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On $N(k)$-contact metric manifolds satisfying certain conditions

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Abstract. We classify $N(k)$-contact metric manifolds satisfying the conditions $\mathcal{Z}(\xi,X)\cdot C_0=0$, $C_0(\xi,X)\cdot\mathcal{Z}=0$ and $C_e(\xi,X)\cdot\mathcal{Z}=0$, where $\mathcal{Z},C_0$ and C_e denote the concircular curvature tensor, the contact conformal curvature tensor and the extended contact conformal curvature tensor, respectively.

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Introduction

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a *concircular transformation* [15]. An invariant of a concircular transformation is the *concircular curvature tensor* $\mathcal{Z}$. It is defined by [15]

$$\mathcal{Z}=R-\frac{r}{n\,(n-1)}R_0, \tag{0.1}$$

where R is the curvature tensor, r is the scalar curvature and

$$R_0\,(X,Y)\,W=g\,(Y,W)\,X-g\,(X,W)\,Y, \qquad X,Y,W\in TM.$$

It is easy to see that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature.

In [4], the classification of $N(k)$-contact metric manifolds satisfying the condition $\mathcal{Z}\,(\xi,X)\cdot\mathcal{Z}=0$ was given by Blair, Kim and Tripathi (see also [3]). In [14], Tripathi and Kim studied the concircular curvature tensor of a (k,μ)-contact metric manifold and they classified (k,μ)-contact metric manifolds satisfying the condition $\mathcal{Z}(\xi, X) \cdot S = 0$. Contact Riemannian manifolds satisfying $R(\xi, X) \cdot R = 0$ and $\xi \in (k, \mu)$-nullity distribution was studied by Papantoniou in [5].

In [9], Kitahara, Matsuo and Pak defined a tensor field B_0 on a Hermitian manifold which is conformally invariant and studied some of its properties. They called this tensor field the *conformal invariant curvature tensor*. By using the Boothby-Wang fibration [7], Jeong, Lee, Oh and Pak constructed a *contact conformal curvature tensor* C_0 [10] on a Sasakian manifold from the conformal invariant curvature tensor. In a $(2n+1)$-dimensional contact metric manifold $(M, \varphi, \xi, \eta, g)$, it is defined by

$$
\begin{aligned}
C_0(X,Y)Z = {} & R(X,Y)Z \\
& + \frac{1}{2n}\{-g(QY,Z)\varphi^2 X + g(QX,Z)\varphi^2 Y \\
& + g(\varphi Y, \varphi Z)QX - g(\varphi X, \varphi Z)QY \\
& + g(Q\varphi X, Z)\varphi Y - g(Q\varphi Y, Z)\varphi X + 2g(Q\varphi X, Y)\varphi Z \\
& + g(\varphi X, Z)QY - g(\varphi Y, Z)QX + 2g(\varphi X, Y)QZ\} \\
& + \frac{1}{2n(n+1)}\left(2n^2 - n - 2 + \frac{(n+2)r}{2n}\right) \times \\
& \times \{g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\} \\
& + \frac{1}{2n(n+1)}\left(n + 2 - \frac{(3n+2)r}{2n}\right)(g(Y,Z)X - g(X,Z)Y) \\
& - \frac{1}{2n(n+1)}\left(4n^2 + 5n + 2 - \frac{(3n+2)r}{2n}\right) \times \\
& \times \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\
& + \eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi\},
\end{aligned}
\tag{0.2}
$$

where R, Q, r are the curvature tensor, the Ricci operator and the scalar curvature, respectively. In [11], Pak and Shin showed that every contact metric manifold with vanishing contact conformal curvature tensor is a Sasakian space form. In [8], Kim, Choi, the first author and Tripathi extended the concept of contact conformal curvature tensor to an *extended contact conformal curvature tensor* C_e. It is defined by

$$
\begin{aligned}
C_e(X,Y)Z = {} & C_0(X,Y)Z - \eta(X)C_0(\xi, Y)Z \\
& - \eta(Y)C_0(X, \xi)Z - \eta(Z)C_0(X,Y)\xi.
\end{aligned}
\tag{0.3}
$$

In [8], it was proved that an $N(k)$-contact metric manifold with vanishing extended contact conformal curvature tensor is a Sasakian manifold.

Motivated by the studies of the above authors, in this study, we consider $N(k)$-contact metric manifolds satisfying the conditions $\mathcal{Z}(\xi, X) \cdot C_0 = 0$, $C_0(\xi, X) \cdot \mathcal{Z} = 0$ and $C_e(\xi, X) \cdot \mathcal{Z} = 0$.

§1. Preliminaries

An odd-dimensional differentiable manifold M is called an *almost contact manifold* [2] if there is an almost contact structure (φ, ξ, η) consisting of a tensor field φ type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \text{and (one of)} \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{1.1}$$

If the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable then the structure (φ, ξ, η) is said to be normal, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. M becomes an *almost contact metric manifold* with an almost contact metric structure (φ, ξ, η, g), if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

or equivalently

$$g(X, \varphi Y) = -g(\varphi X, Y) \quad \text{and} \quad g\ (X, \xi) = \eta(X)$$

for all $X, Y \in TM$, where g is a Riemannian metric tensor of M.

An almost contact metric structure is called a *contact metric structure* if

$$g(X, \varphi Y) = d\eta(X, Y)$$

holds on M for $X, Y \in TM$.

A normal contact metric manifold is a *Sasakian manifold*. However an almost contact metric manifold is Sasakian if and only if

$$\nabla_X \varphi = R_0(\xi, X), \quad X \in TM,$$

where ∇ is Levi-Civita connection. Also a contact metric manifold M is Sasakian if and only if the curvature tensor R satisfies

$$R(X, Y)\xi = R_0(X, Y)\xi, \quad X, Y \in TM,$$

(see [2], Proposition 7.6).

The tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [2]. The (k, μ)-nullity condition on a contact metric manifold is considered as a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case. The (k, μ)-*nullity distribution* $N(k, \mu)$ [5] of a contact metric manifold M^{2n+1} is defined by

$$N(k, \mu) : p \to N_p(k, \mu) = \{W \in T_pM \mid R(X, Y)W = (kI + \mu h)R_0(X, Y)W\},$$

for all $X, Y \in TM$ where $(k,\mu) \in \mathbb{R}^2$ and the tensor field h is defined by $h = \frac{1}{2}L_\xi\varphi$, here L_ξ denotes Lie differentiation in the direction of ξ. If ξ belongs to (k,μ)-nullity distribution $N(k,\mu)$ then a contact metric manifold $M^{2n+1}$is called a (k,μ)-*contact metric manifold*. In particular the condition

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

holds on a (k,μ)-contact metric manifold. On a (k,μ)-manifold $k \leq 1$. If $k = 1$, the structure is Sasakian and if $k < 1$, the (k,μ)-nullity condition determines the curvature of M^{2n+1}completely [5]. For a (k,μ) contact metric manifold, the conditions of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$ are all equivalent. Also h and φ are related by

$$h^2 = (k-1)\varphi^2.$$

If $\mu = 0$, the (k,μ)-nullity distribution $N(k,\mu)$ is reduced to the k-nullity distribution $N(k)$ [13], where the k-*nullity distribution* $N(k)$ of a Riemannian manifold M is defined by

$$N(k) : p \to N_p(k) = \{W \in T_pM \mid R(X,Y)W = kR_0(X,Y)W\};$$

k being a constant. If $\xi \in N(k)$, then we call a contact metric manifold M an $N(k)$-*contact metric manifold*. If $k = 1$, an $N(k)$-contact metric manifold is Sasakian. If $k < 1$, the scalar curvature is $r = 2n(2n-2+k)$. Also in an $N(k)$-contact metric manifold the following conditions hold:

$$S(X,\xi) = 2nk\eta(X), \quad Q\xi = 2nk\xi, \tag{1.2}$$

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{1.3}$$

and

$$R(\xi,X)Y = k(g(X,Y)\xi - \eta(Y)X), \tag{1.4}$$

(see [5]). For an extended contact conformal curvature tensor we find the following equations in an $N(k)$-contact metric manifold:

$$\begin{aligned} C_e(X,Y)Z &= C_0(X,Y)Z - 2(k-1)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi \\ &\quad -4(k-1)\eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &\quad +k\{\eta(X)g(\varphi Y,Z) - \eta(Y)g(\varphi X,Z) - 2\eta(Z)g(\varphi X,Y)\}\xi, \end{aligned} \tag{1.5}$$

$$C_e(X,Y)\xi = -2(k-1)\{\eta(Y)X - \eta(X)Y\} = -2(k-1)R_0(X,Y)\xi$$

and

$$C_e(\xi,X)Y = 2(k-1)\eta(Y)\{X - \eta(X)\xi\} = -2(k-1)\eta(Y)R_0(\xi,X)\xi.$$

Consequently we have

$$C_0(X,Y)\xi = 2(k-1)\{\eta(Y)X - \eta(X)Y\} + 2kg(\varphi X,Y)\xi, \tag{1.6}$$

$$C_0(\xi,X)Y = 2(k-1)\{g(X,Y)\xi - \eta(Y)X\} - kg(\varphi X,Y)\xi = -C_0(X,\xi)Y. \tag{1.7}$$

From (1.5), in a Sasakian manifold, the extended contact conformal curvature tensor and the contact conformal curvature tensor are related by

$$\begin{aligned} C_e(X,Y)Z &= C_0(X,Y)Z + \eta(X)g(\varphi Y,Z)\xi \\ &\quad -\eta(Y)g(\varphi X,Z)\xi - 2\eta(Z)g(\varphi X,Y)\xi, \end{aligned} \tag{1.8}$$

(see [8]).

The standard contact metric structure on the tangent sphere bundle T_1M satisfies the (k,μ)-nullity condition if and only if the base manifold M is of constant curvature. If M has constant curvature c, then $k = c(2-c)$ and $\mu = -2c$.

For a given contact metric structure (φ,ξ,η,g), *$\mathcal{D}$-homothetic deformation* is the structure defined by

$$\overline{\eta} = a\eta, \quad \overline{\xi} = \frac{1}{a}\xi, \quad \overline{\varphi} = \varphi, \quad \overline{g} = ag + a(a-1)\eta\otimes\eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric, K-contact, Sasakian or strongly pseudo-convex CR, it destroys a condition like $R(X,Y)\xi = 0$ or $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y)$. However, the form of the (k,μ)-nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\overline{k} = \frac{k+a^2-1}{a^2}, \quad \overline{\mu} = \frac{\mu+2a-2}{a}.$$

Given a non-Sasakian (k,μ)-manifold M, in [6] an invariant

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-k}}$$

was introduced by E. Boeckx. He showed that for two non-Sasakian (k,μ)-manifolds $(M_i,\varphi_i,\xi_i,\eta_i,g_i)$, $i = 1,2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a $\mathcal{D}$-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Hence we know all non-Sasakian (k,μ)-manifolds locally as soon as we have, for every odd dimension $2n+1$ and for every possible value of the invariant I, one (k,μ)-manifold (M,φ,ξ,η,g) with $I_M = I$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = \frac{1+c}{|1-c|}$ [6].

Using this invariant, an example of a $(2n+1)$-dimensional $N(1-\frac{1}{n})$-contact metric manifold, $n > 1$, was constructed by Blair, Kim and Tripathi in [4] as follows:

Example 1. *Since the Boeckx invariant for a $(1-\frac{1}{n}, 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting $\mathcal{D}$-homothetic deformation will be a $(1-\frac{1}{n}, 0)$-manifold. That is, for $k = c(2-c)$ and $\mu = -2c$ we solve*

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c. The result is

$$c = \frac{(\sqrt{n} \pm 1)^2}{n-1}, \quad a = 1 + c$$

and taking c and a to be these values it is obtained an $N(1-\frac{1}{n})$-contact metric manifold.

We need the following theorems in Section 2.

Theorem 1. *A contact metric manifold M^{2n+1} satisfying the condition $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$ ([2], Theorem 7.5).*

Theorem 2. *If a contact metric manifold M^{2n+1} is of constant curvature c and dimension ≥ 5, then $c = 1$ and the structure is Sasakian ([2], Theorem 7.3).*

§2. Main Results

In this section, we give the main results of the study. Now we begin with the following:

Theorem 3. *Let M be a $(2n+1)$-dimensional non-Sasakian $N(k)$-contact metric manifold. Then M satisfies the condition $\mathcal{Z}(\xi, X) \cdot C_0 = 0$ if and only if either M is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$ or locally isometric to the Example 1.*

Proof. If M is a non-Sasakian $N(k)$-contact metric manifold then the equation (0.1) can be written as

$$\mathcal{Z}(\xi, X) = \frac{2n}{2n+1}\left(k - 1 + \frac{1}{n}\right) R_0(\xi, X), \tag{2.1}$$

which implies that

$$\mathcal{Z}(\xi, X) \cdot C_0 = \frac{2n}{2n+1}\left(k - 1 + \frac{1}{n}\right) R_0(\xi, X) \cdot C_0.$$

Therefore $\mathcal{Z}(\xi, X) \cdot C_0 = 0$ is equivalent to $k = 1 - \frac{1}{n}$ or $R_0(\xi, X) \cdot C_0 = 0$. If $k = 1 - \frac{1}{n}$, then M is locally isometric to the Example 1.

If $R_0(\xi, X) \cdot C_0 = 0$ we can write

$$\begin{aligned} 0 &= R_0(\xi, X)C_0(Y, V)U - C_0(R_0(\xi, X)Y, V)U \\ &\quad - C_0(Y, R_0(\xi, X)V)U - C_0(Y, V)R_0(\xi, X)U \end{aligned}$$

for all $X, Y, V, U \in TM$. So using the definition of R_0 we get

$$\begin{aligned} 0 &= C_0(Y, V, U, X)\xi - \eta(C_0(Y, V)U)X \\ &\quad - g(X, Y)C_0(\xi, V)U + \eta(Y)C_0(X, V)U \\ &\quad - g(X, V)C_0(Y, \xi)U + \eta(V)C_0(Y, X)U \\ &\quad - g(X, U)C_0(Y, V)\xi + \eta(U)C_0(Y, V)X, \end{aligned} \tag{2.2}$$

where $C_0(Y, V, U, X) = g(C_0(Y, V)U, X)$. Putting $U = \xi$ in (2.2) and by the use of (1.6) and (1.7) in (2.2) we obtain

$$\begin{aligned} C_0(Y, V)X &= 2(k-1)[g(X, V)Y - g(X, Y)V] \\ &\quad + 2k[g(\varphi Y, V)X - \eta(Y)g(\varphi X, V)\xi \\ &\quad - \eta(V)g(\varphi Y, X)\xi]. \end{aligned} \tag{2.3}$$

Taking $Y = \xi$ in (2.3) we find

$$C_0(\xi, V)X = 2(k-1)[g(X, V)\xi - \eta(X)V] + 2kg(\varphi V, X)\xi.$$

In view of (1.7), we know that

$$C_0(\xi, V)X = 2(k-1)[g(X, V)\xi - \eta(X)V] - kg(\varphi V, X)\xi.$$

Comparing last two equations we find $kg(\varphi V, X)\xi = 0$. Since $g(\varphi V, X) \neq 0$, we get $k = 0$. Hence from Theorem 1, M is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for dimension 3. The converse statement is trivial. This completes the proof of the theorem. □

Theorem 4. *Let M be a $(2n+1)$-dimensional non-Sasakian $N(k)$-contact metric manifold. If M satisfies the condition $C_0(\xi, X) \cdot \mathcal{Z} = 0$ then either it is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$ or locally isometric to the Example 1.*

Proof. Since M satisfies the condition $C_0(\xi, X) \cdot \mathcal{Z} = 0$, we can write

$$\begin{aligned} 0 &= C_0(\xi, X)\mathcal{Z}(Y, V)U - \mathcal{Z}(C_0(\xi, X)Y, V)U \\ &\quad -\mathcal{Z}(Y, C_0(\xi, X)V)U - \mathcal{Z}(Y, V)C_0(\xi, X)U \end{aligned} \tag{2.4}$$

for all $X, Y, V, U \in TM$. So using (1.7) we have

$$\begin{aligned} 0 &= 2(k-1)\{\mathcal{Z}(Y, V, U, X)\xi - \mathcal{Z}(Y, V, U, \xi)X \\ &\quad -g(X, Y)\mathcal{Z}(\xi, V)U + \eta(Y)\mathcal{Z}(X, V)U \\ &\quad -g(X, V)\mathcal{Z}(Y, \xi)U + \eta(V)\mathcal{Z}(Y, X)U \\ &\quad -g(X, U)\mathcal{Z}(Y, V)\xi + \eta(U)\mathcal{Z}(Y, V)X\} \\ &\quad +k\{-g(\varphi X, \mathcal{Z}(Y, V)U)\xi + g(\varphi X, Y)\mathcal{Z}(\xi, V)U \\ &\quad +g(\varphi X, V)\mathcal{Z}(Y, \xi)U + g(\varphi X, U)\mathcal{Z}(Y, V)\xi\}, \end{aligned} \tag{2.5}$$

where $\mathcal{Z}(Y, V, U, X) = g(\mathcal{Z}(Y, V)U, X)$. Taking $U = \xi$ in (2.5) we get

$$\begin{aligned} 0 &= 2(k-1)\{\mathcal{Z}(Y, V, \xi, X)\xi - g(X, Y)\mathcal{Z}(\xi, V)\xi \\ &\quad +\eta(Y)\mathcal{Z}(X, V)\xi - g(X, V)\mathcal{Z}(Y, \xi)\xi \\ &\quad +\eta(V)\mathcal{Z}(Y, X)\xi - \eta(X)\mathcal{Z}(Y, V)\xi + \mathcal{Z}(Y, V)X\} \\ &\quad +k\{-g(\varphi X, \mathcal{Z}(Y, V)\xi)\xi + g(\varphi X, Y)\mathcal{Z}(\xi, V)\xi \\ &\quad +g(\varphi X, V)\mathcal{Z}(Y, \xi)\xi\}. \end{aligned}$$

Since M is a non-Sasakian $N(k)$-contact metric manifold, using (0.1), the above equation can be written as

$$\begin{aligned} 0 &= \frac{2n}{2n+1}\left(k-1+\frac{1}{n}\right)[2(k-1)\{R_0(Y, V, \xi, X)\xi \\ &\quad -g(X, Y)R_0(\xi, V)\xi + \eta(Y)R_0(X, V)\xi \\ &\quad -g(X, V)R_0(Y, \xi)\xi + \eta(V)R_0(Y, X)\xi - \eta(X)R_0(Y, V)\xi\} \\ &\quad +k\{-g(\varphi X, R_0(Y, V)\xi)\xi + g(\varphi X, Y)R_0(\xi, V)\xi \\ &\quad +g(\varphi X, V)R_0(Y, \xi)\xi\}] + 2(k-1)\mathcal{Z}(Y, V)X. \end{aligned}$$

So by virtue of the definition of R_0 we obtain

$$\begin{aligned} (k-1)\mathcal{Z}(Y, V)X &= \frac{n}{2n+1}\left(k-1+\frac{1}{n}\right)[2(k-1)\{g(X, V)Y \\ &\quad -g(X, Y)V\} + k\{g(\varphi X, Y)V - g(\varphi X, V)Y\}]. \end{aligned} \tag{2.6}$$

Putting $Y = \xi$ in (2.6) we find

$$\begin{aligned} (k-1)\mathcal{Z}(\xi, V)X &= \frac{n}{2n+1}\left(k-1+\frac{1}{n}\right)[(2(k-1))\{g(X, V)\xi \\ &\quad -\eta(X)V\} - kg(\varphi X, V)\xi]. \end{aligned}$$

Hence in view of (0.1) and the definition of R_0 we have

$$k\left(k-1+\frac{1}{n}\right)g(\varphi X,V)\xi=0.$$

Since $g(\varphi X,V)\neq 0$ then we obtain either $k=0$ or $k-1+\frac{1}{n}=0$. If $k=0$ from Theorem 1, M is locally isometric to the $E^{n+1}\times S^n(4)$ for $n>1$ and flat for dimension 3. If $k-1+\frac{1}{n}=0$, then M is locally isometric to the Example 1.

Thus the proof of the theorem is completed. □

Theorem 5. *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold, $n>1$. Then M satisfies the condition $C_e(\xi,X)\cdot\mathcal{Z}=0$ if and only if it is a Sasakian manifold.*

Proof. For all $X,Y,V,U\in TM$, from (0.3) and (1.5), we can write

$$\begin{aligned}\left(C_e(\xi,X)\cdot\mathcal{Z}\right)(Y,V)U &= C_e(\xi,X)\mathcal{Z}(Y,V)U-\mathcal{Z}(C_e(\xi,X)Y,V)U\\ &\quad -\mathcal{Z}(Y,C_e(\xi,X)V)U-\mathcal{Z}(Y,V)C_e(\xi,X)U\\ &= 2(k-1)[-\eta(X)\mathcal{Z}(Y,V,U,\xi)\xi+\mathcal{Z}(Y,V,U,\xi)X\\ &\quad +\eta(X)\eta(Y)\mathcal{Z}(\xi,V)U-\eta(Y)\mathcal{Z}(X,V)U\\ &\quad +\eta(X)\eta(V)\mathcal{Z}(Y,\xi)U-\eta(V)\mathcal{Z}(Y,X)U\\ &\quad +\eta(U)\eta(X)\mathcal{Z}(Y,V)\xi-\eta(U)\mathcal{Z}(Y,V)X].\end{aligned}$$

Therefore $C_e(\xi,X)\cdot\mathcal{Z}=0$ is equivalent to $k=1$ or

$$\begin{aligned}0 &= -\eta(X)\mathcal{Z}(Y,V,U,\xi)\xi+\mathcal{Z}(Y,V,U,\xi)X+\eta(X)\eta(Y)\mathcal{Z}(\xi,V)U\\ &\quad -\eta(Y)\mathcal{Z}(X,V)U+\eta(X)\eta(V)\mathcal{Z}(Y,\xi)U-\eta(V)\mathcal{Z}(Y,X)U\\ &\quad +\eta(U)\eta(X)\mathcal{Z}(Y,V)\xi-\eta(U)\mathcal{Z}(Y,V)X.\end{aligned}\tag{2.7}$$

If $k=1$, then M is a Sasakian manifold. Putting $U=\xi$ in (2.7) we obtain

$$\begin{aligned}0 &= \eta(X)\eta(Y)\mathcal{Z}(\xi,V)\xi-\eta(Y)\mathcal{Z}(X,V)\xi\\ &\quad +\eta(X)\eta(V)\mathcal{Z}(Y,\xi)\xi-\eta(V)\mathcal{Z}(Y,X)\xi\\ &\quad +\eta(X)\mathcal{Z}(Y,V)\xi-\mathcal{Z}(Y,V)X.\end{aligned}\tag{2.8}$$

Since M is an $N(k)$-contact metric manifold, using (0.1) in (2.8) we can write

$$\begin{aligned}0 &= \left(k-\frac{r}{2n(2n+1)}\right)[\eta(X)\eta(Y)R_0(\xi,V)\xi-\eta(Y)R_0(X,V)\xi\\ &\quad +\eta(X)\eta(V)R_0(Y,\xi)\xi-\eta(V)R_0(Y,X)\xi+\eta(X)R_0(Y,V)\xi]-\mathcal{Z}(Y,V)X.\end{aligned}$$

So by virtue of the definition of R_0 we have

$$\mathcal{Z}(Y,V)X = \left(k - \frac{r}{2n(2n+1)}\right)[\eta(X)\eta(V)Y - \eta(X)\eta(Y)V]. \tag{2.9}$$

Then by the use of (0.1), the equation (2.9) can be written as

$$\begin{aligned} R(Y,V)X &= \left(k - \frac{r}{2n(2n+1)}\right)[\eta(X)\eta(V)Y - \eta(X)\eta(Y)V] \\ &\quad + \frac{r}{2n(2n+1)}\{g(X,V)Y - g(Y,X)V\}. \end{aligned} \tag{2.10}$$

Hence from (2.10), by a contraction, we obtain

$$S(X,V) = \frac{r}{2n+1}g(X,V) + \left(2nk - \frac{r}{2n+1}\right)\eta(X)\eta(V). \tag{2.11}$$

From (2.11), by a contraction, we get

$$r = 2nk(2n+1).$$

Then putting $r = 2nk(2n+1)$ into (2.10) we obtain

$$R(Y,V)X = k(g(X,V)Y - g(Y,X)V).$$

So M is a space of constant curvature k. Since $n > 1$, hence from Theorem 2, it is necessarily a Sasakian manifold of constant curvature $+1$, $n > 1$. From (1.8), since $C_e(\xi, X)Y = 0$ for all Sasakian manifolds, the converse statement is trivial. Hence we get the result as required. □

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