\nabla}_X \widetilde{R})(Y,Z)W = 0$. So by a suitable contraction of this equation we have

$$(\widetilde{\nabla}_X \widetilde{S})(Z,W) = \widetilde{\nabla}_X \widetilde{S}(Z,W) - \widetilde{S}(\widetilde{\nabla}_X Z, W) - \widetilde{S}(Z, \widetilde{\nabla}_X W) = 0.$$

Taking $W = \xi$ in above equation we have

$$(\widetilde{\nabla}_X \widetilde{S})(Z,\xi) = \widetilde{\nabla}_X \widetilde{S}(Z,\xi) - \widetilde{S}(\widetilde{\nabla}_X Z, \xi) - \widetilde{S}(Z, \widetilde{\nabla}_X \xi) = 0.$$

By making use of (1.7), (1.10), (2.6) and (2.12) we get

$$(1-n)g(X,Z) - S(X,Z) - g(\varphi Z, X) = 0.$$

Then contracting the last equation over X and Z we obtain

$$r = n(1-n).$$

Thus the proof of the theorem is completed. □

§4. Non-existence of certain kinds of Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Theorem 3. *There is no generalized recurrent Kenmotsu manifold with respect to the quarter-symmetric metric connection* $\widetilde{\nabla}$.

Proof. Suppose that there exists a generalized recurrent Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then from (0.4), we have

$$(\widetilde{\nabla}_X \widetilde{R})(Y,Z)W = \alpha(X)\widetilde{R}(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z] \tag{4.1}$$

for all vector fields X, Y, Z, W on M. Putting $Y = W = \xi$ in (4.1) we have

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = \alpha(X)\widetilde{R}(\xi,Z)\xi + \beta(X)[\eta(Z)\xi - Z].$$

By making use of (2.7) and (1.9) we get

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = [\beta(X) - \alpha(X)]\{\eta(Z)\xi - Z\} - \alpha(X)\varphi Z. \tag{4.2}$$

On the other hand, in view of (1.7), (1.9), (2.6), (2.8), (2.9) and (2.10) we have

$$(\widetilde{\nabla}_X \widetilde{R})(\xi,Z)\xi = 0. \tag{4.3}$$

Hence comparing the right hand sides of the equations (4.2) and (4.3) we obtain

$$[\beta(X) - \alpha(X)]\{\eta(Z)\xi - Z\} - \alpha(X)\varphi Z = 0. \tag{4.4}$$

Replacing Z by φZ in (4.4) we get

$$[\beta(X) - \alpha(X)]\varphi Z + \alpha(X)\{\eta(Z)\xi - Z\} = 0. \tag{4.5}$$

From (4.4) and (4.5) we have

$$[\alpha(X)]^2 + [\beta(X) - \alpha(X)]^2 = 0,$$

which implies that $\alpha = \beta = 0$. This contradicts $\beta \neq 0$. Therefore the statement of this theorem follows. □

Theorem 4. *There is no φ-recurrent Kenmotsu manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$.*

Proof. Suppose that there exists a φ-recurrent Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then from (0.5), we have

$$\varphi^2((\widetilde{\nabla}_X \widetilde{R})(Y, Z)W) = \alpha(X)\widetilde{R}(Y, Z)W$$

for all vector fields X, Y, Z, W on M. Using (1.3) we get

$$-(\widetilde{\nabla}_X \widetilde{R})(Y, Z)W + \eta((\widetilde{\nabla}_X \widetilde{R})(Y, Z)W)\xi = \alpha(X)\widetilde{R}(Y, Z)W. \tag{4.6}$$

Replacing Y and W with ξ in (4.6) we have

$$-(\widetilde{\nabla}_X \widetilde{R})(\xi, Z)\xi + \eta((\widetilde{\nabla}_X \widetilde{R})(\xi, Z)\xi)\xi = \alpha(X)\widetilde{R}(\xi, Z)\xi. \tag{4.7}$$

On the other hand, from (4.3) we have $(\widetilde{\nabla}_X \widetilde{R})(\xi, Z)\xi = 0$. So the equation (4.7) turns into

$$\alpha(X)\widetilde{R}(\xi, Z)\xi = 0.$$

Then by virtue of (2.10), it is obvious that

$$\alpha(X)[Z - \varphi Z - \eta(Z)\xi] = 0,$$

which implies $\alpha(X) = 0$ for any vector field X on M. This contradicts $\alpha \neq 0$. Therefore the statement of this theorem follows. □

Theorem 5. *There is no pseudosymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$.*

Proof. Suppose that there exists a pseudosymmetric Kenmotsu manifold M with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. Then from (0.6) we have

$$(\widetilde{\nabla}_X\widetilde{R})(Y,Z)W = 2\alpha(X)\widetilde{R}(Y,Z)W + \alpha(Y)\widetilde{R}(X,Z)W + \alpha(Z)\widetilde{R}(Y,X)W + \alpha(W)\widetilde{R}(Y,Z)X + g(\widetilde{R}(Y,Z)W,X)A. \tag{4.8}$$

So by a suitable contraction of (4.8) we get

$$\begin{aligned}(\widetilde{\nabla}_X\widetilde{S})(Z,W) = {} & 2\alpha(X)\widetilde{S}(Z,W) + \alpha(\widetilde{R}(X,Z)W) \\ & + \alpha(Z)\widetilde{S}(X,W) + \alpha(W)\widetilde{S}(Z,X) \\ & - \alpha\,(R(W,X)Z) + \alpha(\xi)\eta(X)g(\varphi Z,W) \\ & + \eta(X)\eta(Z)\alpha(\varphi W) - \alpha(\xi)\eta(W)g(\varphi Z,X) \\ & - \eta(Z)\eta(W)\alpha(\varphi X).\end{aligned} \tag{4.9}$$

Taking $W=\xi$ in (4.9) and using (1.10), (2.8), (2.9) and (2.12) we obtain

$$\begin{aligned}(\widetilde{\nabla}_X\widetilde{S})(Z,\xi) = {} & -2n\alpha(X)\eta(Z) + (2-n)\eta(X)\alpha(Z) - \eta(X)\alpha(\varphi Z) \\ & + \alpha(\xi)g(X,Z) + \alpha(\xi)S(X,Z).\end{aligned} \tag{4.10}$$

On the other hand, by the covariant differentiation of the Ricci tensor $\widetilde{S}$ with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$, we have

$$(\widetilde{\nabla}_X\widetilde{S})(Z,W) = \widetilde{\nabla}_X\widetilde{S}(Z,W) - \widetilde{S}(\widetilde{\nabla}_XZ,W) - \widetilde{S}(Z,\widetilde{\nabla}_XW). \tag{4.11}$$

So putting $W=\xi$ in (4.11) and using (2.12), (2.6) and (1.7) we get

$$(\widetilde{\nabla}_X\widetilde{S})(Z,\xi) = (1-n)g(X,Z) - S(X,Z) - g(X,\varphi Z). \tag{4.12}$$

Then comparing the right hand sides of the equations (4.10) and (4.12), we obtain

$$\begin{aligned}& (1-n)g(X,Z) - S(X,Z) - g(X,\varphi Z) \\ & = -2n\alpha(X)\eta(Z) + (2-n)\eta(X)\alpha(Z) - \eta(X)\alpha(\varphi Z) \\ & \quad + \alpha(\xi)g(X,Z) + \alpha(\xi)S(X,Z).\end{aligned}$$

Replacing X and Z with ξ in above equation we find (since $n>3$)

$$\alpha(\xi) = 0. \tag{4.13}$$

Now we show that $\alpha=0$ holds for any vector field on M. Taking $Z=\xi$ in (4.9) and using (4.13) we get

$$\begin{aligned}(\widetilde{\nabla}_X\widetilde{S})(\xi,W) = {} & 2\alpha(X)\widetilde{S}(\xi,W) + \alpha(\widetilde{R}(X,\xi)W) \\ & + \alpha(W)\widetilde{S}(\xi,X) - \alpha(R(W,X)\xi) \\ & + \eta(X)\alpha(\varphi W) - \eta(W)\alpha(\varphi X).\end{aligned} \tag{4.14}$$

By the use of (1.10), (2.8), (2.9), (4.11) and (4.13) in (4.14) we obtain

$$\begin{aligned}&(1-n)g(X,W)-S(X,W)-g(\varphi X,W)\\&=-2n\alpha(X)\eta(W)+(2-n)\alpha(W)\eta(X)+\eta(X)\alpha(\varphi W).\end{aligned}\tag{4.15}$$

Taking $W=\xi$ in (4.15) we find $\alpha(X)=0$ for every vector field X on M, which implies that $\alpha=0$ on M. This contradicts to the definition of pseudosymmetry. Thus our theorem is proved. □

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