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Gaussian estimates of order α and L^p-spectral independence of generators of C_0-semigroups II

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Abstract. Without any assumptions on the space dimension or boundedness of the region, we prove L^p-spectral independence of generators of C_0-semigroups estimated by the positive C_0-semigroup $e^{-t(-\triangle)^\alpha}$ $(0 < \alpha \le 1)$. In particular, if the semigroup is self-adjoint in L^2, it is shown that only the estimate by $e^{-t(-\Delta)^\alpha}$ is sufficient for L^p-spectral independence. The proof depends on the idea of considering the spectra of the operators $e^{-t(-A)^\beta}$ $(0 < \beta < 1)$ and applying the spectral independence result of B.A. Barnes for integral operators, where A is the generator of the semigroup in question.

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§1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set, and suppose that a C_0-semigroup $T_p = \big(T_p(t)\big)_{t \ge 0}$ on $L^p(\Omega)$ with generator A_p is given for each $1 \le p < \infty$. Assume further that T_p's are consistent in the sense that

$$T_p(t) = T_q(t) \qquad \text{on } L^p(\Omega) \cap L^q(\Omega)$$

for all $t \ge 0$. Under these assumptions, it is natural to expect L^p-spectral independence of generators, that is to say,

$$\sigma(A_p) = \sigma(A_2) \tag{1.1}$$

for all $1 \le p < \infty$. However, W. Arendt [1, Section 3] revealed that this equality is not necessarily true. Nonetheless, there are important cases where

L^p-spectral independence (1.1) does hold. In fact, R. Hempel and J. Voigt [5, Theorem] proved that, for a potential V belonging to a large class including a Kato class, the spectrum of Schrödinger operator $-\Delta/2+V$ acting in $L^p(\mathbb{R}^N)$ is independent of $p \in [1,\infty)$. They used the Feynman–Kac formula to obtain their result and so their method of proof is peculiar to the perturbation $-\Delta/2+V$. However, Arendt [1] found that if a C_0-semigroup $T=(T(t))_{t\geq 0}$ on $L^2(\Omega)$ is dominated by the heat semigroup $e^{t\Delta}$ (for details, see (1.2) below), then T naturally induces a C_0-semigroup T_p on $L^p(\Omega)$ for each $p \in [1,\infty)$ and the spectrum of the generator A_p of T_p is independent of p provided $T(t)$ is self-adjoint. Roughly speaking, his proof relies on an subtle argument to obtain an estimate of the integral kernel of the resolvent of T. He also shows the p-independence of the connected component of the resolvent set of A_p containing a right half-plane for non-self-adjoint semigroups. We should note here that Arendt's result contains L^p-spectral independence for the case of $-\Delta/2+V$ with a positive potential V. After the work of Arendt, P.C. Kunstmann [6] proved that a weaker estimate of the integral kernel of the resolvents implies L^p-spectral independence of the generators, and he generalized and completed, in a sense, the work of Arendt.

Arendt's results were generalized in a different direction in [8] and [9]. To state in more details, let $T=(T(t))_{t\geq 0}$ be a C_0-semigroup on $L^2(\Omega)$ with generator A and $\alpha \in (0,1]$. We say that T satisfies a Gaussian estimate of order α if there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ and $b>0$ such that

$$|T(t)f| \leq Me^{\omega t}e^{-bt(-A)^\alpha}|f| \tag{1.2}$$

for all $t \geq 0$ and $f \in L^2(\Omega)$. Here, Δ denotes the usual Laplacian in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$, and we identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^N)$ by considering the elements of $L^2(\Omega)$ to have value 0 on $\mathbb{R}^N \setminus \Omega$. In the case of $\alpha=1$, (1.2) is equivalent to an upper Gaussian estimate defined by Arendt [1, Definition 4.1]. If T satisfies the stronger estimate obtained by replacing $e^{-bt(-\Delta)^\alpha}$ in (1.2) with $e^{-bt(I-\Delta)^\alpha}$, then the resolvent of A satisfies an estimate assumed in [6, Theorem 1.1] and accordingly the spectrum of A_p is independent of $p \in [1,\infty)$, where A_p is the generator of a version of T on $L^p(\Omega)$ ([10, Theorem 3.17]). In the case of $\alpha=1$, this result coincides with that of Arendt. On the other hand, as long as we assume only the estimate (1.2), we could not prove L^p-spectral independence except for the case of bounded Ω or of space dimension 1 ([9]).

It is the purpose of this paper to prove L^p-spectral independence without limitations mentioned above. A crucial tool for this purpose is the result of B.A. Barnes [3] which gives a sufficient condition for L^p-spectral independence of integral operators by using the theory of Banach algebras. More precisely, he gave an estimate for a measurable function $K\colon \Omega\times\Omega \to \mathbb{C}$ that guarantees that K defines a bounded linear operator K_p on $L^p(\Omega)$ for each $p \in [1,\infty)$ and the spectrum of K_p is independent of $p \in [1, \infty)$ ([3, Theorem 3.8]). Suppose that a C_0-semigroup $T = (T(t))_{t \geq 0}$ on $L^2(\Omega)$ with generator A satisfies the estimate (1.2). Then it can be verified that the integral kernel of $T(t)$ $(t > 0)$ satisfies the condition of Barnes, while the resolvent of A does not in general. Therefore, by Barnes' theorem, we can prove that if a C_0-semigroup $T = (T(t))_{t \geq 0} = (e^{tA})_{t \geq 0}$ on $L^2(\Omega)$ satisfies a Gaussian estimate of order α for an $\alpha \in (0, 1]$ and a resolvent of the generator of T is normal, then the spectrum of $T_p(t)$ is independent of $p \in [1, \infty)$, where T_p is a version of T on $L^p(\Omega)$. However, in general, L^p-spectral independence of semigroups does not imply that of their generators. But we can fill this gap by considering simultaneously the spectrum of the semigroups generated by fractional powers $(-A)^\beta$ $(\beta \in (0, 1))$ of the generator A in question (Lemma 2.9). Noting that $e^{-t(-A)^\beta}$ satisfies an Gaussian estimate of order $\alpha\beta$ and combining the observations above, we obtain the desired L^p-spectral independence of the generators of T_p (Theorem 2.11).

§2. Gaussian estimates of order α and L^p-spectral independence

Hereafter Ω denotes an open subset of $\mathbb{R}^N$. In this section, we treat C_0-semigroups that satisfy the following estimates.

Definition 2.1. Let $T = \big(T(t)\big)_{t \geq 0}$ be a C_0-semigroup on $L^2(\Omega)$ and $\alpha \in (0, 1]$. Then we say that T satisfies a Gaussian estimate of order α if there exist $M \geq 1, \omega \in \mathbb{R}$ and $b > 0$ such that

$$|T(t)f| \leq M e^{\omega t} e^{-bt(-\Delta)^\alpha} |f| \tag{2.1}$$

holds for all $t \geq 0$ and $f \in L^2(\Omega)$. Here, Δ denotes the usual Laplacian in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$, and we identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^N)$ by considering the elements of $L^2(\Omega)$ to have value 0 on $\mathbb{R}^N \setminus \Omega$.

We collect some basic facts concerning the C_0-semigroups which satisfy Gaussian estimates of order α.

Proposition 2.2. *Let $T = \big(T(t)\big)_{t \geq 0}$ be a C_0-semigroup on $L^2(\Omega)$. Assume that T satisfies a Gaussian estimate of order α for an $\alpha \in (0, 1]$. Then the following assertions hold.*

(i) *For all $t > 0$, there exists a measurable function $K_t \colon \Omega \times \Omega \to \mathbb{C}$ such that for all $f \in L^2(\Omega)$,*

$$\big(T(t)f\big)(x) = \int_\Omega K_t(x, y) f(y)\, dy$$

for a.e. $x \in \Omega$.

(ii) *There exists a constant* $C > 0$ *such that the function* K_t *in* (i) *satisfies the estimate*

$$|K_t(x,y)| \le Ce^{\omega t}\frac{bt}{((bt)^{\frac{1}{\alpha}} + |x-y|^2)^{\frac{N}{2}+\alpha}}$$

for all $t > 0$ *and a.e.* $(x,y) \in \Omega \times \Omega$.

(iii) *For each* $p \in [1,\infty)$, *there exists a unique* C_0*-semigroup* $T_p = \big(T_p(t)\big)_{t\ge 0}$ *on* $L^p(\Omega)$ *such that for all* $t > 0$ *and* $f \in L^p(\Omega)$,

$$\big(T_p(t)f\big)(x) = \int_\Omega K_t(x,y)f(y)\,dy$$

for a.e. $x \in \Omega$. *(Note that* K_t *is independent of* $p \in [1,\infty)$.*)*

Proof. (i) and (iii) are proved in Proposition 3.5 in [9].

(ii) follows from the estimates (3.5) and (3.4) in [9]. □

In this paper, we use an abstract result by Barnes in [3]. To state his result, we define some function spaces and weight functions.

Definition 2.3 (*cf.* [3, pp. 122, 123])**.** (i) A_1 is defined as the space consisting of all measurable functions $K\colon \Omega \times \Omega \to \mathbb{C}$ such that

$$\|K\|_1 := \max\Big\{\operatorname*{ess.sup}_{x\in\Omega}\int_\Omega |K(x,y)|\,dy, \operatorname*{ess.sup}_{y\in\Omega}\int_\Omega |K(x,y)|\,dx\Big\} < \infty.$$

Similarly, A_2 is defined as the linear space of all measurable functions $K\colon \Omega \times \Omega \to \mathbb{C}$ such that

$$\|K\|_2 := \max\Big\{\operatorname*{ess.sup}_{x\in\Omega}\Big(\int_\Omega |K(x,y)|^2\,dy\Big)^{\frac{1}{2}}, \operatorname*{ess.sup}_{y\in\Omega}\Big(\int_\Omega |K(x,y)|^2\,dx\Big)^{\frac{1}{2}}\Big\} < \infty.$$

The space $(A_1, \|\cdot\|_1)$ and $(A_2, \|\cdot\|_2)$ are Banach spaces. Moreover, A_1 is a Banach $*$-algebra with the following involution $K \mapsto K^*$ and multiplication:

$$K^*(x,y) := \overline{K(y,x)} \quad \big((x,y) \in \Omega \times \Omega\big),$$
$$(K * J)(x,y) := \int_\Omega K(x,z)J(z,y)\,dz \quad (K, J \in A_1).$$

(ii) The weight function w_δ is defined by

$$w_\delta(x,y) := (1 + |x-y|)^\delta \quad \big((x,y) \in \mathbb{R}^N \times \mathbb{R}^N\big)$$

for each $\delta \in (0,1]$. Let A_{w_δ} be the linear space of all measurable functions $K\colon \Omega\times\Omega \to \mathbb{C}$ such that $Kw_\delta \in A_1$ and $\|\cdot\|_{w_\delta}$ be defined by $\|K\|_{w_\delta} := \|Kw_\delta\|_1$ for each $\delta \in (0,1]$, where Kw_δ denotes the pointwise product of K and w_δ.

Then, A_{w_δ} is a $*$-subalgebra of A_1 and $(A_{w_\delta}, \|\cdot\|_{w_\delta})$ is a Banach $*$-algebra (*cf.* [3, Note 4.3]).

(iii) Let $\Gamma[m]$ be the set

$$\Gamma[m] := \big\{(x,y) \in \Omega \ \times \Omega \ \big| \ |x-y| \le m\big\}$$

for each $m \in \mathbb{N}$ and $\chi(\Gamma)$ be the characteristic function of $\Gamma \subset \mathbb{R}^N$. A_1^0 is defined as the linear subspace of all $K \in A_1$ such that

$$\lim_{m\to\infty} \|\chi(\Gamma[m]^c)K\|_1 = 0.$$

A_1^0 is a closed $*$-subalgebra of A_1. In addition, A_2^0 and $A_{w_\delta}^0$ are defined as subspaces of A_2 and A_{w_δ} by replacing $\|\cdot\|_1$ with $\|\cdot\|_2$ and $\|\cdot\|_{w_\delta}$, respectively in the definition of A_1^0.

(iv) Let $A_{w_\delta,2} := A_{w_\delta} \cap A_2$, $A_{w_\delta,2}^{0,0} := A_{w_\delta}^0 \cap A_2^0$ for each $\delta \in (0,1]$ and $\|K\|_{w_\delta,2} := \max\big\{\|K\|_{w_\delta}, \|K\|_2\big\}$. Then, $(A_{w_\delta,2}, \|\cdot\|_{w_\delta,2})$ is a Banach $*$-algebra (*cf.* [3, Lemma 4.4]) and $A_{w_\delta,2}^{0,0}$ is a closed $*$-subalgebra of A_{w_δ}.

Remark 2.4. As is stated in [3, p. 122], any $K \in A_1$ defines the bounded linear operator K_p on $L^p(\Omega)$ by

$$\big(K_p f\big)(x) := \int_\Omega K(x,y) f(y)\, dy \quad \big(f \in L^p(\Omega), x \in \Omega\big)$$

for each $p \in [1,\infty]$.

Now, we introduce a result by Barnes in [3]. For the reason described in Remark 2.6 below, we state it in a form where its "assumption part" is a little strengthened.

Theorem 2.5 (Barnes, *cf.* [3, Theorem 4.8])**.** *Assume that K is in $A_{w_\delta,2}^{0,0}$ for some $\delta \in (0,1]$. Then the following assertions hold:*

(i) *$\sigma_{w_\delta,2}(K) = \sigma(K_p)$ for all $p \in [1,\infty]$ when K is normal (i.e., $K^* * K = K * K^*$).*

(ii) *$\sigma_{w_\delta,2}(K) = \sigma(K_p) \cup \overline{\sigma\big((K^*)_p\big)}$ for all $p \in [1,\infty]$ in general.*

In these assertions, $\sigma_{w_\delta,2}(K)$ denotes the spectrum of K in $A_{w_\delta,2}$ and K_p is as in Remark 2.4.

Remark 2.6. Let $A_{w_\delta,2}^0 := A_{w_\delta}^0 \cap A_2$. Theorem 4.8 in [3] states that the same conclusions (i), (ii) in Theorem 2.5 hold for all $K \in A_{w_\delta,2}^0$. Moreover, in the proof of Theorem 4.8 in [3], it is claimed that if $K = K^* \in A_{w_\delta,2}^0$, then we have

$$\|\chi(\Gamma[m])K - K\|_{w_\delta,2} \to 0$$

as $m \to \infty$, in other words, $K \in A^{0,0}_{w_\delta,2}$. However, let K be defined by

$$K(x,y) := \begin{cases} \sqrt{y} & (y \geq 2, 2y \leq x \leq 2y + 1/y) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, $K + K^*$ is hermitian and belongs to $A^0_{w_\delta,2}$ for each $\delta \in (0, 1/2)$ but does not belong to $A^{0,0}_{w_\delta,2}$ for any $\delta \in (0, 1/2)$. We will give a detailed proof of this fact in Section 3. For this reason, we replaced $A^0_{w_\delta,2}$ in Theorem 4.8 in [3] with $A^{0,0}_{w_\delta,2}$. Once this replacement is made, Theorem 2.5 can be proved in exactly the same way as in [3] except for the part concerning the assertion $K \in A^{0,0}_{w_\delta,2}$.

Remark 2.7. It is easy to see that for all $K \in A_1$,

$$\big((K^*)_p\big)' f = \overline{K_{p'} \overline{f}} \quad \big(f \in L^p(\Omega)\big)$$

for each $p \in [1, \infty)$, where $\big((K^*)_p\big)'$ is the conjugate operator of $(K^*)_p$ and p' is the conjugate exponent of p. Hence, it follows from assertion (ii) that

$$\sigma_{w_\delta,2}(K) = \sigma(K_p) \cup \sigma(K_{p'})$$

holds for each $p \in [1, \infty)$.

Here we would like to note the following relation between a C_0-semigroup T on $L^2(\Omega)$ satisfying a Gaussian estimate of order α and the Banach $*$-algebra $A^{0,0}_{w_\delta,2}$ in Barnes' theorem.

Lemma 2.8. *Let $T = \big(T(t)\big)_{t \geq 0}$ be a C_0-semigroup on $L^2(\Omega)$ and $\alpha \in (0, 1]$ and suppose that T satisfies a Gaussian estimate of order α. Moreover, let $K_t \colon \Omega \times \Omega \to \mathbb{C}$ be the integral kernel of $T(t)$ for each $t > 0$ as in Proposition 2.2. Then, $K_t \in A^{0,0}_{w_\delta,2}$ holds for each $\delta \in (0, 2\alpha)$.*

Proof. This assertion readily follows from Proposition 2.2 (ii). □

Now, let $T = \big(T(t)\big)_{t \geq 0}$ be a C_0-semigroup on $L^2(\Omega)$ satisfying a Gaussian estimate of order α for an $\alpha \in (0, 1]$. Lemma 2.8 and Barnes' theorem imply that if T is normal, the spectrum of $T_p(t)$ is independent of $p \in [1, \infty)$. However, it is not evident that the spectrum of the generator A_p of T_p is independent of $p \in [1, \infty)$. The next lemma connects L^p-spectral independence of T_p's to that of A_p's, which is the key in this paper. The lemma depends heavily on the theory of fractional powers of a generator of a C_0-semigroup and the spectral mapping theorem.

Lemma 2.9. *Let $T_p = \big(T_p(t)\big)_{t \geq 0}$ be a bounded C_0-semigroup on $L^p(\Omega)$ with generator A_p for each $p \in [1, \infty)$. Then the following assertions hold.*

(i) *Assume that there exists a* $t_0 > 0$ *such that for all* $\beta \in (0,1)$ *the spectrum of* $e^{-t_0(-A_p)^\beta}$ *is independent of* $p \in [1,\infty)$. *Then the spectrum of* A_p *is independent of* $p \in [1,\infty)$.

(ii) *Assume that there exists a* $t_0 > 0$ *such that for all* $\beta \in (0,1)$, *the union* $\sigma\big(e^{-t_0(-A_p)^\beta}\big) \cup \sigma\big(e^{-t_0(-A_{p'})^\beta}\big)$ *is independent of* $p \in (1,\infty)$. *Then* $\sigma(A_p) \cup \sigma(A_{p'})$ *is independent of* $p \in (1,\infty)$.

Proof. (i) As is well-known, for each $p \in [1,\infty)$ and $\beta \in (0,1)$, the fractional power $-(-A_p)^\beta$ generates a bounded analytic semigroup with angle $\pi(1-\beta)/2$. Hence, $\sigma\big((-A_p)^\beta\big)$ is included in the sector $\big\{\lambda \in \mathbb{C} \bigm| |\arg\lambda| < \pi\beta/2\big\}$. Keeping this in mind, let p and q be in $[1,\infty)$ and $\lambda \in \sigma(A_p)$. We use the spectral mapping theorem

$$\sigma\big((-A_p)^\beta\big) = \big[\sigma(-A_p)\big]^\beta \big(= \big\{(-\lambda)^\beta \bigm| \lambda \in \sigma(A_p)\big\}\big)$$

by Theorem 3.1 in [2] or Theorem 5.3.1 in [7], where β is an arbitrary number in $(0,1)$ and $(-\lambda)^\beta$ denotes the principal value of $e^{\beta\log(-\lambda)}$ for $\lambda \neq 0$ and denotes 0 for $\lambda = 0$. This equality means that in the case of $0 \in \sigma(A_p)$, we have $0 \in \sigma\big((-A_p)^\beta\big)$. The spectral mapping theorem implies that

$$e^{-t_0(-\lambda)^\beta} \in e^{-t_0\sigma((-A_p)^\beta)}$$

for all $\beta \in (0,1)$. In addition, since $e^{-t_0(-A_p)^\beta}$ is a bounded analytic semigroup as stated above, the spectral mapping theorem

$$e^{-t_0\sigma((-A_p)^\beta)} = \sigma\big(e^{-t_0(-A_p)^\beta}\big) \setminus \{0\} \tag{2.2}$$

holds for all $\beta \in (0,1)$ (*cf.* Corollary 3.12 in [4]). Thus, we have

$$e^{-t_0(-\lambda)^\beta} \in \sigma\big(e^{-t_0(-A_p)^\beta}\big) \setminus \{0\}$$

for all $\beta \in (0,1)$. Since $\sigma\big(e^{-t_0(-A_p)^\beta}\big) \setminus \{0\} = \sigma\big(e^{-t_0(-A_q)^\beta}\big) \setminus \{0\}$ by the assumption and (2.2) holds also in the case where p is replaced with q,

$$e^{-t_0(-\lambda)^\beta} \in e^{-t_0\sigma((-A_q)^\beta)}$$

for all $\beta \in (0,1)$.

Hence for all $\beta \in (0,1)$ there exists an $n_\beta \in \mathbb{Z}$ such that

$$(-\lambda)^\beta + \frac{2n_\beta\pi i}{t_0} \in \sigma\big((-A_q)^\beta\big).$$

In the case of $\lambda = 0$, $n_\beta \neq 0$ implies $(-\lambda)^\beta + 2n_\beta\pi i/t_0 \in i\mathbb{R}\setminus\{0\}$, hence $(-\lambda)^\beta + 2n_\beta\pi i/t_0 \notin \sigma\big((-A_q)^\beta\big)$. Therefore $n_\beta = 0$ and hence $(-\lambda)^\beta \in \sigma\big((-A_q)^\beta\big)$

holds in this case. So let $\lambda \neq 0$ in what follows. Suppose that $\beta \in (0,1)$ is sufficiently small so that

$$\mathrm{Re}\,(-\lambda)^{\beta} \tan\Big(\frac{\pi}{2}\beta\Big) < \frac{\pi}{t_0}.$$

If $n_\beta \neq 0$, then

$$\frac{\pi}{2} > \Big|\arg\Big((-\lambda)^{\beta} + \frac{2n_\beta \pi i}{t_0}\Big)\Big| > \frac{\pi}{2}\beta,$$

hence, $(-\lambda)^{\beta} + 2n_\beta \pi i/t_0 \notin \sigma\big((-A_q)^{\beta}\big)$. Therefore $n_\beta = 0$ and $(-\lambda)^{\beta} \in \sigma\big((-A_q)^{\beta}\big)$, hence $\lambda \in \sigma(A_q)$ by Theorem 3.1 in [2].

(ii) This assertion is proved in a similar way as in the proof of (i). □

We need the next proposition to use Lemma 2.9.

Proposition 2.10. *Let $T = \big(T(t)\big)_{t\geq 0}$ be a C_0-semigroup on $L^2(\Omega)$ with generator A and suppose that T satisfies a Gaussian estimate of order α for an $\alpha \in (0,1]$ with $\omega = 0$ in* (2.1). *Then the following assertions hold.*

(i) *For all $\beta \in (0,1)$, the C_0-semigroup $e^{-t(-A)^{\beta}}$ satisfies a Gaussian estimate of order $\alpha\beta$. In addition, for all $\beta \in (0,1)$ and $t > 0$, $e^{-t(-A)^{\beta}}$ is an integral operator and its kernel $K_{t,\beta}(x,y)$ satisfies the following estimate: There exists a constant $C_\beta > 0$ such that for all $t > 0$*

$$|K_{t,\beta}(x,y)| \leq C_\beta \frac{b^{\beta} t}{(b^{\frac{1}{\alpha}} t^{\frac{1}{\alpha\beta}} + |x-y|^2)^{\frac{N}{2}+\alpha\beta}} \tag{2.3}$$

for a.e. $(x,y) \in \Omega \times \Omega$, where b is as in (2.1).

(ii) *For all $\beta \in (0,1)$ and $p \in [1,\infty)$, there exists a C_0-semigroup $T_{\beta,p} = \big(T_{\beta,p}(t)\big)_{t\geq 0}$ on $L^p(\Omega)$ such that $T_{\beta,p}$ is consistent with $e^{-t(-A)^{\beta}}$ (i.e., $T_{\beta,p}(t) = e^{-t(-A)^{\beta}}$ on $L^p(\Omega) \cap L^2(\Omega)$ for all $t \geq 0$). Moreover, $T_{\beta,p}(t)$ coincides with $e^{-t(-A_p)^{\beta}}$ for all $\beta \in (0,1), t \geq 0$ and $p \in [1,\infty)$, where A_p is the generator of T_p in Proposition* 2.2.

Proof. By the formula (2) in [11, Chapter IX, Section 11], for all $\beta \in (0,1), t > 0$ and $f \in L^2(\Omega)$,

$$\begin{aligned}
\big|e^{-t(-A)^{\beta}} f\big| &= \Big|\int_0^\infty f_{t,\beta}(s) e^{sA} f\, ds\Big| \\
&\leq M \int_0^\infty f_{t,\beta}(s) e^{-bs(-\Delta)^{\alpha}} |f|\, ds \\
&= M e^{-b^{\beta} t((-\Delta)^{\alpha})^{\beta}} |f| \\
&= M e^{-b^{\beta} t(-\Delta)^{\alpha\beta}} |f|.
\end{aligned}$$

(The function $f_{t,\beta} \geq 0$ is defined in [11, Chapter IX, Section 11 (1)].) Thus, $e^{-t(-A)^{\beta}}$ satisfies a Gaussian estimate of order $\alpha\beta$ with $\omega = 0$.

The latter assertion of (i) readily follows from Proposition 2.2.

Now we prove (ii). By assertion (i) and Proposition 2.2, there exists a C_0-semigroup $T_{\beta,p} = \big(T_{\beta,p}(t)\big)_{t\geq 0}$ such that $T_{\beta,p}$ is consistent with $e^{-t(-A)^{\beta}}$. On the other hand, since e^{tA_p} is consistent with e^{tA}, the formula in [11, Chapter IX, Section 11] implies that $e^{-t(-A_p)^{\beta}}$ is consistent with $e^{-t(-A)^{\beta}}$. Since the C_0-semigroup on $L^p(\Omega)$ that is consistent with $e^{-t(-A)^{\beta}}$ is unique, we have $e^{-t(-A_p)^{\beta}} = T_{\beta,p}(t)$. Thus the proof is completed. $\square$

Now we are in a position to prove our main result. The authors would like to emphasize that the following Theorem 2.11 considerably improves our former results (Theorem 3.18, 3.19 and 3.20 in [9]).

Theorem 2.11. *Let $T = \big(T(t)\big)_{t\geq 0}$ be a C_0-semigroup on $L^2(\Omega)$ with generator A and suppose that T satisfies a Gaussian estimate of order α for some $\alpha \in (0,1]$. Moreover, let $T_p = \big(T_p(t)\big)_{t\geq 0}$ be the C_0-semigroup naturally defined by T on $L^p(\Omega)$ for each $p \in [1,\infty)$ as in Proposition 2.2. Then, for the generator A_p of T_p, the following assertions hold.*

(i) *Let ω be as in (2.1). Assume that there exists a $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ such that $(\lambda - A)^{-1}$ is normal. Then $\sigma(A_p)$ is independent of $p \in [1,\infty)$.*

(ii) *$\sigma(A_p) \cup \sigma(A_{p'})$ is independent of $p \in (1,\infty)$ in general.*

Proof. (i) We may assume $\omega = 0$ in (2.1) (if necessary, consider $A - \omega$). We first show that $e^{-t(-A)^{\beta}}$ is normal for each $\beta \in (0,1)$. In fact, by the assumption, there exists a $\lambda \in \rho(A) = \overline{\rho(A^*)}$ with $\operatorname{Re}\lambda > 0$, where A^* is the adjoint operator of A, such that $(\lambda - A)^{-1}$ and $\big((\lambda - A)^{-1}\big)^*\big(= (\overline{\lambda} - A^*)^{-1}\big)$ are commutative. (Note that λ with $\operatorname{Re}\lambda > 0$ belongings to $\rho(A)$ since e^{tA} is a bounded C_0-semigroup.) If $|\mu - \lambda|$ and $|\nu - \overline{\lambda}|$ are sufficiently small, then $(\mu - A)^{-1}$ and $(\nu - A^*)^{-1}$ can be expanded into the infinite series at λ and $\overline{\lambda}$, respectively. Hence, for such μ and ν, $(\mu - A)^{-1}$ and $(\nu - A^*)^{-1}$ are commutative:

$$(\mu - A)^{-1}(\nu - A^*)^{-1} = (\nu - A^*)^{-1}(\mu - A)^{-1}. \tag{2.4}$$

Since both sides of this equality are holomorphic in $\mu \in \rho(A)$ for each $\nu \in \rho(A^*)$, by unique continuation, (2.4) holds for each $\mu \in \rho_\infty(A)$ and ν in a neighborhood of $\overline{\lambda}$, where $\rho_\infty(A)$ is the connected component of $\rho(A)$ including the right half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$. Accordingly since both sides of (2.4) are holomorphic in $\nu \in \rho(A^*)$ for each $\mu \in \rho_\infty(A)$, by unique continuation, (2.4) holds for each $\mu \in \rho_\infty(A)$ and $\nu \in \rho_\infty(A^*)$. In particular, $(\mu - A)^{-1}$ and $(\nu - A^*)^{-1}$ are commutative for each $\mu, \nu > 0$. By using the well-known formula

$$\big(\lambda_1 + (-A)^\beta\big)^{-1} = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \frac{\mu^\beta(\mu - A)^{-1}}{\mu^{2\beta} + 2\lambda_1\mu^\beta\cos(\pi\beta) + \lambda_1^2}\, d\mu$$

for all $\lambda_1 > 0$ (*cf.* [7, (5.24)]) and the resulting equality

$$\begin{aligned}\Big[\big(\lambda_2 + (-A)^\beta\big)^{-1}\Big]^* &= \frac{\sin(\pi\beta)}{\pi}\Big[\int_0^\infty \frac{\nu^\beta(\nu - A)^{-1}}{\nu^{2\beta} + 2\lambda_2\nu^\beta\cos(\pi\beta) + \lambda_2^2}\, d\nu\Big]^* \\ &= \frac{\sin(\pi\beta)}{\pi}\int_0^\infty \frac{\nu^\beta(\nu - A^*)^{-1}}{\nu^{2\beta} + 2\lambda_2\nu^\beta\cos(\pi\beta) + \lambda_2^2}\, d\nu\end{aligned}$$

for all $\lambda_2 > 0$, we obtain that $\big(\lambda_1 + (-A)^\beta\big)^{-1}$ and $\big[\big(\lambda_2 + (-A)^\beta\big)^{-1}\big]^*$ are commutative for all $\lambda_1, \lambda_2 > 0$. Since $\big(\frac{n}{t}\big(\frac{n}{t} + (-A)^\beta\big)^{-1}\big)^n$ strongly converges to $e^{-t(-A)^\beta}$ as $n \to \infty$ and $\big[\big(\frac{m}{t}\big(\frac{m}{t} + (-A)^\beta\big)^{-1}\big)^m\big]^*$ strongly converges to $\big(e^{-t(-A)^\beta}\big)^*$ as $m \to \infty$, we conclude that $e^{-t(-A)^\beta}$ and $\big(e^{-t(-A)^\beta}\big)^*$ are commutative. i.e., $e^{-t(-A)^\beta}$ is normal.

Next fix an arbitrary $t_0 > 0$. Let $\beta \in (0,1)$ and $p \in [1,\infty)$. Then, by Proposition 2.10, there exists a C_0-semigroup $T_{\beta,p} = \big(T_{\beta,p}(t)\big)_{t\geq 0}$ on $L^p(\Omega)$ such that $T_{\beta,p}$ is consistent with $e^{-t(-A)^\beta}$. In addition, $T_{\beta,p}(t_0)$ is an integral operator, and its kernel $K_{t_0,\beta}$ is independent of $p \in [1,\infty)$ and $K_{t_0,\beta} \in A^{0,0}_{w_\delta,2}$ for each $\delta \in (0, 2\alpha\beta)$ by Lemma 2.8. Since $e^{-t_0(-A)^\beta}$ is normal and so is $K_{t_0,\beta}$, by applying Barnes' theorem, $\sigma\big(T_{\beta,p}(t_0)\big)$ is proved to be independent of $p \in [1,\infty)$. Hence, $\sigma\big(e^{-t_0(-A_p)^\beta}\big)$ is independent of $p \in [1,\infty)$ (*cf.* Proposition 2.10 (ii)). By Lemma 2.9 (i), $\sigma(A_p)$ in independent of $p \in [1,\infty)$.

(ii) is proved by using Lemma 2.9 (ii) instead of Lemma 2.9 (i) in the proof of assertion (i). □

Now, we give a corollary to Theorem 2.11, which partly improves Theorem 4.2 in [10]. For each $\alpha \in (0,1]$, H_α and $U_\alpha(t)$ denotes $(-\Delta)^\alpha$ and e^{-tH_α}, respectively, and let $V\colon \mathbb{R}^N \to \mathbb{R}$ be a bounded non-negative measurable function. We verify L^p-spectral independence of a version $H_\alpha + V$ in $L^p(\mathbb{R}^N)$, where we used the same symbol for the function V and also for the associated maximal multiplication operator in $L^p(\mathbb{R}^N)$ defined by V.

Corollary 2.12. *The operator sum $-(H_\alpha + V)$ generates a C_0-semigroup $U_{\alpha,V} = \big(U_{\alpha,V}(t)\big)_{t\geq 0}$ on $L^2(\mathbb{R}^N)$ and there exists a C_0-semigroup $U_{\alpha,V,p} = \big(U_{\alpha,V,p}(t)\big)_{t\geq 0}$ on $L^p(\mathbb{R}^N)$ such that $U_{\alpha,V,p}$ is consistent with $U_{\alpha,V}$ for each $p \in [1,\infty)$. The generator $-H_{\alpha,V,p}$ of $U_{\alpha,V,p}$ coincides with $-(H_{\alpha,p} + V)$ for each $p \in [1,\infty)$, where $-H_{\alpha,p}$ is the generator of the C_0-semigroup naturally* *defined by U_α on $L^p(\mathbb{R}^N)$ for each $p \in [1,\infty)$ as in Proposition 2.2. Moreover, the spectrum*

$$\sigma(H_{\alpha,p} + V)$$

is independent of $p \in [1,\infty)$.

Proof. It is clear that $-(H_\alpha + V)$ generates a positive C_0-semigroup $U_{\alpha,V} = \big(U_{\alpha,V}(t)\big)_{t\geq 0}$ on $L^2(\mathbb{R}^N)$ and $U_{\alpha,V}$ satisfies a Gaussian estimate of order α. More precisely,

$$0 \leq U_{\alpha,V}(t) \leq U_\alpha(t)$$

is obtained for all $t \geq 0$ by using Trotter product formula. Hence, by Proposition 2.2, there exists a C_0-semigroup $U_{\alpha,V,p} = \big(U_{\alpha,V,p}(t)\big)_{t\geq 0}$ on $L^p(\mathbb{R}^N)$ such that $U_{\alpha,V,p}$ is consistent with $U_{\alpha,V}$ for each $p \in [1,\infty)$. Since Trotter product formula implies that the C_0-semigroup $\exp(-(H_{\alpha,p} + V))$ is consistent with $U_{\alpha,V}$, we have $U_{\alpha,V,p}$ coincides with $\exp(-(H_\alpha, p + V))$. Hence, $H_{\alpha,V,p} = H_{\alpha,p} + V$, where $H_{\alpha,V,p}$ is the generator of $U_{\alpha,V,p}$.

Since the generator of $U_{\alpha,V}$ is self-adjoint, Theorem 2.11 implies that the spectrum of $H_{\alpha,V,p}$ is independent of $p \in [1,\infty)$. Thus, the proof is completed. □

§3. Appendix

We prove the statement in Remark 2.6. We first recall what we should prove.

Proposition 3.1. *Let K be defined by*

$$K(x,y) := \begin{cases} \sqrt{y} & (y \geq 2, 2y \leq x \leq 2y + 1/y) \\ 0 & (\textit{otherwise}). \end{cases}$$

Then, $K + K^*$ is hermitian and belongs to $A^0_{w_\delta,2}$ for each $\delta \in (0,1/2)$ but does not belong to $A^{0,0}_{w_\delta,2}$ for any $\delta \in (0,1/2)$.

Proof. Let $\delta \in (0,1/2)$. We prove that $K \in A^0_{w_\delta,2}$ and $K \notin A^{0,0}_{w_\delta,2}$, from which the assertion of this proposition follows. In fact, since $(w_\delta K)^* = w_\delta K^*$, $\big(\chi(\Gamma[m])K\big)^* = \chi(\Gamma[m])K^*$ and the involution is isometric in each of the norms of A_1, A_{w_δ} and A_2, we obtain that K^* hence $K + K^*$ belongs to $A^0_{w_\delta,2}$. On the other hand, since $\operatorname{supp} K \cap \operatorname{supp} K^* = \emptyset$, the inequality $\|\chi(\Gamma[m]^c)(K+K^*)\|_2 \geq \|\chi(\Gamma[m]^c)K\|_2$ holds for all $m \in \mathbb{N}$. Hence, by $K \notin A^0_2$, $\|\chi(\Gamma[m]^c)(K + K^*)\|_2$ does not converge to 0 as $m \to \infty$. i.e., $K + K^* \notin A^0_2$. Since it is clear that $K + K^*$ is hermitian, we see that the desired assertion concerning K leads to the assertion of this proposition.

Now, we prove that $K \in A^0_{w_\delta,2}$. We first estimate $w_\delta(x,y)$ for all $(x,y) \in \operatorname{supp} K$. Since each $(x,y) \in \operatorname{supp} K$ satisfies the estimate $2y \le x \le 2y + 1/2$, we have

$$|x-y| \le y + \frac{1}{2} \le \frac{1}{2}(x+1). \tag{3.1}$$

Hence, $w_\delta(x,y) \le (3/2+y)^\delta$ and $w_\delta(x,y) \le 2^{-\delta}(3+x)^\delta$ for all $(x,y) \in \operatorname{supp} K$.

Next, we estimate the integrals $\int_{\mathbb{R}} K(x,y)\,dx$ and $\int_{\mathbb{R}} K(x,y)\,dy$. It is easy to see that

$$\int_{\mathbb{R}} K(x,y)\,dx = \begin{cases} 0 & (y<2) \\ \dfrac{1}{\sqrt{y}} & (y \ge 2), \end{cases}$$

and there exists a constant $C>0$ such that

$$\int_{\mathbb{R}} K(x,y)\,dy \begin{cases} = 0 & (x<4) \\ \le C & (4 \le x \le 9/2). \end{cases}$$

In the case of $x > 9/2$, we have by using the trivial inequality $\sqrt{x^2-8} < x$

$$\begin{aligned}
\int_{\mathbb{R}} K(x,y)\,dy &= \int_{(x+\sqrt{x^2-8})/4}^{x/2} \sqrt{y}\,dy \\
&= \frac{2}{3}\Big\{\Big(\frac{x}{2}\Big)^{\frac{3}{2}} - \Big(\frac{x+\sqrt{x^2-8}}{4}\Big)^{\frac{3}{2}}\Big\} \\
&= \frac{2}{3}\Big\{\Big(\frac{x}{2}\Big)^{\frac{1}{2}} - \Big(\frac{x+\sqrt{x^2-8}}{4}\Big)^{\frac{1}{2}}\Big\} \\
&\qquad \times \Big\{\frac{x}{2} + \Big(\frac{x}{2}\Big)^{\frac{1}{2}}\Big(\frac{x+\sqrt{x^2-8}}{4}\Big)^{\frac{1}{2}} + \frac{x+\sqrt{x^2-8}}{4}\Big\} \\
&< \frac{2}{3} \cdot \frac{x-\sqrt{x^2-8}}{2\big(\sqrt{2x} + (x+\sqrt{x^2-8})^{\frac{1}{2}}\big)} \cdot \frac{3x}{2} \\
&< \frac{\sqrt{x}}{2\sqrt{2}}(x-\sqrt{x^2-8}) \\
&= \frac{\sqrt{x}}{2\sqrt{2}} \cdot \frac{8}{x+\sqrt{x^2-8}} < \frac{2\sqrt{2}}{\sqrt{x}}.
\end{aligned}$$

Hence, $K \in A_{w_\delta}$ is shown for each $\delta \in (0,1/2)$ by the following estimate:

$$\begin{aligned}
&\operatorname*{ess.sup}_{y\in\mathbb{R}} \int_{\mathbb{R}} w_\delta(x,y)K(x,y)\,dx \le \operatorname*{ess.sup}_{y\ge 2}\Big(\frac{3}{2}+y\Big)^\delta \cdot \frac{1}{\sqrt{y}} < \infty, \\
&\operatorname*{ess.sup}_{x\le 9/2} \int_{\mathbb{R}} w_\delta(x,y)K(x,y)\,dy \le 2^{-\delta}C \operatorname*{ess.sup}_{4\le x\le 9/2}(3+x)^\delta = \Big(\frac{15}{4}\Big)^\delta C < \infty, \\
&\operatorname*{ess.sup}_{x>9/2} \int_{\mathbb{R}} w_\delta(x,y)K(x,y)\,dy \le 2^{-\delta} \operatorname*{ess.sup}_{x>9/2}(3+x)^\delta \cdot \frac{2\sqrt{2}}{\sqrt{x}} < \infty.
\end{aligned}$$

By a similar manner, we can prove that $K \in A^0_{w_\delta}$ for each $\delta \in (0, 1/2)$. In fact, if $(x, y) \in \operatorname{supp} K$ satisfies $|x - y| > m$ for an $m \in \mathbb{N}$, then $y \geq m - 1/2$ and $x \geq 2m - 1$ by (3.1). Hence, we have for $m \geq 3$,

$$\begin{aligned}
&\operatorname*{ess.sup}_{y\in\mathbb{R}} \int_{\mathbb{R}} \chi(\Gamma[m]^c)(x,y) w_\delta(x,y) K(x,y)\, dx \\
&\qquad \leq \operatorname*{ess.sup}_{y\geq m-1/2} \int_{\mathbb{R}} w_\delta(x,y) K(x,y)\, dx \\
&\qquad \leq \operatorname*{ess.sup}_{y\geq m-1/2} \left(\frac{3}{2} + y\right)^\delta \cdot \frac{1}{\sqrt{y}}, \\
&\operatorname*{ess.sup}_{x\in\mathbb{R}} \int_{\mathbb{R}} \chi(\Gamma[m]^c)(x,y) w_\delta(x,y) K(x,y)\, dy \\
&\qquad \leq \operatorname*{ess.sup}_{x\geq 2m-1} \int_{\mathbb{R}} w_\delta(x,y) K(x,y)\, dy \\
&\qquad \leq 2^{-\delta} \operatorname*{ess.sup}_{x\geq 2m-1} (3 + x)^\delta \cdot \frac{2\sqrt{2}}{\sqrt{x}}.
\end{aligned}$$

Since the rightmost side of each inequality above converges to 0 as $m \to \infty$, the norm $\|\chi(\Gamma[m]^c)K\|_{w_\delta}$ converges to 0 as $m \to \infty$. i.e., $K \in A^0_{w_\delta}$.

Next, we verify that $K \in A_2$ for the completeness of the proof. It is easy to see that

$$\int_{\mathbb{R}} K(x,y)^2\, dx = \begin{cases} 0 & (y < 2), \\ 1 & (y \geq 2), \end{cases}$$

and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} K(x,y)^2\, dy \begin{cases} = 0 & (x < 4), \\ \leq C & (4 \leq x \leq 9/2). \end{cases}$$

In the case of $x > 9/2$, we have

$$\begin{aligned}
\int_{\mathbb{R}} K(x,y)^2\, dy &= \int_{(x+\sqrt{x^2-8})/4}^{x/2} y\, dy \\
&= \frac{1}{32}\{4x^2 - (x + \sqrt{x^2-8})^2\} \\
&= \frac{1}{32}(x - \sqrt{x^2-8})(3x + \sqrt{x^2-8}) \\
&\leq \frac{x}{8} \cdot \frac{8}{x + \sqrt{x^2-8}} \leq 1.
\end{aligned}$$

Thus, $\|K\|_2 \leq \max\{1, \sqrt{C}\}$, hence, $K \in A_2$.

The remaining assertion is that $K \notin A_2^0$. To prove this assertion, note that if $(x, y) \in \operatorname{supp} K$ satisfies $y > m$, then $x - y \geq y > m$, i.e., $(x, y) \notin \Gamma[m]$. Hence, we have

$$\operatorname*{ess.sup}_{y \in \mathbb{R}} \int_{\mathbb{R}} \chi(\Gamma[m]^c) K(x,y)^2 \, dx \geq \operatorname*{ess.sup}_{y \geq m+1} \int_{\mathbb{R}} K(x,y)^2 \, dx = 1$$

for all $m \in \mathbb{N}$. Thus, we conclude $K \notin A_2^0$. □

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