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On the topology of the complements of quartic and line configurations

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Abstract. For a reduced plane curve C and a line L in $\mathbb{P}^2$, we put $\mathbb{C}^2_L := \mathbb{P}^2 - L$, and $C_L := C - (C \cap L)$. If C and L intersect transversaly and $\pi_1(\mathbb{P}^2 - C, b_0)$ is abelian, it is known that $\pi_1(\mathbb{C}^2_L - C_L)$ is also abelian. In this article, we study $\pi_1(\mathbb{C}^2_L - C_L)$ and the Alexander polynomial for the case when a quartic curve C and a line L do not intersect transversaly.

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§1. Introduction

Let C be a reduced plane curve in $\mathbb{P}^2$. We choose a line $L \subset \mathbb{P}^2$ and we put $\mathbb{C}^2_L := \mathbb{P}^2 - L$, and $C_L := C - (C \cap L)$. The line L is said to be *generic* with respect to C if L intersects C transversaly.

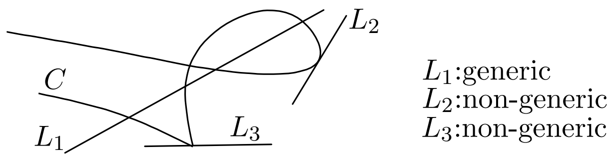


Figure 1.1

An element $\omega \in \pi_1(\mathbb{P}^2 - L, b_0)$ is called a *lasso* for L if it is represented by a loop $l \circ \tau \circ l^{-1}$ where τ is a counter-clockwise oriented boundary of a small disc $D(p)$ of L at a point $p \in L$ and l is a path connecting b_0 and τ. In [O1], Oka proved the following:

Proposition 1.1 ([O1]) *Let ω be a lasso for L and $N(\omega)$ be the subgroup normally generated by ω. Then the following sequence is exact:*

$$1 \to N(\omega) \to \pi_1(\mathbb{C}^2_L - C_L) \to \pi_1(\mathbb{P}^2 - C) \to 1.$$

Moreover, if L is generic with respect to C, then

- *ω is in the center of $\pi_1(\mathbb{C}^2_L - C_L)$ and $N(\omega) \cong \mathbb{Z}$, and*
- *the equality $D(\pi_1(\mathbb{C}^2_L - C_L)) = D(\pi_1(\mathbb{P}^2 - C))$ holds, where $D(\star)$ denotes the commutator group of a group $\star$.*

Note that we assume a base point b_0 is chosen suitably. In the following, we omit the base points unless we need it explicitly.

By Proposition 1.1, when L is generic with respect to C, then $\pi_1(\mathbb{P}^2 - C)$ is abelian if and only if $\pi_1(\mathbb{C}^2_L - C_L)$ is abelian. On the other hand, for a non-generic line L, $\pi_1(\mathbb{C}^2_L - C_L)$ may be non-abelian. For example, when C is the quartic defined by $\{X^3Z + Y^4 = 0\}$ which has an e_6 singularity ($= (3,4)$-cusp) at $[0 : 0 : 1]$, and $L_\infty = \{Z = 0\}$ is the tangent line with multiplicity 4 at [1:0:0], a presentation of the fundamental group $\pi_1(\mathbb{C}^2_L - C_L)$ is given by

$$\pi_1(\mathbb{C}^2_L - C_L) \cong \langle a, b, c \,|\, (abca)a = b(abca),\ c(abca) = (abca)b\ \rangle$$

and its Alexander polynomial is

$$\Delta_C(t, L) = (t^2 - t + 1)(t^4 - t^2 + 1).$$

Hence $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/4\mathbb{Z}$, while $\pi_1(\mathbb{C}^2_L - C_L)$ is non-abelian. More such examples can be found in [O3].

No.	Singularities	L_∞	No.	Singularities	L_∞
(1)	$2a_2$	(i)	(11)	e_6	(i)
(2)	$2a_2$	(iv)	(12)	e_6	(iv)
(3)	$2a_2 + a_1$	(i)	(13)	$a_3 + a_2 + a_1$	(ii), a_3
(4)	$2a_2 + a_1$	(iv)	(14)	$a_5 + a_1$	(i)
(5)	$3a_2$	(i)	(15)	$a_4 + a_2$	(ii), a_2
(6)	$a_2 + a_3$	(ii), a_3	(16)	$a_5 + a_2$	(iii), a_2, a_5
(7)	a_5	(i)	(17)	$2a_3$	(iii), $2a_3$
(8)	a_5	(iv)	(18)	a_7	(v), a_7
(9)	a_6	(ii), a_6	(19)	$2a_3 + a_1$	(iii), $2a_3$
(10)	$a_4 + a_2$	(v), a_4			

Table 1

Our purpose of this article is to study such phenomena for the case when C is a quartic and L is a non-generic line. For simplicity, we call such a configuration a *QL-configuration*. In [T], Tokunaga gave a list of QL-configurations which can be the branch loci of D_{2p}-covers, where D_{2p} is the dihedral group of order $2p$ with p an odd prime number. In particular, the fundamental groups of the complements of such configurations are non-abelian. In this article, we give explicit models for QL-configurations in [T], presentations of $\pi_1(\mathbb{C}^2_L - C_L)$ and the Alexander polynomials of them. We remark that all intersection points of C and L are not transversal by [T, Corollary 3.5].

We introduce some notations in order to state our results. By a suitable projective transformation, we may regard L as the line at infinity L_∞.

Thus we only have to study the fundamental group $\pi_1(\mathbb{C}^2 - C^a)$, where C^a denotes the affine part of C which is defined by $C^a := C - (C \cap L_\infty)$. Table 1 is the list of the possible QL-configurations given in [T].

The numbers (i),. . .,(v) in the column of L_∞ explain how C intersects L_∞ as follows:

(i) L_∞ is bi-tangent to C at two distinct smooth points.

(ii) L_∞ is tangent to a smooth point and passes through a singular point of C.

(iii) L_∞ passes through two distinct singular points.

(iv) L_∞ is tangent to C at a smooth point with intersection multiplicity 4.

(v) L_∞ intersects C at a singular point with intersection multiplicity 4.

For notations for singularities, we use those in [M-P]. We first give an explicit model for each QL-configuration in Table 1. Note that $L = L_\infty$ and quartics whose affine parts C^a are given by $f(x, y) = 0$ in Table 2. Here we put $x = X/Z,\ y = Y/Z$.

No.	$f(x,y)$
(1)	$-6\,y^3 + x^4 - 8\,x^2 + 16 - 2\,y^2x^2 + 8\,y^2 + y^4$
(2)	$-y^3 + x^4 - 2\,x^2 + 1$
(3)	$\frac{27}{4}\,y^4 - 16\,y^3 + 12\,y^2 - 3\,y + \frac{1}{4} - \frac{27}{2}\,x^2y^2 + \frac{27}{4}\,x^4$
(4)	$\frac{16}{9}\,x^3 - 2\,x^2 - 6\,x + \frac{15}{2} - 6\,y^2x - 9\,y^2 - \frac{1}{2}\,y^4$
(5)	$18x^2 + 18y^2 + 24xy^2 - 8x^3 + 2x^2y^2 + y^4 + x^4 - 27$
(6)	$1 + 3\,y - 4\,y^2x^2 + 3\,y^2 - 4\,y^3x + y^3 - y^4$
(7)	$36\,x^4 - 33\,x^3 + 10\,x^2 - x - 12\,y^2x^2 + 10\,y^2x - 2\,y^2 + y^4$
(8)	$-x^3 + x^2 - 2\,xy^2 + y^4$
(9)	$1 - 2\,xy - 4\,y + y^2x^2 + 4\,xy^2 + 6\,y^2 - 2\,xy^3 - 3\,y^3 + y^4$
(10)	$x^3 - \left(x^2 - y + 1\right)^2$
(11)	$(x-1)^3 + y^4$
(12)	$x^3 + (y-x)^2\,(y+x)^2$

(13)	$y\left(81\,x+54+81\,yx^2-54\,yx-54\,y+90\,y^2x+18\,y^2+25\,y^3\right)$
(14)	$\left(y^2-2\,yx-x+x^2\right)\left(y^2+2\,yx-x+x^2\right)$
(15)	$256\,y^4-256\,y^3+96\,y^2-16\,y+1+32\,xy^3-32\,xy^2+10\,xy-x+x^2y^2$
(16)	$(y-x)\left(-y^2x+1+y^3\right)$
(17)	$\left(y^2+yx+2\right)\left(y^2+yx+1\right)$
(18)	$\left(-x-2+y^2\right)\left(-x-1+y^2\right)$
(19)	$(-2\,y+1+2\,x)\,(2\,y+1+2\,x)\left(-y^2+x^2+x\right)$

Table 2: Defining equations of Q

Now we are ready to state our result.

Theorem 1.2 *For the QL-configurations given by the quartic polynomials as above and L_∞, we have presentations of the fundamental groups $\pi_1(\mathbb{C}^2_L - C_L)$ and Alexander polynomials $\Delta_C(t, L)$ as follows:*

No.	presentation of $\pi_1(\mathbb{C}^2_L - C_L)$	$\Delta_C(t,L)$
(1)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(2)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(3)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(4)	$\langle a,b,c \mid aca = cac,\ cbc = bcb,\ ab = ba\rangle$	t^2-t+1
(5)	$\langle a,b,c \mid aba = bab,\ bcb = cbc,$ $c\,(b^{-1}ab)\,c = (b^{-1}ab)\,c\,(b^{-1}ab)\rangle$	$(t^2-t+1)^2$
(6)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(7)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(8)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(9)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(10)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(11)	$\langle a,b \mid aba = bab\rangle$	t^2-t+1
(12)	$\langle a,b,c \mid b(acba) = (acba)a,\ c(acba) = (acba)b\rangle$	$(t^2-t+1)(t^4-t^2+1)$
(13)	$\langle a,b,c \mid bcb = cbc,\ ac = ba,\ accb = cbac\rangle$	$(t-1)(t^2-t+1)$
(14)	$\langle a,b \mid (ab)^3 = (ba)^3,\ babba = abbab\rangle$	$(t-1)(t^2-t+1)$
(15)	$\langle a,b \mid ababa = babab,\ abb = bba\rangle$	1
(16)	$\langle a,b \mid bba = abb\rangle$	t^2-1
(17)	$\langle a,b,c \mid ab = bc\rangle$	1
(18)	$\langle a,b,c \mid ab = bc\rangle$	1
(19)	$\langle a,b,c \mid bc = cb,\ bac = cab\rangle$	$(t-1)^2(t+1)$

Table 3

Remark. Some Alexander polynomials in the above list are computed in [O3] by two ways. They are mainly obtained by the *line degeneration method* and some are done by the *Fox calculus.* And the group represented by $\langle a, b \,|\, aba = bab\rangle$ is isomorphic to the braid group $\mathbb{B}_3$ of 3 strings.

§2. Preliminaries

2.1. Zariski-van Kampen's method

In this subsection, we briefly summarize Zariski-van Kampen's method in computing the fundamental group. For details, see [O3], [S] and [T-S]. Let C^a be a reduced affine curve defined by a polynomial $f(x, y)$ of degree d. By a suitable linear transformation of coordinate system, we may assume that the coefficient of y^d is a non-zero constant.

Let $p : \mathbb{C}^2 - C^a \to \mathbb{C}$ be a map given by $(x, y) \mapsto x$. For $s \in \mathbb{C}$, y-coordinates of the intersection points of the affine line $\{x = s\}$ and C are corresponding to roots of the equation $f(s, y) = 0$. We denote by $D_{(f,y)}(s)$ the discriminant with respect to y and put

$$\Sigma := \{s \in \mathbb{C} \mid D_{(f,y)}(s) = 0\}.$$

We call lines $\{x = s \,|\, s \in \Sigma\}$ *singular lines.* Since C^a is reduced, Σ is a finite set. For all $t \in \mathbb{C} - \Sigma$, $p^{-1}(t)$ is isomorphic to the d-punctured affine line and restriction

$$p|_{p^{-1}(\mathbb{C}-\Sigma)} \;:\; p^{-1}(\mathbb{C} - \Sigma) \to \mathbb{C} - \Sigma$$

is the local trivial fibration.

We choose a sufficiently large positive real number S such that a disc $B_S := \{s \in \mathbb{C} \mid |s| < S\}$ contains Σ. Since the inclusion $p^{-1}(B_S) \hookrightarrow \mathbb{C}^2 - C$ gives a homotopy equivalence, we only have to compute $\pi_1(p^{-1}(B_S))$. Since the coefficients of $y^{d-\nu}$, $(\nu > 0)$ in $f(x, y)$ are polynomials in x and they are bounded on B_S, we can take a base point $b_0' = (b_0, \tilde{b_0}) \in \mathbb{C}^2 - C^a$ where $p(b_0') = b_0$, $b_0 \in B_S$ and $(B_S \times \{\tilde{b_0}\}) \cap C = \emptyset$.

In this setting, the restriction of p

$$p|_{p^{-1}(B_S)} : p^{-1}(B_S) \to B_S$$

has the holomorphic section $s \mapsto (s, \tilde{b_0})$ which passes through b_0'. Using this section, one can define an action of $\pi_1(B_S - \Sigma, b_0)$ on $\pi_1(p^{-1}(b_0), b_0')$. We call this action *the monodromy action.* Let r be a number of points of Σ. Then $\pi_1(B_S - \Sigma, b_0)$ is the free group generated by loops $\gamma_1, \gamma_2, \ldots, \gamma_r$, and $\pi_1(p^{-1}(b_0), b_0')$ is generated by loops $g_1, g_2, \ldots, g_d$. We denote the action of γ_j on g_i by $g_i^{\gamma_j}$. Now, Zariski-van Kampen's theorem can be stated as follows.

Theorem 2.1 ([T-S], [S]) *The inclusion map $p^{-1}(b_0) \hookrightarrow \mathbb{C}^2 - C^a$ induces an isomorphism*

$$\pi_1(p^{-1}(b_0), b_0')/N \to \pi_1(\mathbb{C}^2 - C^a, b_0'),$$

where N is the minimal normal subgroup of $\pi_1(p^{-1}(b_0), b_0')$ which contains

$$\{g_i^{-1} g_i^{\gamma_j} \mid i = 1, 2, \ldots, d,\ j = 1, 2, \ldots, r\}.$$

Thus the presentation of $\pi_1(\mathbb{C}^2 - C^a, b_0')$ is

$$\langle g_1, g_2, \ldots, g_d \mid g_i = g_i^{\gamma_j} \rangle_{\ i=1,2,\ldots,d,\, j=1,2,\ldots,r}.$$

In particular, we call relations $g_i = g_i^{\gamma_j}$ *monodromy relations.*

2.2. Some basic monodromy actions

In this subsection, we recall some basic monodromy actions. We consider curves whose local equations at $(0, 0)$ given by

$$\text{(i) } h_1 = x - y^2, \quad \text{(ii) } h_2 = x^2 - y^2, \quad \text{(iii) } h_3 = x^3 - y^2.$$

Since the line $x = 0$ is a singular line for all cases, we take a base point $x = \epsilon$ and its general fiber $F_\epsilon = p^{-1}(\epsilon)$, where ϵ is a sufficiently small positive number. Since each h_i have degree 2 with respect to y, F_ϵ is isomorphic to the 2-punctured plane. We take meridians g_1 and g_2 as generators of $\pi_1(F_\epsilon)$ as follows:

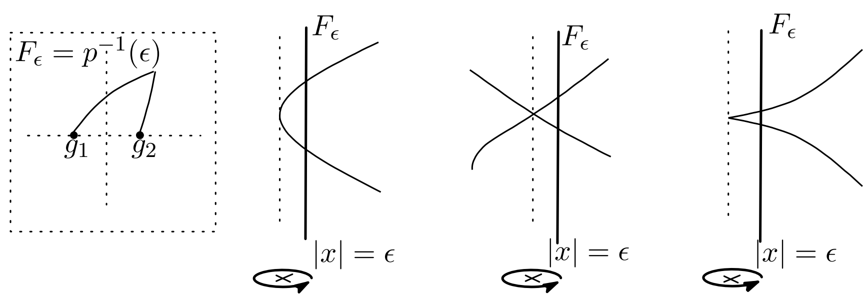


Figure 2.1

When x goes with counter clockwise direction along the circle $|x| = \epsilon$ which is the generator of $\pi_1(\mathbb{C} - \{0\})$, g_1 and g_2 are moved as following figures by the monodromy action.

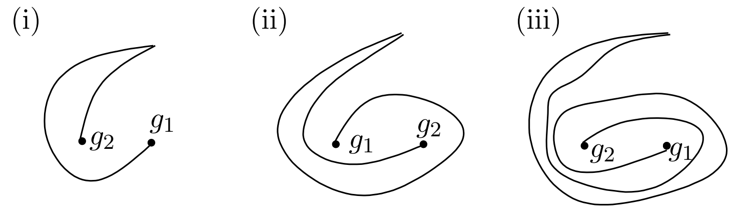


Figure 2.2

By Theorem 2.1, we have monodromy relations of each cases:

$$\text{(i) } g_1 = g_2 \quad \text{(ii) } g_1 g_2 = g_2 g_1 \quad \text{(iii) } g_1 g_2 g_1 = g_2 g_1 g_2.$$

We call these relations *the tangential relation, the nodal relation* and *the cuspidal relation* respectively.

2.3. Alexander polynomial

In this subsection, we briefly summarize Fox calculus in computing the Alexander polynomial. For details, see [C-F, 119p]. Suppose that $G := \pi_1(\mathbb{C}^2 - C^a, b_0')$ is given by the following finite representation:

$$G = \langle g_1, \ldots, g_n \mid R_1, \ldots, R_m \rangle,$$

where g_i are generators of $\pi_1(p^{-1}(b_0))$ and R_i denotes the monodromy relations. Let $F(n)$ be a free group of rank $= n$ which is generated by $g_1, \ldots, g_n$. Moreover we consider the group ring $\mathbb{C}\big[F(n)\big]$ of $F(n)$ with $\mathbb{C}$-coefficient. The *Fox differentials* $\frac{\partial}{\partial g_i} : \mathbb{C}\big[F(n)\big] \to \mathbb{C}\big[F(n)\big]$ are a $\mathbb{C}$-linear operator which satisfies the following two properties:

$$\text{(i) } \frac{\partial}{\partial g_j}(g_i) = \delta_{ij}, \quad \text{(ii) } \frac{\partial}{\partial g_j}(uv) = \frac{\partial u}{\partial g_j} + u\frac{\partial v}{\partial g_j}, \quad u, v \in \mathbb{C}\left[F(n)\right].$$

Let $\gamma : \mathbb{C}\left[F(n)\right] \to \mathbb{C}[t, t^{-1}]$ be a ring homomorphism defined by $g_i, g_i^{-1} \longmapsto t, t^{-1}$ for all i. Now we get $(m \times n)$-matrix whose elements are in $\mathbb{C}[t, t^{-1}]$. We put

$$A := \left(\gamma\left(\frac{\partial R_i}{\partial g_j}\right)\right)$$

and call A the *Alexander matrix.* Then *Alexander polynomial* $\Delta_C(t)$ is given by the greatest common divisor of the determinants of all $(n-1) \times (n-1)$ submatrices of the Alexander matrix A if $m \geq n-1$, and it is understood that

$$\Delta_C(t) = 0 \quad \text{if} \quad n - 1 > m,$$
$$\Delta_C(t) = 1 \quad \text{if} \quad n - 1 \leq 0.$$

Example 2.2 *For the group presentad by* $\langle a,b \mid aba=bab\rangle$*, we put the monodromy relation as* $R=abab^{-1}a^{-1}b^{-1}$*. Then we have*

$$\frac{\partial R}{\partial a}=1+ab-abab^{-1}a^{-1},\ \frac{\partial R}{\partial b}=a-abab^{-1}-abab^{-1}a^{-1}b^{-1}.$$

Moreover the Alexander matrix A *and Alexander polynomial* $\Delta(t)$ *are given as follows.*

$$A=\left[\begin{array}{cc} 1+t^2-t & t-t^2-1 \end{array}\right],\ \Delta(t)=t^2-t+1.$$

§3. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. Since our proof is done by case by case computation and most of them are similar, we give complete proofs for the cases (5) and (9), and rough sketches for the remaining cases.

Case (5): In this case, Q is a quartic with $3a_2$ and L is its unique bi-tangent line. As we have seen in the introduction, such an example is given by an equation

$$F(X,Y,Z):=18X^2Z^2+18Y^2Z^2+24XY^2Z-8X^3Z+2X^2Y^2+Y^4+X^4-27Z^4.$$

When we put $C=\{F(X,Y,Z)=0\}$ and its affine part $C^a=\{f(x,y)=F(x,y,1)=0\}$, then

(i) C has three cusps at $[3:0:1]$ and $[-3/2:\pm 3\sqrt{3}/2:1]$,

(ii) $L_\infty=\{Z=0\}$ is bi-tangent to C, and

(iii) the discriminant of f with respect to y is

$$4096(x+1)(x-3)^3(2x+3)^6.$$

Figure 3.1 shows the graph of the real part of C^a. Put $\Sigma=\{x\in\mathbb{C}\mid x=-3/2,\ -1,\ 3\}$ and take the base point $b_0:=-1-\epsilon$, where ϵ is a small positive real number. We denote by F_0 the fiber $p^{-1}(b_0)$ which is isomorphic to 4-punctured plane.

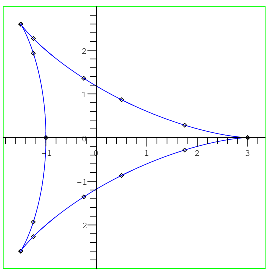


Figure 3.1

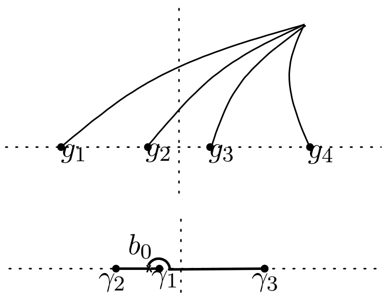


Figure 3.2

To compute the monodromy action by the fundamental group of base space $\pi_1(\mathbb{C} - \Sigma, b_0)$, we fix g_1, g_2, g_3, g_4 as generators of $\pi_1(F_0, b_0')$, and meridians $\gamma_1, \gamma_2, \gamma_3$ as generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$, where γ_1, γ_2 and γ_3 are given by the following:

- γ_1: the meridian of $x = -1$ from b_0
- γ_2: the meridian of $x = -3/2$ from b_0
- γ_3: the meridian of $x = 3$ from b_0 stepping aside $x = -1$.

Figure 3.2 shows our choice of g_1, g_2, g_3, g_4 and $\gamma_1, \gamma_2, \gamma_3$. Note that • denotes a circle. In this setting, we see each monodromy actions explicitly.

Monodromy action of γ_1: Since g_2 and g_3 have the tangential relation, we obtain $g_2 = g_3$.

Monodromy action of γ_2: Since g_1, g_2 and g_3, g_4 have the cuspidal relations each other, we obtain $g_1 g_2 g_1 = g_2 g_1 g_2$ and $g_3 g_4 g_3 = g_4 g_3 g_4$.

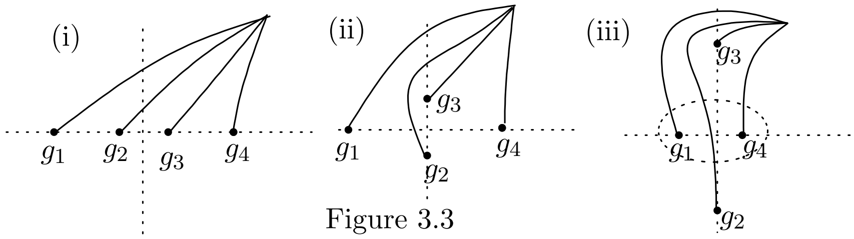


Figure 3.3

Monodromy action of γ_3: Figure 3.3 corresponds each steps of the monodromy action with respect to γ_2 as follows:

(i) start position,

(ii) z comes close to -1 and steps aside -1,

(iii) z comes close to 3.

By Figure 3.3, we can observe g_1 and g_4 have the cuspidal relation.

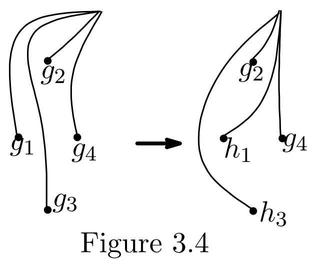


Figure 3.4

For simplicity, we replace two generators g_1 and g_3 by h_1 and h_3 as in Figure 3.4. Namely we put

$$h_1 = g_2^{-1} g_1 g_2, \ \ h_3 = g_1 g_3 g_1^{-1}.$$

h_1 and g_4 have the cuspidal relation, we just only obtain $g_4 h_1 g_4 = h_1 g_4 h_1$. By Theorem 2.1,

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2, g_4, h_1 \mid g_1 g_2 g_1 = g_2 g_1 g_2, \ g_2 g_4 g_2 = g_4 g_2 g_4, \ g_4 h_1 g_4 = h_1 g_4 h_1, \ h_1 = g_2^{-1} g_1 g_2 \rangle.$$

Now the Alexander matrix A and its Alexander polynomial are

$$A = \begin{bmatrix} t^2 - t + 1 & -t^2 + t - 1 & 0 \\ 0 & t^2 - t + 1 & -t^2 + t - 1 \\ t - t^2 + 1 & (1 - t)(t^2 - t + 1) & t^2 - t + 1 \end{bmatrix}, \ \Delta_C(t) = (t^2 - t + 1)^2.$$

Case (9): In this case, Q is a quartic with an a_6 singularity and L is its unique tangent line which intersects C at the a_6 singular point. As we have seen in the introduction, such an example is given by an equation

$$F := X^4 - 2\, Z^2 Y X - 4\, Z^3 Y + Y^2 X^2 + 4\, Z Y^2 X + 6\, Y^2 Z^2 - 2\, X Y^3 - 3\, Z Y^3 + Y^4.$$

When we put $C = \{F(X, Y, Z) = 0\}$ and its affine part $C^a = \{f(x, y) = F(x, y, 1) = 0\}$, then

(i) C has an a_6 singularity at $[1 : 0 : 0]$,

(ii) $L_\infty = \{Z = 0\}$ is a simple tangent line which intersects C at the a_6 singular point, and

(iii) The discriminant of f with respect to y is

$$D_{(f,y)}(x) = 16\, x^4 + 124\, x^3 + 232\, x^2 + 96\, x + 229.$$

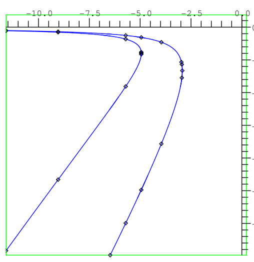


Figure 3.5

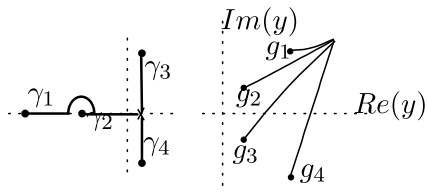


Figure 3.6

Figure 3.5 shows the graph of the real part of C^a. We put $\Sigma := \{\alpha_1, \alpha_2, \beta, \overline{\beta}\}$, where α_i are real roots of $D_{(f,y)}(x) = 0$ and $\beta, \overline{\beta}$ are complex roots. We assume that $\alpha_1 < \alpha_2$. Take the base point b_0 of $\mathbb{C} - \Sigma$ at $b_0 = Re(\beta) = Re(\overline{\beta}) \in \mathbb{C}$

and meridians $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$. For $F_0 = p^{-1}(b_0)$, we also fix meridians g_1, g_2, g_3, g_4 as generators of $\pi_1(F_0, b_0')$. Figure 3.6 shows our setting of generators.

Monodromy action of γ_3: By Figure 3.7, we have $g_2 = g_4$ immediately. Similarly, we have monodromy relation $g_1 = g_3$ with γ_4.

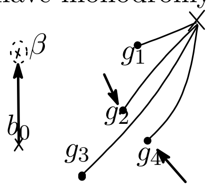


Figure 3.7

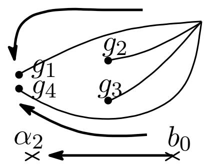


Figure 3.8

Monodromy action of γ_2: By Figure 3.8, g_1 and g_4 have the tangential relation. Considering the homotopy equivarence of roops, we have the monodromy relation,

$$g_1 g_2 g_3 = g_2 g_3 g_4.$$

By using previous relations $g_2 = g_4$ and $g_1 = g_3$, this relation implies $g_1 g_2 g_1 = g_2 g_1 g_2$.

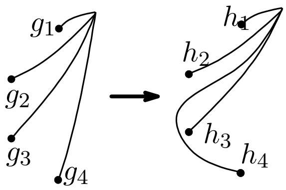


Figure 3.9

Monodromy action of γ_1: For simplicity, we replace generators as follows:

$$h_1 = g_1, \ h_2 = g_2, \ h_3 = g_3, \ h_4 = g_3 g_4 g_3^{-1}.$$

Figure 3.9 explains how to replace the base position of generators, and Figure 3.10 corresponds each steps of the monodromy action with respect to γ_1:

(i) z comes close to α_2, (ii) z steps aside α_2, (iii) z rotates arround α_1.

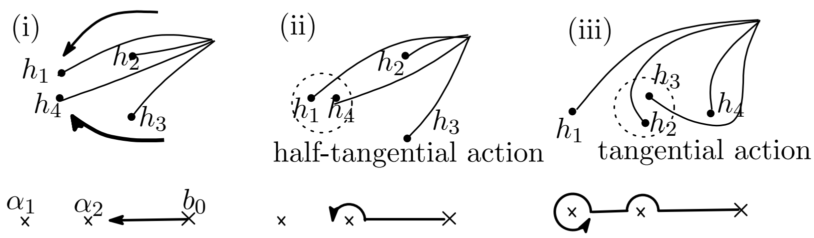


Figure 3.10

By Figure 3.10, we have just one monodromy relation with γ_1:

$$h_2 = h_4 h_3 h_4^{-1} \Leftrightarrow h_2 h_4 = h_4 h_3.$$

Rewriting this relation into g_1, g_2, g_3, g_4, and $g_1 = g_3,\ g_2 = g_4$, we obtain

$$g_2 g_1 g_2 = g_1 g_2 g_1.$$

By Theorem 2.1, we have

$$\pi_1(\mathbb{C}^2_L - C_L) \cong \langle g_1, g_2 \mid g_2 g_1 g_2 = g_1 g_2 g_1 \rangle.$$

Moreover the Alexander polynomial is equal to $t^2 - t + 1$ by Example 2.2.

For simplicity, we sometimes denote the monodromy action of each γ_i and its monodromy relations by γ_i-*action* and γ_i-*relation* in the following sketch of proofs.

Case (1): We consider the following homogenized polynomial:

$$F(X,Y,Z) := 16\,Z^4 - 8\,X^2Z^2 + X^4 + 8\,Y^2Z^2 - 2\,Y^2X^2 - 6\,Y^3Z + Y^4.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has two cusps at $[\pm 2 : 0 : 1]$,

(ii) $L_\infty = \{Z = 0\}$ is a bi-tangent line, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = -144\,(x-2)^4\,(x+2)^4\,\left(64\,x^2 - 13\right).$$

Figure 3.11 shows the real part of the affine curve C^a and the setting of generators.

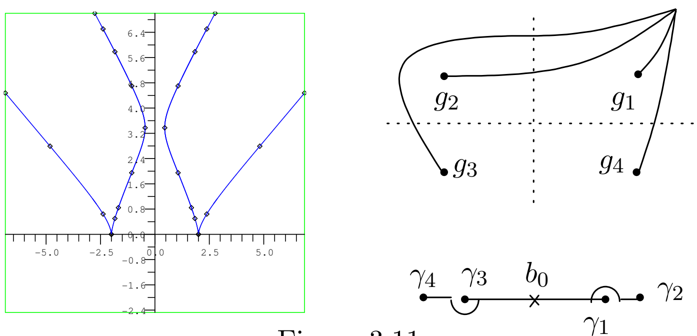


Figure 3.11

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = 0 \in \mathbb{C} - \Sigma$. Furthermore, we take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.12. Monodromy relations with each γ_i are

$$(\gamma_1)\ \ g_1 = g_4,\ \ (\gamma_2)\ \ g_1 = g_3,\ g_1 g_2 g_1 = g_2 g_1 g_2.$$

Since $f(x,y) = f(-x,y)$, monodromy relations with γ_3 and γ_4 are same to γ_1-relation and γ_2-relation. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2 \rangle, \ \Delta_C(t,L) = t^2 - t + 1.$$

Case (2): We consider the following homogenized polynomial:

$$F(X,Y,Z) := -Y^3Z + X^4 - 2\,X^2Z^2 + Z^4.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has two cusps at $[\pm 1 : 0 : 1]$,

(ii) $L_\infty = \{Z = 0\}$ is a tangent line with multiplicity 4, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = -27\,(x-1)^4\,(x+1)^4\,.$$

Figure 3.12 show the real part of the affine curve C^a and the setting of generators.

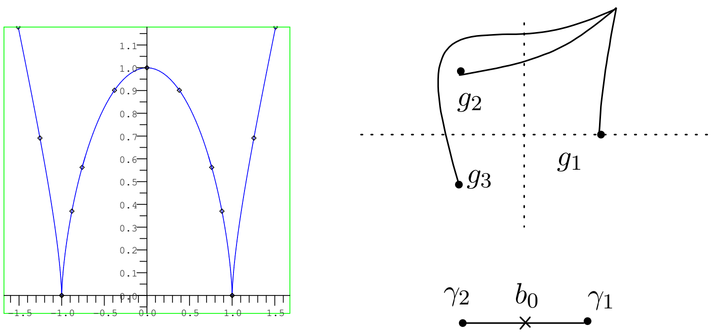


Figure 3.12

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = 0 \in \mathbb{C} - \Sigma$. Moreover, we take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.13. Monodromy relations with γ_1 are

$$g_1 = g_3, \ g_1g_2g_1 = g_2g_1g_2.$$

Since $f(x,y) = f(-x,y)$, the monodromy relation with γ_2 is same to the γ_1-relation. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2 \rangle, \ \Delta_C(t,L) = t^2 - t + 1.$$

Case (3): We consider the following homogenized polynomial:

$$F(X,Y,Z) := \frac{27}{4}\,Y^4 - 16\,Y^3Z + 12\,Y^2Z^2 - 3\,YZ^3 + \frac{1}{4}\,Z^4 - \frac{27}{2}\,X^2Y^2 + \frac{27}{4}\,X^4.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has two cusps and a_1 singular point at $[\pm 1/4 : 1/4 : 1]$, $[0 : 1 : 1]$ respectively,

(ii) $L_\infty = \{Z = 0\}$ is a bi-tangent line, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = -\frac{19683}{16}x^2\left(27x^2-2\right)(4x-1)^3(4x+1)^3.$$

Figure 3.13 shows the real part of the affine curve C^a and the setting of generators.

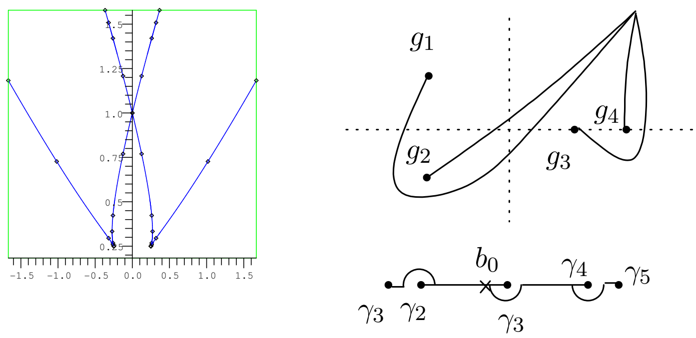


Figure 3.13

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = -\epsilon \in \mathbb{C} - \Sigma$, where ϵ is a sufficiently small positive real number. Moreover, we take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.14. Monodromy relations with each γ_i are

$$(\gamma_1)\quad g_3g_4 = g_4g_3, \qquad (\gamma_2)\quad g_1g_2g_1 = g_2g_1g_2,$$
$$(\gamma_3)\quad g_1 = g_4g_3g_4^{-1}, \quad (\gamma_5)\quad g_1 = g_4.$$

Since $f(x,y) = f(-x,y)$, γ_4-action is same to γ_2-action. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2\rangle,\ \Delta_C(t,L) = t^2 - t + 1.$$

Case (4): We consider the following homogenized polynomial:

$$F(X,Y,Z) := \frac{16}{9}X^3Z - 2X^2Z^2 - 6XZ^3 + \frac{15}{2}Z^4 - 6XY^2Z - 9Y^2Z^2 - \frac{1}{2}Y^4.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has two cusps and an a_1 singular point at $[-3 : \pm 3 : 1]$, $[3/2 : 0 : 1]$,

(ii) $L_\infty = \{Z = 0\}$ is a tangent line with multiplicity 4, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = -\frac{4096}{729}(8x+15)(2x-3)^2(x+3)^6.$$

Figure 3.14 shows the real part of the affine curve C^a and the setting of generators.

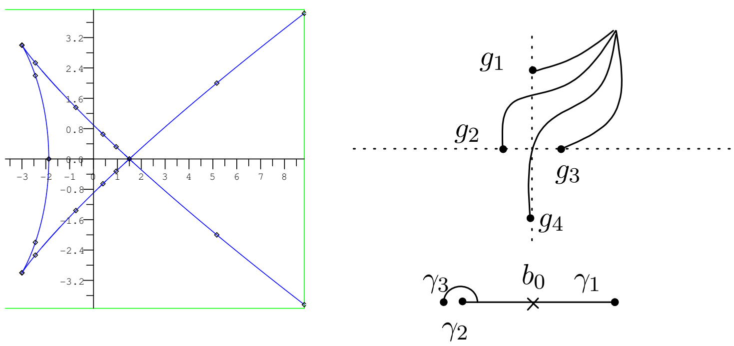


Figure 3.14

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = 3/2 - \epsilon \in \mathbb{C} - \Sigma$, where ϵ is a sufficiently small positive real number. We also take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.15. Monodromy relations with γ_i are

$$(\gamma_1)\ g_2g_3 = g_3g_2,\ (\gamma_2)\ g_1 = g_4,\ (\gamma_3)\ g_1g_2g_1 = g_2g_1g_2,\ g_3g_4g_3 = g_4g_3g_4.$$

By Theorem 2.1, we have

$$\pi_1(\mathbb{C}^2_L - C_L) \cong \langle g_1, g_2, g_3 \mid g_1g_2g_1 = g_2g_1g_2,\ g_1g_3g_1 = g_3g_1g_3,\ g_2g_3 = g_3g_2\rangle.$$

By this presentation, the Alexander matrix A and its Alexander polynomial are given by

$$A = \begin{bmatrix} t^2-t+1 & -t^2+t-1 & 0 \\ t^2-t+1 & 0 & -t^2+t-1 \\ 0 & 1-t & t-1 \end{bmatrix},\ \Delta_C(t,L) = t^2 - t + 1.$$

Case (6): We consider the following homogenized polynomial:

$$F(X,Y,Z) := Z^4 + 3YZ^3 - 4X^2Y^2 + 3Y^2Z^2 - 4XY^3 + Y^3Z - Y^4.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has one a_3 and one a_2 singular points at $[1:0:0]$, $[1/2,-1,1]$ respectively,

(ii) $L_\infty = \{Z = 0\}$ is a tangent line which passes through a_3 , and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = -\left(256x^4 - 32x^3 - 480x^2 + 822x - 283\right)(2x-1)^3.$$

Figure 3.15 shows the real part of the affine curve C^a and the setting of generators.

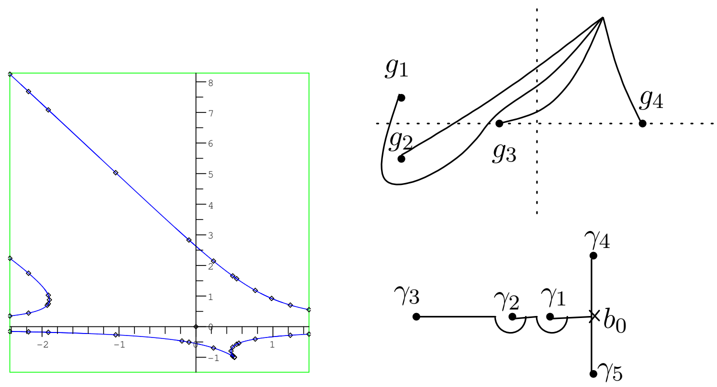


Figure 3.15

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = \alpha \in \mathbb{C} - \Sigma$, where α is the real part of the complex root of $256\,x^4 - 32\,x^3 - 480\,x^2 + 822\,x - 283 = 0$. We take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.16. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_1g_2g_1 = g_2g_1g_2, \quad (\gamma_2)\ g_1 = g_3,$$
$$(\gamma_3)\ g_1 = g_3, \quad (\gamma_4)\ g_2 = g_3g_4g_3^{-1},$$
$$(\gamma_5)\ g_4 = g_2g_1g_2^{-1}.$$

Computing these relations, we have $g_1g_2g_1 = g_2g_1g_2$. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}^2_L - C_L) \cong \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2 \rangle,\ \Delta_C(t,L) = t^2 - t + 1.$$

Case (7): We consider the following homogenized polynomial:

$$F(X,Y,Z) := 36X^4 - 33X^3Z + 10X^2Y^2 - XZ^3 - 12X^2Y^2 + 10XY^2Z - 2Y^2Z^2 + Y^4.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has an a_5 singular point at $[1/3 : 0 : 1]$,

(ii) $L_\infty = \{Z = 0\}$ is a bi-tangent line, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = 256x\,(4x-1)\,(-1+3x)^8\,.$$

The Figure 3.16 shows the real part of the affine curve C^a and the setting of generators.

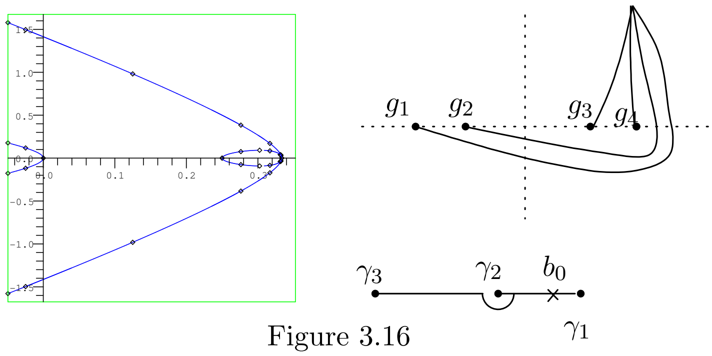


Figure 3.16

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = 1/3 - \epsilon \in \mathbb{C} - \Sigma$, where ϵ is a sufficiently small positive real number. Moreover, we take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.17. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_1 = g_3,\ g_2 = g_3g_4g_3^{-1},\ g_4 = (g_3g_4g_2g_1)g_2(g_3g_4g_2g_1)^{-1},$$
$$(\gamma_2)\ g_3 = g_3g_4g_2(g_3g_4)^{-1}.$$

Since the relative position of g_1 and g_4 about γ_3 is fixed on real axis, the γ_3-relation is same to that of γ_2-relation. Computing these relations, we have $g_1g_2g_1 = g_2g_1g_2$. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}^2_L - C_L) \cong \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2\rangle,\ \Delta_C(t,L) = t^2 - t + 1.$$

Case (8): We consider the following homogenized polynomial:

$$F(X,Y,Z) := -X^3Z + X^2Z^2 - 2XY^2Z + Y^4.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has an a_5 singular point at $[0:0:1]$,

(ii) $L_\infty = \{Z=0\}$ is a tangent line with multiplicity 4, and

(iii) the discriminant of f with respect to y is $D_{(f,y)}(x) = -256x^8(x-1)$.

Figure 3.17 shows the real part of the affine curve C^a and the setting of generators.

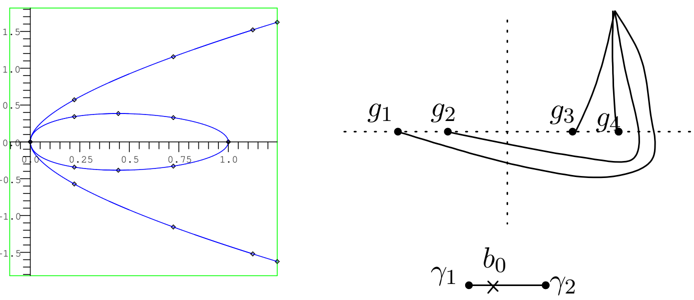


Figure 3.17

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = \epsilon \in \mathbb{C} - \Sigma$, where ϵ is a sufficiently small positive real number. We take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.17. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_1 = g_3,\ g_2 = g_3 g_4 g_3^{-1},\ g_4 = (g_3 g_4 g_2 g_1) g_2 (g_3 g_4 g_2 g_1)^{-1},$$
$$(\gamma_2)\ g_3 = g_3 g_4 g_2 (g_3 g_4)^{-1}.$$

Computing these relations, we have $g_1 g_2 g_1 = g_2 g_1 g_2$. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle,\ \Delta_C(t, L) = t^2 - t + 1.$$

Case (10): We consider the following homogenized polynomial:

$$F(X,Y,Z) := X^3 Z - \left(X^2 - YZ + Z^2\right)^2.$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has one a_4 and one a_2 at $[0:1:0]$, $[0,1,1]$ respectively,

(ii) $L_\infty = \{Z=0\}$ is a line which intersects at the a_4 singular, and

(iii) The discriminant of f with respect to y is $D_{(f,y)}(x) = 4x^3$.

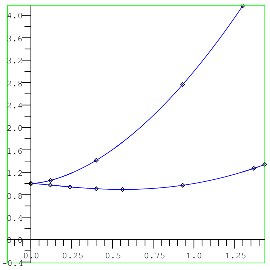


Figure 3.18

Figure 3.18 shows the real part of the affine curve C^a. By the graph and the discriminant with respect to y, we can clearly see that all monodromy relations are only one cuspidal relation. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2 \rangle,$$
$$\Delta_C(t, L) = t^2 - t + 1.$$

Case (11): We consider the following homogenized polynomial:

$$F(X, Y, Z) := X^3Z + (Y - X)^2 (Y + X)^2 .$$

When we put $C = \{F(X, Y, Z) = 0\}$ and its affine part $C^a = \{f(x, y) = F(x, y, 1) = 0\}$, then

(i) C has an e_6 singular point at $[0 : 0 : 1]$,

(ii) $L_\infty = \{Z = 0\}$ is a bi-tangent line, and

(iii) the discriminant of f with respect to y is $D_{(f,y)}(x) = 256x^9 (1 + x)$.

Figure 3.19 shows the real part of the affine curve C^a and the setting of generators.

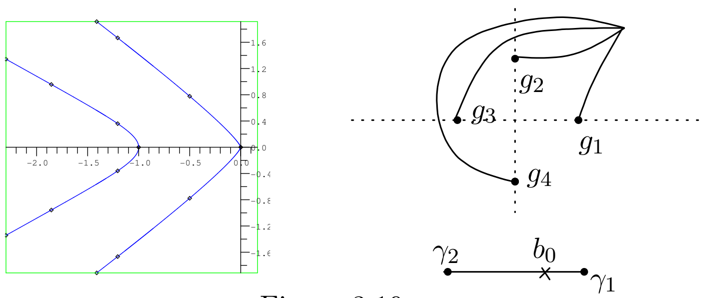


Figure 3.19

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = -1 + \epsilon \in \mathbb{C} - \Sigma$, where ϵ is a small number. Moreover, we take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and

$\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.19. Monodromy relations with each γ_i are

(γ_1) $g_1 = g_4,\ g_2 = (g_4g_3g_2)g_1(g_4g_3g_2)^{-1},\ g_3 = (g_4g_3g_2g_1)g_2(g_4g_3g_2g_1)^{-1}$,

(γ_2) $g_2 = (g_3g_2)^{-1}g_4(g_3g_2)$.

Computing these relations, we have $g_1g_3g_1 = g_3g_1g_3$. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_3 \mid g_1g_3g_1 = g_3g_1g_3\rangle,\ \Delta_C(t, L) = t^2 - t + 1.$$

Case (12): We consider the following homogenized polynomial:

$$F(X, Y, Z) := (X - Z)^3\, Z + Y^4.$$

When we put $C = \{F(X, Y, Z) = 0\}$ and its affine part $C^a = \{f(x, y) = F(x, y, 1) = 0\}$, then

(i) C has an e_6 singular point at $[1 : 0 : 1]$,

(ii) $L_\infty = \{Z = 0\}$ is a tangent line with multiplicity 4, and

(iii) The discriminant of f with respect to y is $256\ (x - 1)^9$.

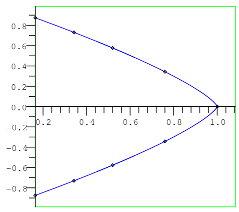


Figure 3.20

Figure 3.20 shows the real part of the affine curve C^a. As we can observe clearly by the graph and the discriminant with respect to y, all monodromy relations are given by the e_6-action of γ_1. Monodromy relations are

$$g_1 = g_4,\ g_2 = (g_4g_3g_2)g_1(g_4g_3g_2)^{-1},\ g_3 = (g_4g_3g_2g_1)g_2(g_4g_3g_2g_1)^{-1}.$$

Computing these relations and replacing $w = g_1g_3g_2g_1$. By Theorem 2.1, we have

$$\langle g_1, g_2, g_3 \mid wg_1 = g_2w,\ wg_2 = g_3w\rangle.$$

Then the Alexander matrix and the Alexander polynomial are given by

$$A = \begin{bmatrix} t - t^3 - 1 & t^3 - t^2 + 1 & t(t-1) \\ t^4 - t^3 + t - 1 & -t^2(t^2 - t + 1) & t^2 - t + 1 \end{bmatrix}$$

$$\Delta_C(t, L) = (t^2 - t + 1)(t^4 - t^2 + 1).$$

Case (13): We consider the following homogenized polynomial:

$$F(X, Y, Z) := Y(81XZ^2 + 54Z^4 + 81X^2Y - 54XYZ - 54YZ^2 + 90XY^2 + 18Y^2Z + 25Y^3).$$

When we put $C = \{F(X, Y, Z) = 0\}$ and its affine part $C^a = \{f(x, y) = F(x, y, 1) = 0\}$, then

(i) C has one a_1, one a_2 and one a_3 sigular points at $[-2/3 : 0 : 1]$, $[-3/4 : 3/4 : 1]$, $[1 : 0 : 0]$ respectively,

(ii) $L_\infty = \{Z = 0\}$ is a tangent line which intersects C at the a_3 singular point, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = 243\,(3x+2)^2\left(5x^2 - 12x - 12\right)(4x+3)^3.$$

Figure 3.21 shows the real part of the affine curve C^a and the setting of generators.

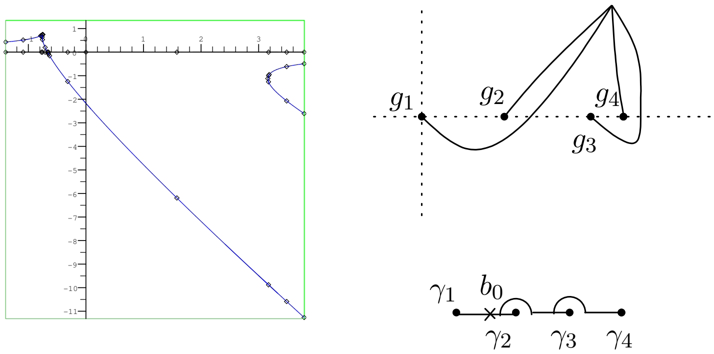


Figure 3.21

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = -3/4 - \epsilon \in \mathbb{C} - \Sigma$, where ϵ is a sufficiently small positive real number. We take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.21. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_4g_3g_4^{-1} = g_2, \quad (\gamma_2)\ g_3g_4g_3 = g_4g_3g_4,$$
$$(\gamma_3)\ g_1g_2 = g_2g_1, \quad (\gamma_4)\ g_1g_4 = g_3g_1.$$

By Theorem 2.1, we have

$$\langle g_1, g_3, g_4 \mid g_3g_4g_3 = g_4g_3g_4,\ g_1g_4 = g_3g_1,\ (g_1g_4)(g_4g_3) = (g_4g_3)(g_1g_4)\rangle.$$

Then the Alexander matrix and the Alexander polynomial are

$$A = \begin{bmatrix} 0 & t^2 - t + 1 & -t^2 + t - 1 \\ 1 - t^2 & t(t^2 - 1) & -t^3 + t^2 + t - 1 \\ 1 - t & -1 & t \end{bmatrix}, \ \Delta_C(t, L) = (t^2 - t + 1)(t - 1).$$

Case (14): We consider the following homogenized polynomial:

$$F(X, Y, Z) := \left(Y^2 - 2XY - XZ + X^2\right)\left(Y^2 + 2XY - XZ + X^2\right).$$

When we put $C = \{F(X, Y, Z) = 0\}$ and its affine part $C^a = \{f(x, y) = F(x, y, 1) = 0\}$, then

(i) C has one a_5 and one $a1$ singular points at $[0 : 0 : 1]$, $[1 : 0 : 1]$,

(ii) $L_\infty = \{Z = 0\}$ is a bi-tangent line, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = 4096x^8 (-1 + x)^2.$$

Figure 3.22 shows the real part of the affine curve C^a and the setting of generators.

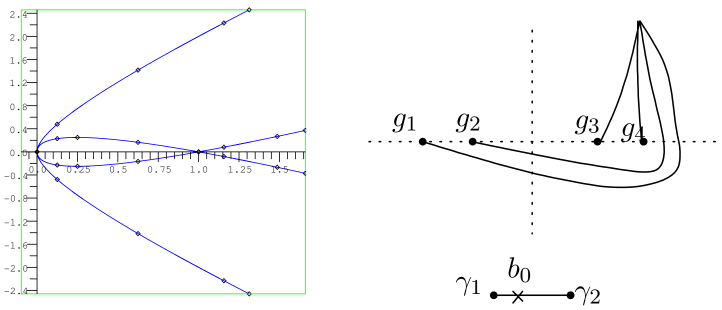


Figure 3.22

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = \epsilon \in \mathbb{C} - \Sigma$, where ϵ is a sufficiently small positive real number. We also take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.22. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_1 = g_3,\ g_2 = g_3 g_4 g_3^{-1},\ g_4 = (g_3 g_4 g_2 g_1) g_2 (g_3 g_4 g_2 g_1)^{-1},$$
$$(\gamma_2)\ g_3 g_3 g_4 g_2 (g_3 g_4)^{-1} = g_3 g_4 g_2 (g_3 g_4)^{-1} g_3.$$

Computing these relations, we have $g_2 g_1 g_2 g_2 g_1 = g_1 g_2 g_2 g_1 g_2$ and $(g_1 g_2)^3 = (g_2 g_1)^3$. By theorem 2.1, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid (g_1 g_2)^3 = (g_2 g_1)^3,\ g_2 g_1 g_2 g_2 g_1 = g_1 g_2 g_2 g_1 g_2 \rangle.$$

Then the Alexander matrix and the Alexander polynomial are given by

$$A = \begin{bmatrix} -t^5+t^4-t^3+t^2-t+1 & -(-t^5+t^4-t^3+t^2-t+1) \\ t^4-t^3+t-1 & -(t^4-t^3+t-1) \end{bmatrix},$$

$$\Delta_C(t,L) = (t^2-t+1)(t-1).$$

Case (15): We consider the following homogenized polynomial:

$$F(X,Y,Z) := 256Y^4 - 256Y^3Z + 96Y^2Z^2 - 16YZ^3 + 1 + 32XY^3 \\ - 32XY^2Z + 10XYZ^2 - XZ^3 + X^2Y^2.$$

When we put $C = \{F(X,Y,Z)=0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has one a_2 and one a_4 singular points at $[1:0:0], [0:1/4:1]$,

(ii) $L_\infty = \{Z=0\}$ is a tangent line which passes through a cusp, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = 65536x^7\,(x-64)\,.$$

Figure 3.23 shows the real part of the affine curve C^a and the setting of generators.

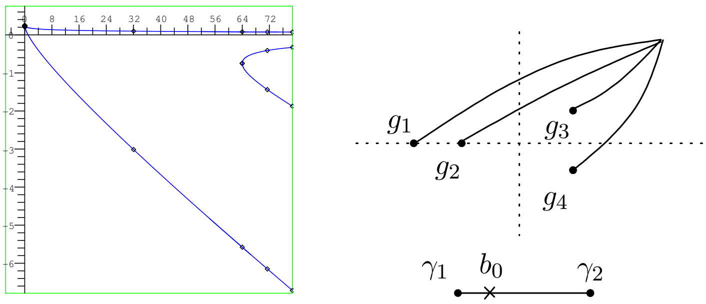


Figure 3.23

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = \epsilon \in \mathbb{C} - \Sigma$. We take generators of $\pi_1(\mathbb{C}-\Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.23. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_1 = g_4,\ g_3 = g_1g_2g_1^{-1},\ g_2 = (g_1g_2g_3)g_4(g_1g_2g_3)^{-1},$$
$$(\gamma_2)\ (g_3^{-1}g_2g_3)g_4(g_3^{-1}g_4g_3)^{-1} = g_3.$$

Computing these relations, we have two relations $g_2g_1g_2g_1g_2 = g_1g_2g_1g_2g_1$ and $g_1g_2g_2 = g_2g_2g_1$. By Theorem 2.1, we have

$$\langle g_1, g_2 \mid g_2g_1g_2g_1g_2 = g_1g_2g_1g_2g_1,\ g_1g_2g_2 = g_2g_2g_1\rangle.$$

The Alexander matrix and its Alexander polynomial are given by

$$A = \begin{bmatrix} -t^4+t^3-t^2+t-1 & t^4-t^3+t^2-t+1 \\ 1-t^2 & t^2-1 \end{bmatrix}, \ \Delta_C(t,L) = 1.$$

Case (16): We consider the following homogenized polynomial:

$$F(X,Y,Z) := (Y-X)\left(-XY^2+Z^3+Y^3\right).$$

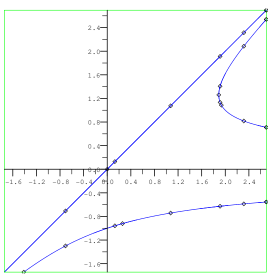
When we put $C = \{F(X,Y,Z)=0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1)=0\}$, then

(i) C has one a_5 and one a_2 singular points at $[1:1:0]$, $[1:0:0]$ respectively,

(ii) $L_\infty = \{Z=0\}$ is a line which passes through each a_2 and a_5, and

(iii) the discriminant of f with respect to y is $D_{(f,y)}(x) = 4x^3-27$.

Figure 3.24 shows the real part of the affine curve C^a and the setting of generators.



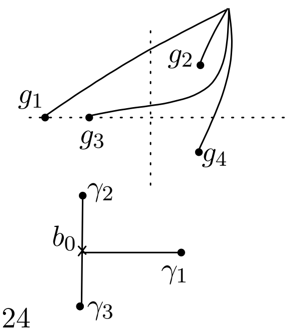


Figure 3.24

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = \epsilon \in \mathbb{C}-\Sigma$. Moreover, we take generators of $\pi_1(\mathbb{C}-\Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.24. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_2 = g_3g_4g_3^{-1},\ (\gamma_2)\ (g_2g_3g_2^{-1})^{-1}g_1(g_2g_3g_2^{-1}) = g_2,\ (\gamma_3)\ g_2^{-1}g_1g_2 = g_4.$$

Computing these relations, we have $g_3g_3g_2 = g_2g_3g_3$. By Theorem 2.1, we have

$$\langle g_2, g_3 \mid g_3g_3g_2 = g_2g_3g_3\rangle.$$

Then the Alexander matrix and its Alexander polynomial are

$$A = \begin{bmatrix} t^2-1 & 1-t^2 \end{bmatrix},\ \Delta_C(t,L) = t^2-1.$$

Case (17): We consider the following homogenized polynomial:

$$F(X,Y,Z) := \left(Y^2+XY+2Z^2\right)\left(Y^2+XY+Z^2\right)$$

When we put $C = \{F(X,Y,Z)=0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1)=0\}$, then

(i) C has two a_3 singular points at $[1:0:0]$, $[-1:1:0]$ respectively,

(ii) $L_\infty = \{Z=0\}$ is a line which intersects C at two a_3 singularities, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = (x-2)(x+2)\left(x^2-8\right).$$

Figures 3.25 shows the real part of the affine curve C^a and the setting of generators.

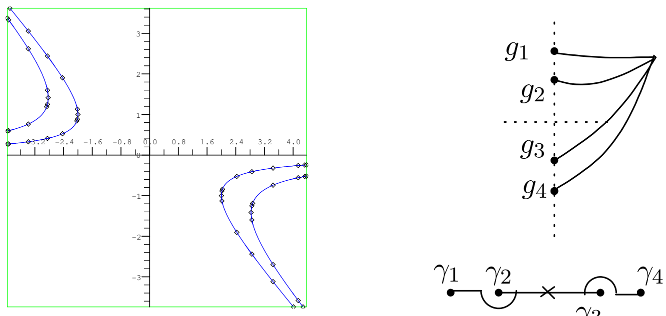


Figure 3.25

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x)=0\}$ and take a base point $b_0 = 0 \in \mathbb{C}-\Sigma$. We take generators of $\pi_1(\mathbb{C}-\Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.25. Monodromy relations with each γ_i are

$$(\gamma_1 \text{ and } \gamma_3)\ g_2 = g_3,\ \ (\gamma_2 \text{ and } \gamma_4)\ g_3^{-1}g_4g_3 = g_1.$$

Computing these relations, we have $g_1g_2 = g_2g_4$. By Theorem 2.1, we have

$$\langle g_1, g_2, g_4 \mid g_1g_2 = g_2g_4\rangle.$$

The Alexander matrix and its Alexander polynomial are:

$$A = \begin{bmatrix} 1 & t-1 & -t \end{bmatrix},\ \Delta_C(t,L) = 1.$$

Case (18): We consider the following homogenized polynomial:

$$F(X,Y,Z) := \left(-XZ-2Z^2+Y^2\right)\left(-XZ-Z^2+Y^2\right).$$

When we put $C = \{F(X,Y,Z)=0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1)=0\}$, then

(i) C has an a_7 singular point at $[1:0:0]$,

(ii) $L_\infty = \{Z=0\}$ is a line which intersects C at the a_7 singular point, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = 16(x+2)(x+1).$$

Figure 3.26 shows the real part of the affine curve C^a and the setting of generators.

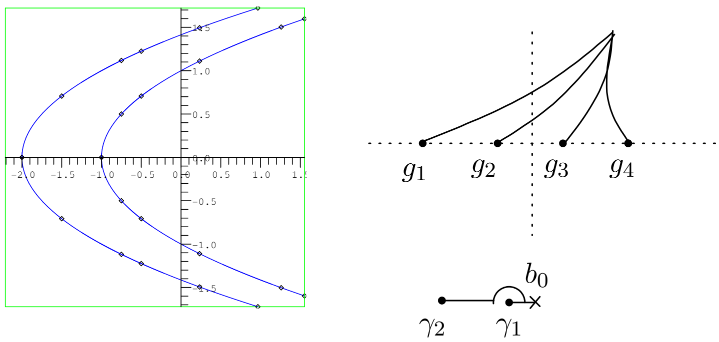


Figure 3.26

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = -1+\epsilon \in \mathbb{C} - \Sigma$, where ϵ is a small number. We take generators of $\pi_1(\mathbb{C}-\Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.26. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_2 = g_3,\ \ (\gamma_2)\ g_3^{-1} g_4 g_3 = g_1.$$

Computing these relations, we have $g_1 g_2 = g_2 g_4$. By Theorem 2.1, we have

$$\langle g_1, g_2, g_4 \mid g_1 g_2 = g_2 g_4 \rangle.$$

The Alexander matrix and its Alexander polynomial are:

$$A = \begin{bmatrix} 1 & t-1 & -t \end{bmatrix},\ \ \Delta_C(t, L) = 1.$$

Case (19): We consider the following homogenized polynomial:

$$F(X,Y,Z) := (-2Y+Z+2X)(2Y+Z+2X)\left(-Y^2+X^2+XZ\right).$$

When we put $C = \{F(X,Y,Z) = 0\}$ and its affine part $C^a = \{f(x,y) = F(x,y,1) = 0\}$, then

(i) C has two a_3 and one a_1 singular points at $[1:1:0]$, $[-1:1:0]$, $[-\frac{1}{2}:0:1]$ respectively,

(ii) $L_\infty = \{Z = 0\}$ is a line which intersects C at two a_3, and

(iii) the discriminant of f with respect to y is

$$D_{(f,y)}(x) = 64x\,(x+1)\,(1+2x)^2\,.$$

Figure 3.27 shows the real part of C^a and the setting of generators.

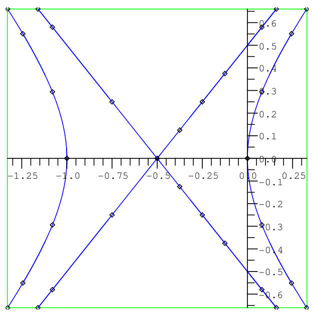
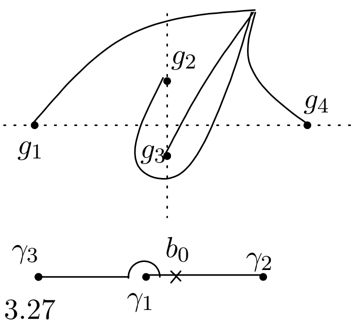



Figure 3.27

Put $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$ and take a base point $b_0 = -1/2 + \epsilon \in \mathbb{C} - \Sigma$, where ϵ is a small number. We take generators of $\pi_1(\mathbb{C} - \Sigma, b_0)$ and $\pi_1(p^{-1}(b_0), b_0')$ as Figure 3.27. Monodromy relations with each γ_i are

$$(\gamma_1)\ g_2g_3 = g_3g_2,\ \ (\gamma_2)\ (g_3g_2g_3^{-1})g_1(g_3g_2g_3^{-1}) = g_4,\ \ (\gamma_3)\ g_1g_2 = g_2g_4.$$

Computing these relations, we have $g_2g_3 = g_3g_2$ and $g_2g_1g_3 = g_3g_1g_2$. By Theorem 2.1, we have

$$\langle g_1, g_2, g_3 \mid g_2g_3 = g_3g_2,\ g_2g_1g_3 = g_3g_1g_2\rangle.$$

Then the Alexander matrix and its Alexander polynomial are

$$A = \begin{bmatrix} 1-t & t-1 & 0 \\ 0 & 1-t^2 & t^2-1 \end{bmatrix},\ \Delta_C(t, L) = (t-1)^2(t+1).$$

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