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Propagation of singularities for a system of semilinear wave equations with null condition

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Abstract. We study the propagation of singularities for a system of semilinear wave equations satisfying the null condition in one space dimension. We show that if a solution (u, v) to the system is in $H^s_{loc}(\Omega) \cap H^r_{ml}(0, x_0, \tau_0, \xi_0)$, then $(u, v) \in H^r_{ml}(\Gamma)$ as long as $3/2 < s < r < 2s - 1$, where $\Omega \subset \mathbb{R}^2$ is an open set and Γ is a null bicharacteristic of $\Box$ passing through $(0, x_0, \tau_0, \xi_0)$.

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§1. Introduction

In this paper, we consider the propagation of singularities of solutions to the following system of semilinear wave equations with the null condition in one space dimension,

(1.1)
$$\begin{cases} \Box u = h_1(u,v)Q_0(u,u) + h_2(u,v)Q_0(u,v) + h_3(u,v)Q_0(v,v) + h_4(u,v)Q_1(u,v), \\ \Box v = h_5(u,v)Q_0(u,u) + h_6(u,v)Q_0(u,v) + h_7(u,v)Q_0(v,v) + h_8(u,v)Q_1(u,v), \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \\ v(0,x) = v_0(x),\ \partial_t v(0,x) = v_1(x), \end{cases}$$

where $(t, x) \in \mathbb{R}^2$, $u(t, x)$, $v(t, x)$, $u_0(x)$, $u_1(x)$, $v_0(x)$ and $v_1(x)$ are real valued functions, $h_j(u, v)$ are polynomials of u and v for $j = 1, 2, \ldots, 8$ and Q_0, Q_1 are the null forms

$$Q_0(f, g) = (\partial_t f)(\partial_t g) - (\partial_x f)(\partial_x g) \tag{1.2}$$

and

$$Q_1(f,g) = (\partial_t f)(\partial_x g) - (\partial_x f)(\partial_t g). \tag{1.3}$$

We assume that $3/2 < s \le 2$, initial data u_0 and v_0 are in $H^s(\mathbb{R})$ and u_1 and v_1 are in $H^{s-1}(\mathbb{R})$.

Notation. $\langle\xi\rangle$ and $\langle\tau,\xi\rangle$ denote $(1+|\xi|^2)^{1/2}$ and $(1+|\tau|^2+|\xi|^2)^{1/2}$ respectively. If $\Omega \subset \mathbb{R}^n$ is open, $H^s_{loc}(\Omega)$ is the standard Sobolev space of distributions u such that $\langle\xi\rangle^s \widehat{\phi u} \in L^2(\mathbb{R}^n)$ for all $\phi \in C_0^\infty(\Omega)$. Let $(x_0,\xi_0) \in \Omega \times (\mathbb{R}^n_\xi \backslash \{0\})$. We say that u is in microlocally H^r at (x_0,ξ_0) and write $u \in H^r_{ml}(x_0,\xi_0)$ if there exists a $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) = 1$ and a conic neighborhood K of ξ_0 in $\mathbb{R}^n\backslash\{0\}$ such that $\langle\xi\rangle^r \chi_K(\xi)|\widehat{\phi u}(\xi)| \in L^2(\mathbb{R}^n)$, where χ_K is the characteristic function of K. If Γ is a closed conic set in $\Omega \times (\mathbb{R}^n_\xi\backslash\{0\})$, we say that $u \in H^r_{ml}(\Gamma)$ if $u \in H^r_{ml}(x,\xi)$ for all $(x,\xi) \in \Gamma$. Let $p(x,\xi)$ is a characteristic polynomial of differential operator P of order m and homogeneous of degree m in ξ. For a point (x_0,ξ_0) with $p(x_0,\xi_0) = 0$, the null bicharacteristic through (x_0,ξ_0) is the curve defined by $\frac{dx}{ds} = \frac{\partial p}{\partial \xi}$, $\frac{d\xi}{ds} = -\frac{\partial p}{\partial x}$ with $x(0) = x_0$, $\xi(0) = \xi_0$. Throughout this paper, C_s serves as a generalized positive constant depending only on s if the precise value of which is not needed.

In the case of the linear wave equation $\Box u = 0$, Hörmander [9] has shown that the wave front set of u propagates along the null bicharacteristic for $\Box$. Generally, in the case of nonlinear wave equations, such a result cannot be obtained. However, it is known that if we consider the microlocal Sobolev regularity and assume the suitable range of the Sobolev exponent, then a phenomenon similar to the linear case is observed. In [20], Rauch first analyzed such result for the solutions to $\Box u = f(u)$ where f is a polynomial of u. Let $f \in C^\infty$, $\Omega \subset \mathbb{R}^n$ be an open set, $u \in H^s_{loc}(\Omega) \cap H^r_{ml}(t_0,x_0,\tau_0,\xi_0)$ be a solution to

$$\Box u = f(u, Du) \tag{1.4}$$

and (t_0,x_0,τ_0,ξ_0) is a point in the null bicharacteristic $\Gamma \subset \Omega \times \mathbb{R}^n$ of $\Box$. Bony [7] and Beals-Reed [6] have shown that u is in H^r_{ml} at all points of Γ as long as for $n/2+1 < s \leqq r < 2s-1-n/2$ by the different way, respectively. For a second order strictly hyperbolic differential operator $p_2(x,D)$, Beals [5] has shown that solutions $u \in H^s_{loc}(\Omega) \cap H^r_{ml}(x_0,\xi_0)$ to $p_2(x,D)u = f(u,Du)$ is in H^r_{ml} at all points of a null bicharacteristic of p_2 starting from (x_0,ξ_0) as long as for $n/2+1 < s \leqq r < 3s-n-2$ by using a simple commutator lemma, Rauch's lemma and the standard calculus of pseudo differential operators. The technique used in Beals [5] plays an important role in this paper. In [18], Linqi Liu has shown that the same result holds in the case of $n/2+1 < s \leqq$

$r < 3s - n - 1$ for $\Box u = f(u, Du)$ by using a particular kind of weighted Sobolev spaces. In the case of a system, H.Michael [19] has shown that for $U = (u_1, \ldots, u_m) \in H^s_{loc}(\Omega) \cap H^r_{ml}(x_0, \xi_0)$ which are solutions to $p_2(x, D)U = G(U)$ $(G \in C^\infty)$, U is in H^r_{ml} at all points of the null bicharacteristic of Γ as long as for $n/2 < s \leqq r < 3s - n + 1$. On the other hand, if r is sufficiently large, new singularities are observed (refer to [3], [4], [21] and [22]). We are interested in the threshold of r and s. Although numerous attempts have been made to study these analysis, the threshold of r has not been determined exactly. In [11] and [12], we consider the case which nonlinearity satisfying the null condition. The null condition is defined by Klainerman [14]. Klainerman introduced the null condition as a sufficient condition for a global existence of smooth solutions to $\Box u = F(u, u', u'')$, which is defined as follows.

Definition 1.1. *(Klainerman [14]) Let $F(u, v_1, \ldots, v_n)$ a real valued function, smoothly defined in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R}^n$. We say that $F(u, Du)$ (where Du denote the first partial derivatives of u) satisfies the null condition if, for any u, v and any vector $X = (X_1, \ldots, X_n)$ such that $X_1^2 - \sum_{i=2}^n X_i^2 = 0$, the following identity holds;*

$$\sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial v_i \partial v_j} X_i X_j = 0. \tag{1.5}$$

The attempt to lead the global and local existence theorem has been studied by a lot of people who have improved the null condition (refer to [1], [2], [8], [10], [13], [16], [17] and [23]). The null condition of semilinear wave equation which we consider are restricted to (1.1). In [12], we improved a lower bound of the threshold of s and r in the case that the nonlinear term satisfies the null condition. We have shown that if $n/2 < s \leqq r < 3s - n$ then u is in H^r_{ml} at all points of a null bicharacteristic of $\Box$. The key of the proof is to make the Cole-Hopf type transformation to u. This transformation makes nonlinearity of (1.4) change to a polynomial of first degree with respect to Du. Then, we can apply the result of Beals [5] directly. This feature is obtained when nonlinearity satisfies the null condition.

The result of this paper is an extension of [12] to the system (1.1) and we show that the same result is true for a time local solution (u, v) of the system (1.1) as long as $3/2 < s \leqq r < 2s - 1$. In the case of the system (1.1), the Cole-Hopf type transformation doesn't work. To avoid this problem, we estimate the microlocal regularity of the solution in the function space used by Klainerman and Machedon [15]. Firstly, we construct a time local solution of the initial value problem (1.1) in the function space associated to $\Box$ introduce in [15]. Secondly, we prove a propagation of a singularities to the constructed solution in the above function space by using the idea of Beals [5].

In order to state the main results precisely, we define several function spaces. We put

(1.6)
$$X^s = \left\{ f \in H^s_{loc}(\mathbb{R}^2) \;\middle|\; \begin{array}{l} {}^\forall a(t) \in C_0^\infty(\mathbb{R}) \text{ such that } a(t)f \in H^s(\mathbb{R}^2), \\ f|_{t=0} \in H^s(\mathbb{R}),\ \partial_t f|_{t=0} \in H^{s-1}(\mathbb{R}), \\ \|\Box f\|_{X_1^{s-1}} < \infty \end{array} \right\}$$

with norm

$$\|f\|_{X^s} = \|f|_{t=0}\|_{H^s} + \|\partial_t f|_{t=0}\|_{H^{s-1}} + \|\Box f\|_{X_1^{s-1}} \tag{1.7}$$

and $\|F\|_{X_1^s} = \|\langle D_x\rangle^s F\|_{L^2_{t,x}}$.

Proposition 1.2. *Let $3/2 < s \le 2$. Then for any u_0, $v_0 \in H^s$ and u_1, $v_1 \in H^{s-1}$, there exists a positive constant T and a unique time local solution (u, v) of the initial value problem* (1.1) *satisfying*

$$(u, v) \in \{X^s \cap L^\infty([-T, T]; H^s_x)\} \times \{X^s \cap L^\infty([-T, T]; H^s_x)\}. \tag{1.8}$$

Our main result is the following.

Theorem 1.3. *Let $3/2 < s \le 2$ and $(u, v) \in \{X^s \cap L^\infty([-T, T]; H^s_x)\} \times \{X^s \cap L^\infty([-T, T]; H^s_x)\}$ be a time local solution constructed in Proposition 1.2. If $\Gamma \subset \mathbb{R}^2_{t,x} \times (\mathbb{R}^2_{\tau,\xi}\backslash\{0\})$ denotes a null bicharacteristic of $\Box$ and $(u, v) \in H^r_{ml}(0, x_0, \tau_0, \xi_0) \times H^r_{ml}(0, x_0, \tau_0, \xi_0)$ for a point $(0, x_0, \tau_0, \xi_0)$ on Γ, then $(u, v) \in H^r_{ml}(\Gamma) \times H^r_{ml}(\Gamma)$ for $|t| < T$ as long as $r < 2s - 1$.*

Remark 1.4. *For the case of $s > 2$, the same result holds but this case is treated in Beals [5]. So we do not treat this case.*

Remark 1.5. *Simple calculation shows that the null bicharacteristic of $\Box$ through the point $(0, x_0, \tau_0, \xi_0) \in \mathbb{R}^2_{t,x} \times (\mathbb{R}^2_{\tau,\xi}\backslash\{0\})$ with $\tau_0 = \pm|\xi_0|$ is the straight line $\Gamma = \{(t, x, \tau_0, \xi_0) \mid x = x_0 - (\xi_0/\tau_0)t\}$.*

In section 2, we prepare the null form estimate which is necessary for proving the existence of a time local solution. In section 3, we prove the existence of a time local solution. In section 4, we prove a propagation of a singularity theorem.

§2. Estimate for the Null form

In this section, we give the estimates for the X_1^{s-1} norm of the null forms. From the definition (1.2) and (1.3) of the null forms, we can rewrite

$$Q_0(u, v) = \frac{1}{2}\{(\partial_t + \partial_x)u \cdot (\partial_t - \partial_x)v + (\partial_t - \partial_x)u \cdot (\partial_t + \partial_x)v\} \tag{2.1}$$

and

$$Q_1(u,v) = \frac{1}{2}\{(\partial_t - \partial_x)u \cdot (\partial_t + \partial_x)v - (\partial_t + \partial_x)u \cdot (\partial_t - \partial_x)v\}. \tag{2.2}$$

Let $a(t) \in C_0^\infty(\mathbb{R})$ such that $a(t) = 1$ for $|t| \le 1/2$, $a(t) = 0$ for $|t| \ge 1$ and $0 \le a(t) \le 1$ and put $a_T(t) = a(t/T)$ for $T > 0$. The following lemma is prepared in order to prove Proposition 2.2.

Lemma 2.1. *Let $f, g \in H^{s-1}(\mathbb{R})$ for $3/2 < s \le 2$. Then*

$$\|a_T(t)f(x+t)g(x-t)\|_{X_1^{s-1}} \le C\sqrt{T}\,\|f\|_{H^{s-1}}\|g\|_{H^{s-1}}, \tag{2.3}$$

where C is a constant depending on s and $a(t)$.

Proof. Since $s-1 > 1/2$, $fg \in H^{s-1}(\mathbb{R})$. Hence we have

$$\begin{aligned}\|a_T(t)f(x+t)g(x-t)\|_{X_1^{s-1}} &\le \left\| a_T(t)\left\|\langle D_x\rangle^{s-1}(f(x+t)g(x-t))\right\|_{L_x^2}\right\|_{L_t^2} \\ &\le C_s\,\|a_T(t)\|_{L_t^2}\,\|f\|_{H^{s-1}}\|g\|_{H^{s-1}} \\ &\le C_s\sqrt{T}\|a\|_{L^2}\,\|f\|_{H^{s-1}}\|g\|_{H^{s-1}}.\end{aligned}$$

Putting $C = C_s\|a\|_{L^2}$, we have the conclusion. ∎

Proposition 2.2. *Let $(u,v) \in X^s \times X^s$ for $3/2 < s \le 2$. Then we have*

$$\|a_T(t)Q(u,v)\|_{X_1^{s-1}} \le CT'\|u\|_{X^s}\|v\|_{X^s}, \tag{2.4}$$

where $Q(u,v)$ stands either $Q_0(u,v)$ or $Q_1(u,v)$, $T' = max\{T^{3/2}, T, T^{1/2}\}$ and C is a constant depending on s and $a(t)$.

Proof. We prove only the case of $Q_0(u,u)$, since the other cases can be proved similarly. We put $u(0,x) = u_0(x)$, $\partial_t u(0,x) = u_1(x)$ and $f_0(t,x) = \frac{1}{2}\{u_0(x+t) + u_0(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy$. By the density argument, we have

$$a_T(t)(\partial_t \pm \partial_x)u = a_T(t)(\partial_t \pm \partial_x)f_0 + 2a_T(t)\int_0^t \Box u(\alpha, x \pm t \mp \alpha)d\alpha$$

holds in $L^2_{loc}(\mathbb{R}, H_x^{s-1}(\mathbb{R}))$. Putting $f_{0,\pm} = (\partial_t \pm \partial_x)f_0$, $u_\pm = (\partial_t \pm \partial_x)u$ and $\widetilde{u}_\pm = 2\int_0^t \Box u(\alpha, x \pm t \mp \alpha)d\alpha$, we have

$$\begin{aligned}&\|a_T Q_0(u,u)\|_{X_1^{s-1}} \\ &\le \|a_T\widetilde{u}_+\widetilde{u}_-\|_{X_1^{s-1}} + \|a_T\widetilde{u}_+ f_{0,-}\|_{X_1^{s-1}} + \|a_T\widetilde{u}_- f_{0,+}\|_{X_1^{s-1}} + \|a_T f_{0,+}f_{0,-}\|_{X_1^{s-1}}.\end{aligned}$$

We only show that

$$\|a_T\widetilde{u}_+\widetilde{u}_-\|_{X_1^{s-1}} \le C_s T^{3/2}\|a\|_{L^2}\|\Box u\|^2_{X_1^{s-1}} \tag{2.5}$$

since the other terms can be estimated similarly. By Lemma 2.1 and the Schwarz inequality, we obtain

$$\begin{aligned}
&\|a_T(t)\widetilde{u}_+\widetilde{u}_-\|_{X_1^{s-1}} \\
&\le C_s\int_0^T\int_0^T \left\|a_T(t)\langle D_x\rangle^{s-1}\{\Box u(\alpha, x+t-\alpha)\Box u(\beta, x-t+\beta)\}\right\|_{L^2_{t,x}} d\alpha d\beta \\
&\le C_s\int_0^T\int_0^T \sqrt{T}\|a\|_{L^2}\,\|\Box u(\alpha,\cdot)\|_{H^{s-1}}\|\Box u(\beta,\cdot)\|_{H^{s-1}} d\alpha d\beta \\
&\le C_s T^{3/2}\|a\|_{L^2}\|\Box u\|^2_{X_1^{s-1}}.
\end{aligned} \tag{2.6}$$

Similarly, we obtain

$$\|a_T(t)f_{0,+}f_{0,-}\|_{X_1^{s-1}} \le C_s T^{1/2}\|a\|_{L^2}(\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}})^2, \tag{2.7}$$

$$\|a_T(t)\widetilde{u}_+f_{0,-}\|_{X_1^{s-1}} \le C_s T\|a\|_{L^2}(\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}})\|\Box u\|_{X_1^{s-1}} \tag{2.8}$$

and

$$\|a_T(t)\widetilde{u}_-f_{0,+}\|_{X_1^{s-1}} \le C_s T\|a\|_{L^2}(\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}})\|\Box u\|_{X_1^{s-1}}. \tag{2.9}$$

By (2.6), (2.7), (2.8) and (2.9), we have the conclusion. ∎

§3. Existence of solutions

Let

$$X^s_\rho = \{f\in X^s \mid \|f|_{t=0}\|_{H^s}+\|\partial_t f|_{t=0}\|_{H^{s-1}}\le \rho/8,\ \|\Box f\|_{X_1^{s-1}}\le\rho\}$$

and

$$Y^s_{\rho,T} = \{f\in L^\infty([-T,T];H^s_x) \mid \|f\|_{Y^s_T}\le\rho\}$$

with $\|f\|_{Y^s_T} = \|f\|_{L^\infty([-T,T];H^s_x)}$. We define a mapping M formally as follows

$$M(u,v) = {}^t\begin{pmatrix} M_1(u,v)\\ M_2(u,v)\end{pmatrix} = {}^t\begin{pmatrix} f_0+\int_0^t U(t-\alpha)a_T(\alpha)A(u,v;\alpha,x)d\alpha \\ g_0+\int_0^t U(t-\alpha)a_T(\alpha)B(u,v;\alpha,x)d\alpha\end{pmatrix}, \tag{3.1}$$

where U is the evolution operator for $\Box$ defined by $U(t)\psi(x) = \int_{x-t}^{x+t} \psi(y)dy$, $f_0(t,x) = \frac{1}{2}\{u_0(x+t)+u_0(x-t)\}+\frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy$, $g_0(t,x) = \frac{1}{2}\{v_0(x+t)+v_0(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} v_1(y)dy$ with $u(0,x) = u_0(x)$, $\partial_t u(0,x) = u_1(x)$, $v(0,x) = v_0(x)$ and $\partial_t v(0,x) = v_1(x)$, $a_T(t)$ is a function defined in section 2,

$$\begin{aligned} &A(u,v;t,x)\\ &\quad = h_1(u,v)Q_0(u,u) + h_2(u,v)Q_0(u,v) + h_3(u,v)Q_0(v,v) + h_4(u,v)Q_1(u,v) \end{aligned}$$

and

$$\begin{aligned} &B(u,v;t,x)\\ &\quad = h_5(u,v)Q_0(u,u) + h_6(u,v)Q_0(u,v) + h_7(u,v)Q_0(v,v) + h_8(u,v)Q_1(u,v). \end{aligned}$$

We show that the mapping M is a contraction mapping on $(X^s_\rho \cap Y^s_{\rho,T}) \times (X^s_\rho \cap Y^s_{\rho,T})$ for sufficiently small $T > 0$. To prove this, we use the following lemma.

Lemma 3.1. *Let $(u,v) \in (X^s_\rho \cap Y^s_{\rho,T}) \times (X^s_\rho \cap Y^s_{\rho,T})$ for $3/2 < s \leq 2$. Then we have*

$$\|\Box M_\ell(u,v)\|_{X_1^{s-1}} \leq CT' \sum_{0\leq j+k\leq n} \|u\|^j_{Y^s_T} \|v\|^k_{Y^s_T} (\|u\|_{X^s} + \|v\|_{X^s})^2, \tag{3.2}$$

where M_ℓ ($\ell = 1,2$) is the mapping defined in (3.1), $T' = max\{T^{3/2}, T, T^{1/2}\}$, *$C$ is a constant depending on s, $a(t)$ and h_i and n is the maximum of orders of h_i ($i = 1, 2, \ldots, 8$).*

Proof. From the definition (3.1) of the mapping M and the triangle inequality, we have

$$\begin{aligned} \|\Box M_1(u,v)\|_{X_1^{s-1}} &= \|a_T(t)A(u,v;t,x)\|_{X_1^{s-1}}\\ &\leq \|a_T(t)h_1(u,v)Q_0(u,u)\|_{X_1^{s-1}} + \|a_T(t)h_2(u,v)Q_0(u,v)\|_{X_1^{s-1}}\\ &\quad + \|a_T(t)h_3(u,v)Q_0(v,v)\|_{X_1^{s-1}} + \|a_T(t)h_4(u,v)Q_1(u,v)\|_{X_1^{s-1}}. \end{aligned}$$

We put $h_i(u,v) = \sum_{0\leq j+k\leq n_i} c^{(i)}_{j,k} u^j v^k$. By Proposition 2.2 and the assumption $s - 1 > 1/2$, we have

$$\begin{aligned} &\|a_T(t)h_1(u,v)Q_0(u,u)\|_{X_1^{s-1}}\\ &\leq C_s \left\| \|h_1(u,v)\|_{H_x^{s-1}} \|a_T(t)Q_0(u,u)\|_{H_x^{s-1}} \right\|_{L^2_{[-T,T]}}\\ &\leq C_s \left\| \|h_1(u,v)\|_{H_x^{s-1}} \right\|_{L^\infty_{[-T,T]}} \|a_T(t)Q_0(u,u)\|_{X_1^{s-1}}\\ &\leq C_1 T' \sum_{0\leq j+k\leq n_1} \|u\|^j_{Y^s_T} \|v\|^k_{Y^s_T} \|u\|^2_{X^s}, \end{aligned}$$

where C_1 is a constant depending on s, $a(t)$ and h_1. Similarly, we have

$$\|a_T(t)h_2(u,v)Q_0(u,v)\|_{X_1^{s-1}} \le C_2T' \sum_{0\le j+k\le n_2} \|u\|_{Y_T^s}^j \|v\|_{Y_T^s}^k \|u\|_{X^s}\|v\|_{X^s},$$

$$\|a_T(t)h_3(u,v)Q_0(v,v)\|_{X_1^{s-1}} \le C_3T' \sum_{0\le j+k\le n_3} \|u\|_{Y_T^s}^j \|v\|_{Y_T^s}^k \|v\|_{X^s}^2$$

and

$$\|a_T(t)h_4(u,v)Q_1(u,v)\|_{X_1^{s-1}} \le C_4T' \sum_{0\le j+k\le n_4} \|u\|_{Y_T^s}^j \|v\|_{Y_T^s}^k \|u\|_{X^s}\|v\|_{X^s},$$

where C_i are constants depending on s, $a(t)$ and h_i $(i = 2, 3, 4)$. If we put $n = \max_{1\le i\le 4} n_i$ and $C = \max_{1\le i\le 4} C_i$, then we obtain

$$\|\Box M_1(u,v)\|_{X_1^{s-1}} \le CT' \sum_{0\le j+k\le n} \|u\|_{Y_T^s}^j \|v\|_{Y_T^s}^k (\|u\|_{X^s} + \|v\|_{X^s})^2.$$

Similarly, we obtain

$$\|\Box M_2(u,v)\|_{X_1^{s-1}} \le CT' \sum_{0\le j+k\le n} \|u\|_{Y_T^s}^j \|v\|_{Y_T^s}^k (\|u\|_{X^s} + \|v\|_{X^s})^2.$$

∎

Lemma 3.2. *Under the same assumptions in Lemma 3.1, we have*

$$\begin{aligned}\|M_1(u,v)\|_{Y_T^s} &\\ \le CT' \sum_{0\le j+k\le n} &\|u\|_{Y_T^s}^j \|v\|_{Y_T^s}^k (\|u\|_{X^s} + \|v\|_{X^s})^2 + \|u_0\|_{H^s} + 2\|u_1\|_{H^{s-1}}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\|M_2(u,v)\|_{Y_T^s} &\\ \le CT' \sum_{0\le j+k\le n} &\|u\|_{Y_T^s}^j \|v\|_{Y_T^s}^k (\|u\|_{X^s} + \|v\|_{X^s})^2 + \|v_0\|_{H^s} + 2\|v_1\|_{H^{s-1}},\end{aligned} \tag{3.4}$$

where C are constants depending on s, $a(t)$ and h_i $(i = 1, 2, \ldots, 8)$.

Proof. From the definition (3.1) of the mapping M and the triangle inequality, we have

$$\|M_1(u,v)\|_{Y_T^s} = \left\| f_0 + \int_0^t U(t-\alpha)a_T(\alpha)A(u,v;\alpha,x)d\alpha \right\|_{L^\infty([-T,T];H_x^s)}$$
$$\leq \|f_0\|_{L^\infty([-T,T];H_x^s)} + C_s\left(\left\| \int_0^t U(t-\alpha)a_T(\alpha)A(u,v;\alpha,x)d\alpha \right\|_{L^\infty([-T,T];L_x^2)} \right.$$
$$\left. + \left\| |D_x|^s \int_0^t U(t-\alpha)a_T(\alpha)A(u,v;\alpha,x)d\alpha \right\|_{L^\infty([-T,T];L_x^2)} \right).$$

By change of variables and the Schwarz inequality, we obtain

$$\left\| \int_0^t U(t-\alpha)a_T(\alpha)A(u,v;\alpha,x)d\alpha \right\|_{L^\infty([-T,T];L_x^2)}$$
$$\leq \left\| \int_0^T \left\| \int_{x-(t-\alpha)}^{x+t-\alpha} a_T(\alpha)A(u,v;\alpha,y)dy \right\|_{L_x^2} d\alpha \right\|_{L^\infty[-T,T]}$$
$$\leq \left\| \int_0^T \int_{-(t-\alpha)}^{t-\alpha} \|a_T(\alpha)A(u,v;\alpha,x+y)\|_{L_x^2}\, dyd\alpha \right\|_{L^\infty[-T,T]}$$
$$\leq \left\| 2 \sup_{\alpha\in[0,T]} |t-\alpha| \int_0^T \|a_T(\alpha)A(u,v;\alpha,x)\|_{L_x^2}\, d\alpha \right\|_{L^\infty[-T,T]}$$
$$\leq 2 \left\| \sup_{\alpha\in[0,T]} |t-\alpha|\sqrt{T}\|a_T(t)A(u,v;t,x)\|_{X_1^{s-1}} \right\|_{L^\infty[-T,T]}$$
$$\leq 4T^{3/2}\|a_T(t)A(u,v;t,x)\|_{X_1^{s-1}}$$

and

$$\left\| |D_x|^s \int_0^t U(t-\alpha)a_T(\alpha)A(u,v;\alpha,x)d\alpha \right\|_{L^\infty([-T,T];L_x^2)}$$
$$\leq \left\| \left\| \int_0^t |D_x|^{s-1}a_T(\alpha)(A(u,v;\alpha,x+t-\alpha)-A(u,v;\alpha,x-t+\alpha))d\alpha \right\|_{L_x^2} \right\|_{L^\infty[-T,T]}$$
$$\leq \left\| 2\int_0^T \left\| |D_x|^{s-1}a_T(\alpha)A(u,v;\alpha,x) \right\|_{L_x^2} d\alpha \right\|_{L^\infty[-T,T]}$$
$$\leq 2\sqrt{T}\|a_T(t)A(u,v;t,x)\|_{X_1^{s-1}}.$$

By a similar calculation for $f_0 = \frac{1}{2}\{u_0(x+t)+u_0(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy$, we obtain

$$\|f_0\|_{L^\infty([-T,T];H_x^s)} \leq \|u_0\|_{H^s} + 2\,\|u_1\|_{H^{s-1}} + 2T\,\|u_1\|_{H^{s-1}}.$$

By Lemma 3.1, we have (3.3). Similarly, we obtain (3.4). ∎

Lemma 3.3. *Let* $(u,v),(\tilde{u},\tilde{v}) \in (X^s_\rho \cap Y^s_{\rho,T}) \times (X^s_\rho \cap Y^s_{\rho,T})$ *for* $3/2 < s \leq 2$, *$h(u,v)$ is a polynomial of degree n with respect to u and v. Then we have*

$$\begin{aligned}(3.5)\quad & \|a_T(t)\{h(u,v)Q(u,v) - h(\tilde{u},\tilde{v})Q(\tilde{u},\tilde{v})\}\|_{X_1^{s-1}} \\ &\leq CT'\Big\{\Big(K(u,\tilde{u},v)\,\|u-\tilde{u}\|_{Y^s_T} + K(v,\tilde{v},\tilde{u})\,\|v-\tilde{v}\|_{Y^s_T}\Big)\,\|u\|_{X^s}\|v\|_{X^s} \\ &\qquad + h(\|\tilde{u}\|_{Y^s_T},\|\tilde{v}\|_{Y^s_T})\,(\|u-\tilde{u}\|_{X^s}\|v\|_{X^s} + \|\tilde{u}\|_{X^s}\|v-\tilde{v}\|_{X^s})\Big\}\end{aligned}$$

and

$$\begin{aligned}(3.6)\quad & \left\|\int_0^t U(t-\alpha)a_T(\alpha)\{h(u,v)Q(u,v) - h(\tilde{u},\tilde{v})Q(\tilde{u},\tilde{v})\}d\alpha\right\|_{Y^s_T} \\ &\leq CT'\Big\{\Big(K(u,\tilde{u},v)\,\|u-\tilde{u}\|_{Y^s_T} + K(v,\tilde{v},\tilde{u})\,\|v-\tilde{v}\|_{Y^s_T}\Big)\,\|u\|_{X^s}\|v\|_{X^s} \\ &\qquad + h(\|\tilde{u}\|_{Y^s_T},\|\tilde{v}\|_{Y^s_T})\,(\|u-\tilde{u}\|_{X^s}\|v\|_{X^s} + \|\tilde{u}\|_{X^s}\|v-\tilde{v}\|_{X^s})\Big\}\end{aligned}$$

where

$$K(p,q,r) = \sum_{1\leq i+j\leq n} \|r\|^i_{Y^s_T}(\|p\|^{j-1}_{Y^s_T} + \|p\|^{j-2}_{Y^s_T}\|q\|_{Y^s_T} + \|p\|^{j-3}_{Y^s_T}\|q\|^2_{Y^s_T} + \cdots + \|q\|^{j-1}_{Y^s_T}),$$

$T' = max\{T^{3/2}, T, T^{1/2}\}$, *$Q(u,v)$ stands either $Q_0(u,v)$ or $Q_1(u,v)$ and C is a constant depending on s and h.*

Proof. Since $3/2 < s \leq 2$, we have

$$\begin{aligned}(3.7)\quad & \|a_T(t)\{h(u,v)Q(u,v) - h(\tilde{u},\tilde{v})Q(\tilde{u},\tilde{v})\}\|_{X_1^{s-1}} \\ &\leq C(\|h(u,v) - h(\tilde{u},\tilde{v})\|_{Y^s_T}\,\|a_T(t)Q(u,v)\|_{X_1^{s-1}} \\ &\qquad + \|h(\tilde{u},\tilde{v})\|_{Y^s_T}\,\|a_T(t)\{Q(u,v) - Q(\tilde{u},\tilde{v})\}\|_{X_1^{s-1}}).\end{aligned}$$

Putting $h(u,v)=\sum_{0\le i+j\le n} c_{i,j}u^iv^j$, we have

$$
\begin{aligned}
(3.8)\qquad &\|h(u,v)-h(\tilde{u},\tilde{v})\|_{Y_T^s}\\
&\le \|h(u,v)-h(\tilde{u},v)\|_{Y_T^s}+\|h(\tilde{u},v)-h(\tilde{u},\tilde{v})\|_{Y_T^s}\\
&\le \left\|(u-\tilde{u})\sum_{1\le i+j\le n} c_{i,j}v^j(u^{i-1}+\cdots+\tilde{u}^{i-1})\right\|_{Y_T^s}\\
&\qquad + \left\|(v-\tilde{v})\sum_{1\le i+j\le n} c_{i,j}\tilde{u}^i(v^{j-1}+\cdots+\tilde{v}^{j-1})\right\|_{Y_T^s}\\
&\le C\max_{1\le i+j\le n}|c_{i,j}|\left(K(u,\tilde{u},v)\,\|u-\tilde{u}\|_{Y_T^s}+K(v,\tilde{v},\tilde{u})\,\|v-\tilde{v}\|_{Y_T^s}\right).
\end{aligned}
$$

By Proposition 2.2, we have

$$
\begin{aligned}
(3.9)\qquad &\|a_T(t)\{Q(u,v)-Q(\tilde{u},\tilde{v})\}\|_{X_1^{s-1}}\\
&\le \|a_T(t)Q(u-\tilde{u},v)\|_{X_1^{s-1}}+\|a_T(t)Q(\tilde{u},v-\tilde{v})\|_{X_1^{s-1}}\\
&\le CT'\left(\|u-\tilde{u}\|_{X^s}\|v\|_{X^s}+\|\tilde{u}\|_{X^s}\|v-\tilde{v}\|_{X^s}\right).
\end{aligned}
$$

Therefore the conclusion is obtained combining (3.7), (3.8) and (3.9). Similarly we can prove (3.6). ∎

Proof of Proposition 1.2. We show that the nonlinear map M defined in (3.1) is a contraction mapping from $(X_\rho^s\cap Y_{\rho,T}^s)\times(X_\rho^s\cap Y_{\rho,T}^s)$ to itself for sufficiently small $T>0$. That is, for any $(u,v),(\tilde{u},\tilde{v})\in(X_\rho^s\cap Y_{\rho,T}^s)\times(X_\rho^s\cap Y_{\rho,T}^s)$, we show that

$$
(3.10)\qquad (M_1(u,v),M_2(u,v))\in(X_\rho^s\cap Y_{\rho,T}^s)\times(X_\rho^s\cap Y_{\rho,T}^s)
$$

and

$$
\begin{aligned}
(3.11)\quad &\|M(u,v)-M(\tilde{u},\tilde{v})\|_{X^s}+\|M(u,v)-M(\tilde{u},\tilde{v})\|_{Y_T^s}\\
&\qquad\le \frac{1}{2}\|(u,v)-(\tilde{u},\tilde{v})\|_{X^s}+\frac{1}{2}\|(u,v)-(\tilde{u},\tilde{v})\|_{Y_T^s}.
\end{aligned}
$$

The contraction mapping principle yields from (3.10) and (3.11) that there is a fixed point of $(X_\rho^s\cap Y_{\rho,T}^s)\times(X_\rho^s\cap Y_{\rho,T}^s)$. This gives the solution of (1.1).

It is obvious that $M_1(u,v)|_{t=0}=u_0(x)$ and $\partial_t M_1(u,v)|_{t=0}=u_1(x)$. So we have that $\|M_1(u,v)|_{t=0}\|_{H^s}+\|\partial_t M_1(u,v)|_{t=0}\|_{H^{s-1}}\le\rho/8$ for $(u,v)\in(X_\rho^s\cap Y_{\rho,T}^s)\times(X_\rho^s\cap Y_{\rho,T}^s)$. By Lemma 3.1 and Lemma 3.2, we have

$$
\|\Box M_1(u,v)\|_{X_1^{s-1}}\le CT'\sum_{0\le j+k\le n}\rho^{j+k+2}
$$

and

$$\|M_1(u,v)\|_{Y_T^s} \leq CT' \sum_{0\leq j+k\leq n} \rho^{j+k+2} + \frac{3\rho}{8}.$$

If we take sufficiently small $T > 0$, we obtain $M_1(u,v) \in X_\rho^s \cap Y_{\rho,T}^s$. Similarly, we obtain $M_2(u,v) \in X_\rho^s \cap Y_{\rho,T}^s$. Hence we have (3.10).

Similarly, we can prove (3.11) for sufficiently small $T > 0$ by Lemma 3.1, Lemma 3.2 and Lemma 3.3. ∎

§4. Propagation of Singularities

Let $\varphi(x)$ be a C_0^∞ function satisfying $\varphi \equiv 1$ near x_0 and $\chi(\tau,\xi)$ be smooth, homogeneous of degree 0 in $|(\tau,\xi)| > \epsilon$ for some $\epsilon > 0$, with conic support, $\chi \equiv 1$ on conic neighborhood of (τ_0,ξ_0). We consider the operator P with its symbol $p(t,x,\tau,\xi) = \varphi(x + (\xi/\tau)t)\chi(\tau,\xi)$, which is defined by

$$Pf = \int_{\mathbb{R}^2} p(t,x,\tau,\xi)\hat{f}(\tau,\xi)e^{i(t\tau+x\xi)}d\tau d\xi. \tag{4.1}$$

Simple calculation yields that the symbol of the commutator $[\Box, P] = \Box P - P\Box$ is $(\xi^2/\tau^2 - 1)\varphi''(x + (\xi/\tau)t)\chi$, which is a symbol of pseudo differential operator of order 0.

In order to prove Theorem 1.3, we multiply the operator P to the both sides of the equations (1.1) and use the energy estimates. We prepare several lemmas to prove the main theorem. The following well-known result will be used frequently.

Lemma 4.1. *(Rauch and Reed [21]) Suppose that $G(\xi,\eta)$ may be decomposed into finitely many pieces, i.e., $G = \sum_i G_i(\xi,\eta)$, each of which satisfies either*

$$\sup_\xi \int |G_i|^2 d\eta < \infty \quad \text{or} \quad \sup_\eta \int |G_i|^2 d\xi < \infty. \tag{4.2}$$

If $f, g \in L^2$ and $h(\xi) = \int G(\xi,\eta)f(\xi-\eta)g(\eta)d\eta$, then we have $\|h\|_{L^2} \leqq C\|f\|_{L^2}\|g\|_{L^2}$.

Lemma 4.2. *Let P be the operator defined in* (4.1)*. Suppose that $f, g, h \in H^s$, $\Box f, \Box g \in H^{s-1}$ and $3/2 < s \leq 2$. Then, for any $0 \leq \epsilon < s-1$, we have*

$$[P, h(t,x)(\partial_t + \partial_x)f(\partial_t - \partial_x)](\partial_t \pm \partial_x)g \in H^{s-2+\epsilon}$$

and

$$[P, h(t,x)(\partial_t - \partial_x)f(\partial_t + \partial_x)](\partial_t \pm \partial_x)g \in H^{s-2+\epsilon}.$$

Remark 4.3. *To estimate for K defined in the following proof, the inequality*

$$\int_{\mathbb{R}} \frac{1}{(1+|x-\alpha|)^r(1+|x-\beta|)^r}dx \le \frac{C}{(1+|\alpha-\beta|)^r} \quad for \quad r>1 \tag{4.3}$$

is used frequently.

Proof. Assume that p depends only on τ and ξ (the general case requires some obvious modifications). Let $f_\pm = (\partial_t \pm \partial_x)f$ and $g_\pm = (\partial_t \pm \partial_x)g$. For simplicity, we put $\eta = (\tau, \xi)$, $\eta' = (\tau', \xi')$. Then

$$\begin{aligned}
&\mathcal{F}_{t,x}\left[[P,(hf_-)(\partial_t+\partial_x)]g_+\right] \\
&= ip(\eta)\int \widehat{hf_-}(\eta')\{(\tau-\tau')+(\xi-\xi')\}\widehat{g_+}(\eta-\eta')d\eta' \\
&\qquad - i\int \widehat{hf_-}(\eta')\{(\tau-\tau')+(\xi-\xi')\}p(\eta-\eta')\widehat{g_+}(\eta-\eta')d\eta' \\
&= i\int \widehat{hf_-}(\eta')\{p(\eta)-p(\eta-\eta')\}\{(\tau-\tau')+(\xi-\xi')\}\widehat{g_+}(\eta-\eta')d\eta'.
\end{aligned}$$

Write $\theta_1(\eta) = \{\langle\eta\rangle^{s-1}(1+|\tau+\xi|)\}\widehat{hf_-}(\eta)$ and $\theta_2(\eta) = \{\langle\eta\rangle^{s-1}(1+|\tau-\xi|)\}\widehat{g_+}(\eta)$, then simple calculation yields that $\theta_1, \theta_2 \in L^2$. Thus

$$\langle\eta\rangle^{s-2+\epsilon}\mathcal{F}_{t,x}\left[[P,(hf_-)(\partial_t+\partial_x)]g_+\right] = \int K(\eta,\eta')\theta_1(\eta')\theta_2(\eta-\eta')d\eta',$$

where

$$|K(\eta,\eta')| = \frac{\langle\eta\rangle^{s-2+\epsilon}|p(\eta)-p(\eta-\eta')|\cdot|(\tau-\tau')+(\xi-\xi')|}{\langle\eta'\rangle^{s-1}\langle\eta-\eta'\rangle^{s-1}(1+|\tau'+\xi'|)(1+|(\tau-\tau')-(\xi-\xi')|)}.$$

By Lemma 4.1, it suffices to divide K into finitely many pieces K_i such that

$$\sup_{\eta'}\int |K_i|^2 d\eta < \infty \quad \text{or} \quad \sup_{\eta}\int |K_i|^2 d\eta' < \infty. \tag{4.4}$$

(i) For $|\eta'| \ge |\eta|/2$, $|\eta-\eta'| \ge |\eta|/2$ and $\dfrac{|\tau-\tau'+\xi-\xi'|}{2} \le |\tau'+\xi'|$,

$$|K| \le \frac{C}{\langle\eta\rangle^{s-\epsilon}(1+|(\tau-\tau')-(\xi-\xi')|)}.$$

(ii) For $|\eta'| \ge |\eta|/2$, $|\eta-\eta'| \ge |\eta|/2$ and $\dfrac{|\tau-\tau'+\xi-\xi'|}{2} \le |\tau+\xi|$,

$$|K| \le \frac{C}{\langle\eta'\rangle^{s-1-\epsilon}(1+|\tau'+\xi'|)(1+|(\tau-\tau')-(\xi-\xi')|)}.$$

(iii) For $|\eta'| \geq |\eta|/2$, $|\eta - \eta'| \leq |\eta|/2$ and $\dfrac{|\tau-\tau'+\xi-\xi'|}{2} \leq |(\tau-\tau') - (\xi-\xi')|$,

$$|K| \leq \frac{C}{\langle \eta - \eta' \rangle^{s-\epsilon}(1 + |\tau' + \xi'|)}.$$

(iv) For $|\eta'| \geq |\eta|/2$, $|\eta - \eta'| \leq |\eta|/2$ and $\dfrac{|\tau - \tau' + \xi - \xi'|}{2} \leq 2|\xi - \xi'|$,

$$|K| \leq \frac{C}{\langle \eta - \eta' \rangle^{s-1-\epsilon}(1 + |\tau' + \xi'|)(1 + |(\tau - \tau') - (\xi - \xi')|)}.$$

(v) For $|\eta'| < |\eta|/2$ and $\dfrac{|\tau-\tau'+\xi-\xi'|}{2} \leq |\tau' + \xi'|$, thus $|p(\eta) - p(\eta - \eta')| \leq C\langle \eta' \rangle / \langle \eta \rangle$,

$$|K| \leq \frac{C}{\langle \eta' \rangle^{s-\epsilon}(1 + |(\tau - \tau') - (\xi - \xi')|)}.$$

(vi) For $|\eta'| < |\eta|/2$ and $\dfrac{|\tau - \tau' + \xi - \xi'|}{2} \leq |\tau + \xi|$, thus $|p(\eta) - p(\eta - \eta')| \leq C\langle \eta' \rangle / \langle \eta \rangle$,

$$|K| \leq \frac{C}{\langle \eta' \rangle^{s-1-\epsilon}(1 + |\tau' + \xi'|)(1 + |(\tau - \tau') - (\xi - \xi')|)}.$$

Therefore, in all the cases, (4.4) holds since $\epsilon < s - 1$. Here, we used Hölder's inequality and (4.3) in the case of (ii), (iv) and (vi). It is similar in the case of $[P, (hf_-)(\partial_t + \partial_x)]g_-$, $[P, (hf_+)(\partial_t - \partial_x)]g_+$ and $[P, (hf_+)(\partial_t - \partial_x)]g_-$. ∎

Proposition 4.4. *Let $3/2 < s \leq 2$. Suppose that (u, v) is a solution of the initial value problem (1.1) in $(X^s_\rho \cap Y^s_{\rho,T}) \times (X^s_\rho \cap Y^s_{\rho,T})$ for some $T > 0$ with $u_0(x)$, $v_0(x) \in H^s$ and $u_1(x)$, $v_1(x) \in H^{s-1}$. Then $a_T(t)Q(u, v) \in H^{s-1}(\mathbb{R}^2)$, where $Q(u, v)$ stands either $Q_0(u, v)$ or $Q_1(u, v)$.*

Proof. Because of the fact that

$$\|a_T(t)Q_0(u,v)\|_{H^{s-1}} \leq \|a_T(t)Q_0(u,v)\|_{X_1^{s-1}} + \left\|\langle D_x \rangle^{s-2} \langle D_t \rangle a_T(t) Q_0(u,v)\right\|_{L^2_{t,x}}$$

and the fact that $\|a_T(t)Q_0(u, v)\|_{X_1^{s-1}} < \infty$ from Proposition 2.2, it suffices to show that

$$\left\|\langle D_x \rangle^{s-2} \langle D_t \rangle a_T(t) Q_0(u, v)\right\|_{L^2_{t,x}} < \infty.$$

For the solution u and v of (1.1), we write $u = f_0 + \widetilde{u}$ and $v = g_0 + \widetilde{v}$, where

$$f_0(t,x) = \frac{1}{2}\{u_0(x+t) + u_0(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy,$$

$$g_0(t,x) = \frac{1}{2}\{v_0(x+t) + v_0(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} v_1(y)dy,$$

$$\widetilde{u} = \int_0^t U(t-\alpha)a_T(t)A(u,v;\alpha,x)d\alpha$$

and

$$\widetilde{v} = \int_0^t U(t-\beta)a_T(t)B(u,v;\beta,x)d\beta.$$

Then we have $Q_0(u,v) = Q_0(f_0,g_0) + Q_0(f_0,\widetilde{v}) + Q_0(g_0,\widetilde{u}) + Q_0(\widetilde{u},\widetilde{v})$. We show that only

$$\left\|\langle D_x\rangle^{s-2}\langle D_t\rangle a_T(t)Q_0(\widetilde{u},\widetilde{v})\right\|_{L^2_{t,x}} < \infty, \tag{4.5}$$

since the other terms can be estimated similarly. From Leibniz rule and the triangle inequality, we have

$$\begin{aligned}
&\left\|\langle D_x\rangle^{s-2}\langle D_t\rangle a_T(t)Q_0(\widetilde{u},\widetilde{v})\right\|_{L^2_{t,x}} \\
&\qquad \le \|a_T(t)Q_0(\widetilde{u},\widetilde{v})\|_{X_1^{s-1}} + \left\|\langle D_x\rangle^{s-2}(D_t a_T(t))Q_0(\widetilde{u},\widetilde{v})\right\|_{L^2_{t,x}} \\
&\qquad\qquad + \left\|\langle D_x\rangle^{s-2} a_T(t)(D_t Q_0(\widetilde{u},\widetilde{v}))\right\|_{L^2_{t,x}}.
\end{aligned}$$

Since Proposition 2.2 shows that

$$\|a_T(t)Q_0(\widetilde{u},\widetilde{v})\|_{X_1^{s-1}} < \infty \quad \textit{and} \quad \left\|\langle D_x\rangle^{s-2}(D_t a_T(t))Q_0(\widetilde{u},\widetilde{v})\right\|_{L^2_{t,x}} < \infty,$$

we only show

$$\left\|\langle D_x\rangle^{s-2} a_T(t)(D_t Q_0(\widetilde{u},\widetilde{v}))\right\|_{L^2_{t,x}} < \infty.$$

For simplicity, we write $A(u,v;\alpha,x) = A(\alpha,x)$, $B(u,v;\alpha,x) = B(\alpha,x)$. From (2.1), we have

$$\begin{aligned}
Q_0(\widetilde{u},\widetilde{v}) = 2\int_0^t\int_0^t a_T(\alpha)a_T(\beta)\{&A(\alpha,x+t-\alpha)B(\beta,x-t+\beta) \\
&+ A(\alpha,x-t+\alpha)B(\beta,x+t-\beta)\}d\alpha d\beta
\end{aligned}$$

and

$$\begin{aligned}
\partial_t Q_0(\widetilde{u},\widetilde{v}) = 2\int_0^t\int_0^t &a_T(\alpha)a_T(\beta)\partial_t\{A(\alpha,x+t-\alpha)B(\beta,x-t+\beta) \\
&+ A(\alpha,x-t+\alpha)B(\beta,x+t-\beta)\}d\alpha d\beta \\
&+ 2a_T(t)(B(t,x)\partial_t\widetilde{u} + A(t,x)\partial_t\widetilde{v}).
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\left\| \langle D_x\rangle^{s-2} a_T(t) \int_0^t\int_0^t a_T(\alpha)a_T(\beta)\partial_t\{A(\alpha,x+t-\alpha)B(\beta,x-t+\beta)\}d\alpha d\beta \right\|_{L^2_{t,x}} \\
&\le \int_0^T\int_0^T a_T(\alpha)a_T(\beta)\left\|\langle D_x\rangle^{s-2}a_T(t)\partial_t A(\alpha,x+t-\alpha)B(\beta,x-t+\beta)\right\|_{L^2_{t,x}} d\alpha d\beta \\
&\le \int_0^T\int_0^T a_T(\alpha)a_T(\beta)\left\|\langle D_x\rangle^{s-2}\partial_t\{a_T(t)A(\alpha,x+t-\alpha)B(\beta,x-t+\beta)\}\right\|_{L^2_{t,x}} d\alpha d\beta \\
&+ \int_0^T\int_0^T a_T(\alpha)a_T(\beta)\left\|\langle D_x\rangle^{s-2}(\partial_t a_T(t))A(\alpha,x+t-\alpha)B(\beta,x-t+\beta)\}\right\|_{L^2_{t,x}} d\alpha d\beta \\
&\equiv I_1 + I_2.
\end{aligned}
$$

Applying Lemma 4.1 as in the proof of Lemma 4.2, we have

$$I_1 \le C_1 \int_0^T \int_0^T \|A(\alpha,\cdot)\|_{H^{s-1}}\|B(\alpha,\cdot)\|_{H^{s-1}} d\alpha d\beta \le C\|A\|_{X_1^{s-1}}\|B\|_{X_1^{s-1}}$$

and

$$I_2 \le C_2 \int_0^T \int_0^T \|A(\alpha,\cdot)\|_{H^{s-1}}\|B(\alpha,\cdot)\|_{H^{s-1}} d\alpha d\beta \le C\|A\|_{X_1^{s-1}}\|B\|_{X_1^{s-1}},$$

where C_1 and C_2 are constants depending on s and a_T. Since

$$\partial_t\widetilde{u} = \int_0^t a_T(\alpha)\{A(\alpha,x+t-\alpha)+A(\alpha,x-t+\alpha)\}d\alpha$$

and

$$\partial_t\widetilde{v} = \int_0^t a_T(\beta)\{B(\beta,x+t-\beta)+B(\beta,x-t+\beta)\}d\beta,$$

the same calculation as the above yields

$$\left\|\langle D_x\rangle^{s-2}a_T(t)\partial_t\widetilde{u}B(t,x)\right\|_{L^2_{t,x}} \le C\|A\|_{X_1^{s-1}}\|B\|_{Y_T^s}$$

and

$$\left\|\langle D_x\rangle^{s-2}a_T(t)\partial_t\widetilde{v}A(t,x)\right\|_{L^2_{t,x}} \le C\|B\|_{X_1^{s-1}}\|A\|_{Y_T^s}.$$

Therefore the proof is completed. ∎

Lemma 4.5. *Let $3/2 < s \le 2$. Suppose that P is the operator defined in (4.1) and (u,v) is a solution of the initial value problem (1.1) in $(X_\rho^s \cap Y_{\rho,T}^s) \times (X_\rho^s \cap$*

$Y^s_{\rho,T}$) *for some* $T > 0$. *If* $0 < \delta < T$ *and* $0 \leq \epsilon < s - 1$, *then, for* $|t| < T - \delta$, *we have*

$$\left\| a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} \left(P(\partial_t \pm \partial_x) h(u,v) Q(u,v) \right) (t - \tilde{t}) \right\|_{L^2_{\tilde{t},x}} \tag{4.6}$$
$$\leq C_1(\|u\|_{X^s}, \|v\|_{X^s}, \|u\|_{Y^s_T}, \|v\|_{Y^s_T}) \left\| a_\delta(\tilde{t}) \Lambda^{s-1+\epsilon}_{\tilde{t},x} P u_\pm (t - \tilde{t}, x) \right\|_{L^2_{\tilde{t},x}}$$
$$+ C_2(\|u\|_{X^s}, \|v\|_{X^s}, \|u\|_{Y^s_T}, \|v\|_{Y^s_T}) \left\| a_\delta(\tilde{t}) \Lambda^{s-1+\epsilon}_{\tilde{t},x} P v_\pm (t - \tilde{t}, x) \right\|_{L^2_{\tilde{t},x}}$$
$$+ C_3(\|u\|_{X^s}, \|v\|_{X^s}, \|u\|_{Y^s_T}, \|v\|_{Y^s_T}),$$

where $u_\pm = (\partial_t \pm \partial_x)u$, $v_\pm = (\partial_t \pm \partial_x)v$, $C_i(\|u\|_{X^s}, \|v\|_{X^s}, \|u\|_{Y^s_T}, \|v\|_{Y^s_T})$ $(i = 1, 2, 3)$ *are constants depending on* $\|u\|_{X^s}$, $\|v\|_{X^s}$, $\|u\|_{Y^s_T}$ *and* $\|v\|_{Y^s_T}$ *and* $Q(u,v)$ *stands either* $Q_0(u,v)$ *or* $Q_1(u,v)$.

Proof. It is enough to consider only the case $Q(u,v) = Q_0(u,v)$. Let $b(\tilde{t}) \in C_0^\infty$ with $\operatorname{supp} b \subset \{|\tilde{t}| < T\}$ and $b \equiv 1$ for $|\tilde{t}| \leq \delta + \delta'$, where δ' is a positive constant satisfying $|t| < \delta' < T - \delta$. Then

$$\left\| a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} \left(P(\partial_t \pm \partial_x) h(u,v) Q_0(u,v) \right) (t - \tilde{t}) \right\|_{L^2_{\tilde{t},x}} \tag{4.7}$$
$$= \left\| b(t - \tilde{t}) a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} \left(P(\partial_t \pm \partial_x) h(u,v) Q_0(u,v) \right) (t - \tilde{t}) \right\|_{L^2_{\tilde{t},x}}$$
$$\leq \left\| b(t - \tilde{t}) a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} \left(P h_1 u_\pm Q_0(u,v) \right) (t - \tilde{t}) \right\|_{L^2_{\tilde{t},x}}$$
$$+ \left\| b(t - \tilde{t}) a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} (P h_2 v_\pm Q_0(u,v))(t - \tilde{t}) \right\|_{L^2_{\tilde{t},x}}$$
$$+ \left\| b(t - \tilde{t}) a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} (P h Q(u, v_\pm))(t - \tilde{t}) \right\|_{L^2_{\tilde{t},x}}$$
$$+ \left\| b(t - \tilde{t}) a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} (P h Q(u_\pm, v))(t - \tilde{t}) \right\|_{L^2_{\tilde{t},x}}$$

as long as $|t| < \delta'$, where h_j $(j = 1, 2)$ are partial derivatives of h. Here, we have with $[A, B] = AB - BA$,

$$b(t - \tilde{t}) a_\delta(\tilde{t}) \Lambda^{s-2+\epsilon}_{\tilde{t},x} (P h_1 u_\pm Q_0(u,v))(t - \tilde{t}) \tag{4.8}$$
$$= [b(t - \tilde{t}) a_\delta(\tilde{t}), \Lambda^{s-2+\epsilon}_{\tilde{t},x}] (P h_1 u_\pm Q_0(u,v))(t - \tilde{t})$$
$$+ \Lambda^{s-2+\epsilon}_{\tilde{t},x} a_\delta(\tilde{t}) [b(t - \tilde{t}), P] (h_1 Q_0(u,v) u_\pm)(t - \tilde{t})$$
$$+ \Lambda^{s-2+\epsilon}_{\tilde{t},x} a_\delta(\tilde{t}) [P, (b h_1 Q_0(u,v))(t - \tilde{t})] u_\pm(t - \tilde{t})$$
$$+ \Lambda^{s-2+\epsilon}_{\tilde{t},x} a_\delta(\tilde{t}) (h_1 b Q_0(u,v))(t - \tilde{t}) (P u_\pm)(t - \tilde{t}).$$

Since $[b(t-\tilde{t})a_\delta(\tilde{t}), \Lambda_{\tilde{t},x}^{s-2+\epsilon}]$ and $[b, P]$ are of order $s-3+\epsilon$ and -1 respectively and $fgh \in H^{3s-5-\nu}$ for $f, g, h \in H^{s-1}(\mathbb{R}^2)$ and any $\nu > 0$, thus for the first and second terms of the right hand side of (4.8), we have

$$\left\| [b(t-\tilde{t})a_\delta(\tilde{t}), \Lambda_{\tilde{t},x}^{s-2+\epsilon}](Ph_1 u_\pm Q_0(u,v))(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} < \infty$$

and

$$\left\| \Lambda_{\tilde{t},x}^{s-2+\epsilon} a_\delta(\tilde{t})([b(t-\tilde{t}), P]h_1 Q_0(u,v)u_\pm)(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} < \infty.$$

In the same way, as the proof of Lemma 4.2, for the third term of the right hand side of (4.8) we have

$$\left\| \Lambda_{\tilde{t},x}^{s-2+\epsilon} a_\delta(\tilde{t})([P, (bh_1 Q_0(u,v))(t-\tilde{t})]u_\pm)(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} < \infty.$$

Now, we estimate the forth term of (4.8). For simplicity, we put $\eta = (\tilde{\tau}, \xi)$ and $\eta' = (\tilde{\tau}', \xi')$. Then

$$\begin{aligned} &\langle \eta \rangle^{s-2+\epsilon} \mathcal{F}_{\tilde{t},x}\left[(h_1 b Q_0(u,v))(t-\tilde{t}) a_\delta(\tilde{t}) P u_\pm(t-\tilde{t})\right](\eta) \\ &\qquad\qquad = \int K_1(\eta, \eta') \theta_t^{(1)}(\eta') \theta_t^{(2)}(\eta - \eta') d\eta', \end{aligned}$$

where $\theta_t^{(1)}(\eta) = \langle \eta \rangle^{s-1} \mathcal{F}_{\tilde{t},x}[h_1 b Q_0(u,v)(t-\tilde{t})]$, $\theta_t^{(2)}(\eta) = \langle \eta \rangle^{s-1+\epsilon} \mathcal{F}_{\tilde{t},x}[a_\delta(\tilde{t}) P u_\pm(t-\tilde{t})]$ and $|K_1(\eta, \eta')| = \dfrac{\langle \eta \rangle^{s-2+\epsilon}}{\langle \eta' \rangle^{s-1} \langle \eta - \eta' \rangle^{s-1+\epsilon}}$. By Lemma 4.1, we have

$$\begin{aligned} &\left\| \Lambda_{\tilde{t},x}^{s-2+\epsilon} (h_1 b Q_0(u,v))(t-\tilde{t}) a_\delta(\tilde{t}) (P u_\pm)(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} \\ &\le C \|\theta_t^{(1)}\|_{L^2_{\tilde{t},x}} \|\theta_t^{(2)}\|_{L^2_{\tilde{t},x}} \\ &\le C' \|h_1 b Q_0(u,v)\|_{H^{s-1}} \\ &\quad \times \left(\left\| a_\delta(\tilde{t}) \Lambda_{\tilde{t},x}^{s-1+\epsilon} P u_\pm(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} + \left\| [\Lambda_{\tilde{t},x}^{s-1+\epsilon}, a_\delta(\tilde{t})] P u_\pm(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} \right) \end{aligned}$$

as long as $0 \le \epsilon < s-1$. Thus we obtain

$$\begin{aligned} &\left\| b(t-\tilde{t}) a_\delta(\tilde{t}) \Lambda_{\tilde{t},x}^{s-2+\epsilon} (P h_1 u_\pm Q_0(u,v))(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} \\ &\le C_1'(\|u\|_{X^s}, \|v\|_{X^s}, \|u\|_{Y_T^s}, \|v\|_{Y_T^s}) \left\| a_\delta(\tilde{t}) \Lambda_{\tilde{t},x}^{s-1+\epsilon} P u_\pm(t-\tilde{t}) \right\|_{L^2_{\tilde{t},x}} \\ &\qquad + C_2'(\|u\|_{X^s}, \|v\|_{X^s}, \|u\|_{Y_T^s}, \|v\|_{Y_T^s}). \end{aligned} \tag{4.9}$$

Since $Q_0(u, v_\pm) = 1/2\{u_+(\partial_t - \partial_x)v_\pm + u_-(\partial_t + \partial_x)v_\pm\}$, we have

$$
\begin{aligned}
(4.10)\quad 2\left\|b(t-\tilde{t})a_\delta(\tilde{t})\Lambda_{\tilde{t},x}^{s-2+\epsilon}\big(PhQ(u,v_\pm)\big)(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} \\
\le \left\|b(t-\tilde{t})a_\delta(\tilde{t})\Lambda_{\tilde{t},x}^{s-2+\epsilon}\big(Phu_+(\partial_t - \partial_x)v_\pm\big)(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} \\
+ \left\|b(t-\tilde{t})a_\delta(\tilde{t})\Lambda_{\tilde{t},x}^{s-2+\epsilon}\big(Phu_-(\partial_t + \partial_x)v_\pm\big)(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}}.
\end{aligned}
$$

In the same way as the estimates of (4.8), we have

$$
\begin{aligned}
(4.11)\qquad & b(t-\tilde{t})a_\delta(\tilde{t})\Lambda_{\tilde{t},x}^{s-2+\epsilon}\big(Phu_+(\partial_t - \partial_x)v_\pm\big)(t-\tilde{t}) \\
&= [b(t-\tilde{t})a_\delta(\tilde{t}), \Lambda_{\tilde{t},x}^{s-2+\epsilon}]\big(Phu_+(\partial_t - \partial_x)v_\pm\big)(t-\tilde{t}) \\
&\quad + \Lambda_{\tilde{t},x}^{s-2+\epsilon} a_\delta(\tilde{t})([b(t-\tilde{t}), P]hu_+(\partial_t - \partial_x)v_\pm)(t-\tilde{t}) \\
&\quad + \Lambda_{\tilde{t},x}^{s-2+\epsilon} a_\delta(\tilde{t})[P, b(t-\tilde{t})hu_+(\partial_t - \partial_x)]v_\pm \\
&\quad - \Lambda_{\tilde{t},x}^{s-2+\epsilon}(bhu_+)(t-\tilde{t})[a_\delta(\tilde{t}), \partial_{\tilde{t}} + \partial_x](Pv_\pm)(t-\tilde{t}) \\
&\quad - \Lambda_{\tilde{t},x}^{s-2+\epsilon}(bu_+h)(t-\tilde{t})(\partial_{\tilde{t}} + \partial_x)a_\delta(\tilde{t})(Pv_\pm)(t-\tilde{t})
\end{aligned}
$$

and first, second and fourth terms of the right hand side of (4.11) are in $L^2_{\tilde{t},x}$. By Lemma 4.2, third term is in $L^2_{\tilde{t},x}$. Now, we estimate the fifth term of (4.11). Putting $\theta_t^{(3)}(\eta) = \langle\eta\rangle^{s-1}(1 + |\tilde{\tau} - \xi|)\mathcal{F}_{\tilde{t},x}[(bhu_+)(t-\tilde{t})]$, $\theta_t^{(4)}(\eta) = \langle\eta\rangle^{s-1+\epsilon}\mathcal{F}_{\tilde{t},x}[a_\delta(\tilde{t})Pv_\pm(t-\tilde{t})]$ and

$$
|K_2(\eta,\eta')| = \frac{\langle\eta\rangle^{s-2+\epsilon}|\tilde{\tau}' + \xi'|}{\langle\eta'\rangle^{s-1}\langle\eta-\eta'\rangle^{s-1+\epsilon}(1 + |(\tilde{\tau} - \tilde{\tau}') - (\xi - \xi')|)},
$$

we have

$$
\begin{aligned}
\langle\eta\rangle^{s-2+\epsilon}\mathcal{F}_{\tilde{t},x}\left[(bu_+h)(t-\tilde{t})(\partial_{\tilde{t}} + \partial_x)a_\delta(\tilde{t})(Pv_\pm)(t-\tilde{t})\right](\eta) \\
= \int K_2(\eta,\eta')\theta_t^{(3)}(\eta')\theta_t^{(4)}(\eta-\eta')d\eta'.
\end{aligned}
$$

By Lemma 4.1, we have

$$
\begin{aligned}
&\left\|\Lambda_{\tilde{t},x}^{s-2+\epsilon}(bu_+h)(t-\tilde{t})(\partial_{\tilde{t}} + \partial_x)a_\delta(\tilde{t})(Pv_\pm)(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} \\
&\le C\|\theta_t^{(3)}\|_{L^2_{\tilde{t},x}}\|\theta_t^{(4)}\|_{L^2_{\tilde{t},x}} \\
&\le C\,\|h\|_{H^s}(\|u_\pm\|_{H^{s-1}} + \|\partial_{\tilde{t}} b\, u_+\|_{H^{s-1}} + \|b(t)\Box u\|_{H^{s-1}}) \\
&\qquad \times \left(\left\|a_\delta(\tilde{t})\Lambda_{\tilde{t},x}^{s-2+\epsilon}Pv_\pm(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}} + \left\|[\Lambda_{\tilde{t},x}^{s-2+\epsilon}, a_\delta(\tilde{t})]Pv_\pm(t-\tilde{t})\right\|_{L^2_{\tilde{t},x}}\right).
\end{aligned}
$$

Therefore we obtain

$$\begin{aligned}(4.12)\quad &\left\| b(t-\tilde t)a_\delta(\tilde t)\Lambda^{s-2+\epsilon}_{\tilde t,x}\big(Phu_+(\partial_t-\partial_x)v_\pm\big)(t-\tilde t)\right\|_{L^2_{\tilde t,x}} \\ &\le C_3'(\|u\|_{X^s},\|v\|_{X^s},\|u\|_{Y^s_T},\|v\|_{Y^s_T})\left\| a_\delta(\tilde t)\Lambda^{s-2+\epsilon}_{\tilde t,x}Pv_\pm(t-\tilde t)\right\|_{L^2_{\tilde t,x}} \\ &\qquad + C_4'(\|u\|_{X^s},\|v\|_{X^s},\|u\|_{Y^s_T},\|v\|_{Y^s_T}).\end{aligned}$$

By (4.9) and (4.12), we have the conclusion. ∎

Now, we prove the main theorem.

Proof of Theorem 1.3. Since $(\partial_t+\partial_x)$ or $(\partial_t-\partial_x)$ is elliptic near Γ, it suffices to show that $(\partial_t\pm\partial_x)u, (\partial_t\pm\partial_x)v \in H^{r-1}_{ml}(\Gamma)$ for $|t|<T$. Let $u_\pm=(\partial_t\pm\partial_x)u$ and $v_\pm=(\partial_t\pm\partial_x)v$. Multiplying $(\partial_t\pm\partial_x)$ to the both sides of (1.1), we have

$$(4.13)\qquad \Box u_\pm = (\partial_t\pm\partial_x)A(u,v),$$

where

$$\begin{aligned}(4.14)\quad &A(u,v)\\ &=h_1(u,v)Q_0(u,u)+h_2(u,v)Q_0(u,v)+h_3(u,v)Q_0(v,v)+h_4(u,v)Q_1(u,v).\end{aligned}$$

Applying the operator P defined in (4.1) to the both sides of (4.13), We have

$$\begin{aligned}(4.15)\qquad \Box Pu_\pm &=[\Box,P]u_\pm + P\Box u_\pm\\ &=[\Box,P]u_\pm + P(\partial_t\pm\partial_x)A(u,v).\end{aligned}$$

Let

$$\begin{aligned}E(t;u,v)\equiv &\left\|a_\delta(\tilde t)\Lambda^{r-2}_{\tilde t,x}\partial_{\tilde t}Pu_\pm(t-\tilde t)\right\|^2_{L^2_{\tilde t,x}} + \left\|a_\delta(\tilde t)\Lambda^{r-2}_{\tilde t,x}\partial_x Pu_\pm(t-\tilde t)\right\|^2_{L^2_{\tilde t,x}}\\ &+\left\|a_\delta(\tilde t)\Lambda^{r-2}_{\tilde t,x}Pu_\pm(t-\tilde t)\right\|^2_{L^2_{\tilde t,x}} + \left\|a_\delta(\tilde t)\Lambda^{r-2}_{\tilde t,x}\partial_{\tilde t}Pv_\pm(t-\tilde t)\right\|^2_{L^2_{\tilde t,x}}\\ &+\left\|a_\delta(\tilde t)\Lambda^{r-2}_{\tilde t,x}\partial_x Pv_\pm(t-\tilde t)\right\|^2_{L^2_{\tilde t,x}} + \left\|a_\delta(\tilde t)\Lambda^{r-2}_{\tilde t,x}Pv_\pm(t-\tilde t)\right\|^2_{L^2_{\tilde t,x}},\end{aligned}$$

where $\Lambda=\langle D\rangle$ and $0<\delta<T$. Let $\varphi\in C_0^\infty(\mathbb{R})$, $\int\varphi(x)dx=1$, $\varphi_\omega(x)=\frac{1}{\omega}\varphi\left(\frac{x}{\omega}\right)$ and put $u_0^\omega=\varphi_\omega * u_0$, $u_1^\omega=\varphi_\omega * u_1$, $v_0^\omega=\varphi_\omega * v_0$ and $v_1^\omega=\varphi_\omega * v_1$. Let (u_ω,v_ω) be a smooth solution of $u_\omega=f_0^\omega(t,x)+\int_0^t U(t-\alpha)a_T(\alpha)(\varphi_\omega * A(u_\omega,v_\omega))d\alpha$, $v_\omega=g_0^\omega(t,x)+\int_0^t U(t-\alpha)a_T(\alpha)(\varphi_\omega * B(u_\omega,v_\omega))d\alpha$ where $f_0^\omega(t,x)=\frac12\{u_0^\omega(x+t)+u_0^\omega(x-t)\}+\frac12\int_{x-t}^{x+t}u_1^\omega(y)dy$, $g_0^\omega(t,x)=\frac12\{v_0^\omega(x+$

$t) + v_0^\omega(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} v_1^\omega(y)dy$. Using the same technique as in the proof of Lemma 3.2 and Lemma 3.3, we have

$$\|u - u_\omega\|_{X^s} \to 0, \quad \|u - u_\omega\|_{Y_T^s} \to 0 \quad as \quad \omega \to 0.$$

Hence $\|u_\omega\|_{X^s} \le C\|u\|_{X^s}$ and $\|u_\omega\|_{X^s} \le C\|v\|_{X^s}$ for $0 < {}^\forall\omega \le 1$. Similarly, we have $\|v_\omega\|_{X^s} \le C\|u\|_{X^s}$ and $\|v_\omega\|_{X^s} \le C\|v\|_{X^s}$. By the calculus of pseudo differential operators and the Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{dE(t;u_\omega,v_\omega)}{dt} \\
&= -\mathrm{Re}\langle a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\Box Pu_{\omega,\pm}(t-\tilde t), a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_{\tilde t}(Pu_{\omega,\pm}(t-\tilde t))\rangle \\
&\quad + \mathrm{Re}\langle a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_t(Pu_{\omega,\pm}(t-\tilde t)), a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}Pu_{\omega,\pm}(t-\tilde t)\rangle \\
&\quad - \mathrm{Re}\langle a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\Box Pv_{\omega,\pm}(t-\tilde t), a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_{\tilde t}(Pv_{\omega,\pm}(t-\tilde t))\rangle \\
&\quad + \mathrm{Re}\langle a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_t(Pv_{\omega,\pm}(t-\tilde t)), a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}Pv_{\omega,\pm}(t-\tilde t)\rangle \\
&\le \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\Box Pu_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}} \cdot \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_{\tilde t}(Pu_{\omega,\pm}(t-\tilde t))\right\|_{L^2_{\tilde t,x}} \\
&\quad + \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_t(Pu_{\omega,\pm}(t-\tilde t))\right\|_{L^2_{\tilde t,x}} \cdot \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}Pu_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}} \\
&\quad + \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\Box Pv_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}} \cdot \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_{\tilde t}(Pv_{\omega,\pm}(t-\tilde t))\right\|_{L^2_{\tilde t,x}} \\
&\quad + \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\partial_t(Pv_{\omega,\pm}(t-\tilde t))\right\|_{L^2_{\tilde t,x}} \cdot \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}Pv_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}} \\
&\le \frac{1}{2}\left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\Box Pu_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}}^2 + \frac{1}{2}\left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\Box Pv_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}}^2 \\
&\quad + 3E(t;u_\omega,v_\omega),
\end{aligned}$$

where $u_{\omega,\pm} = (\partial_t \pm \partial_x)u_\omega$, $v_{\omega,\pm} = (\partial_t \pm \partial_x)v_\omega$. By (4.15) we have

$$\begin{aligned}
&\left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}\Box Pu_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}}^2 \\
&\qquad \le \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}[\Box,P]u_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}}^2 \\
&\qquad\quad + \left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}P(\partial_t \pm \partial_x)A(u_\omega,v_\omega)(t-\tilde t)\right\|_{L^2_{\tilde t,x}}^2 .
\end{aligned}$$

Since $r < 2s-1$ and $[\Box,P]$ is of order 0 ,

$$\left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}[\Box,P]u_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}}^2 < \infty.$$

By (4.14), Lemma 4.5 and the triangle inequality, we have

(4.16)

$$\begin{aligned}&\left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-2}P(\partial_t \pm \partial_x)A(u_\omega,v_\omega)(t-\tilde t)\right\|_{L^2_{\tilde t,x}}^2\\ &\le \left(C_1\left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-1}Pu_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}}+C_2\left\|a_\delta(\tilde t)\Lambda_{\tilde t,x}^{r-1}Pv_{\omega,\pm}(t-\tilde t)\right\|_{L^2_{\tilde t,x}}+C_3\right)^2\\ &\le C_4E(t;u_\omega,v_\omega)+C_5\end{aligned}$$

for $\delta < T$, $|t| < T-\delta$ and $r < 2s-1$, where C_i $(i = 1,2,3,4,5)$ are positive constants depending on $\|u\|_{X^s}$, $\|v\|_{X^s}$, $\|u\|_{Y_T^s}$ and $\|v\|_{Y_T^s}$. Therefore we obtain

$$\frac{dE(t;u_\omega,v_\omega)}{dt} \le C_1'E(t;u_\omega,v_\omega)+C_2'.$$

By Gronwall's inequality, we have

$$E(t,u_\omega,v_\omega) \le e^{C_1't}\left\{E(0,u_\omega,v_\omega)+\frac{C_2'}{C_1'}(1-e^{-C_1't})\right\}.$$

Taking the limit of the above inequality as $\omega \longrightarrow 0$, we have

$$E(t;u,v) \le e^{C_1't}\left\{E(0,u,v)+\frac{C_2'}{C_1'}(1-e^{-C_1't})\right\} < \infty$$

since $E(0;u,v) < \infty$. Hence $Pu_\pm \in H^{r-1}$ for $|t| < T-\delta$. If we take sufficiently small $\delta > 0$, then we have $Pu_\pm \in H^{r-1}$ for $|t| < T$ and $r < 2s-1$. Hence we have $u_\pm \in H^{r-1}_{ml}(\Gamma)$ for $|t| < T$. Similarly, we have $v_\pm \in H^{r-1}_{ml}(\Gamma)$ for $|t| < T$. Therefore we have $(u,v) \in H^r_{ml}(\Gamma) \times H^r_{ml}(\Gamma)$ for $|t| < T$ and $r < 2s-1$. ∎

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