Quasi-conformally flat manifolds satisfying certain condition on the Ricci tensor

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Abstract. The object of the present paper is to study a non-flat quasi-conformally flat Riemannian manifold whose Ricci tensor $S$ satisfies the condition $S(X, Y) = \gamma T(X)T(Y)$, where $\gamma$ is the scalar curvature and $T$ is a 1-form defined by $T(X) = g(X, \xi)$, $\xi$ is a unit vector field.

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§1. Introduction

The notion of a quasi-conformal curvature tensor was given by Yano and Sawaki [10]. According to them a quasi-conformal curvature tensor $C^*$ is defined by

$$C^*(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{\gamma}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y], \quad (1.1)$$

where $a$ and $b$ are constants and $R$, $Q$ and $\gamma$ are the Riemannian curvature tensor of type $(1, 3)$, the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and the scalar curvature, respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the
form
\[
C^*(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{\gamma}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y]
\] = C(X,Y)Z,

where \( C \) is the conformal curvature tensor \([4]\). Thus the conformal curvature tensor \( C \) is a particular case of the tensor \( C^* \). For this reason \( C^* \) is called the quasi-conformal curvature tensor. A manifold \((M^n, g)\) \((n > 3)\) shall be called quasi-conformally flat if \( C^* = 0 \). It is known \([1]\) that a quasi-conformally flat manifold is either conformally flat if \( a \neq 0 \) or Einstein if \( a = 0 \) and \( b \neq 0 \). Since they give no restrictions for manifolds if \( a = 0 \) and \( b = 0 \), it is essential for us to consider the case of \( a \neq 0 \) or \( b \neq 0 \).

A Riemannian manifold of quasi-constant curvature was given by B. Y. Chen and K. Yano \([3]\) as a conformally flat manifold with the curvature tensor \( \tilde{R} \) of type \((0, 4)\) satisfies the condition
\[
\tilde{R}(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q[g(X,W)T(Y)T(Z) + g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W) - g(Y,W)T(X)T(Z)],
\]
\[(1.2)\]

where \( \tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W) \), \( R \) is the curvature tensor of type \((1, 3)\), \( p, q \) are scalar functions and \( T \) is a non-zero 1-form defined by
\[
g(X,\xi) = T(X),\tag{1.3}
\]

where \( \xi \) is a unit vector field. It can be easily seen that if the curvature tensor \( \tilde{R} \) is of the form \((1.2)\), then the manifold is conformally flat. On the other hand, G. Vrǎnciu \([8]\) defined the notion of almost constant curvature by the same expression \((1.2)\). Later A. L. Mocanu \([6]\) pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Vrǎnciu are the same. Hence a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor \( \tilde{R} \) satisfies the relation \((1.2)\). If \( q = 0 \), then the manifold reduces to a manifold of constant curvature.

The present paper deals with the quasi-conformally flat manifold \((M^n, g)\) \((n > 3)\) whose Ricci tensor \( S \) satisfies
\[
S(X,Y) = \gamma T(X)T(Y),\tag{1.4}
\]

where \( T \) is a non-zero 1-form defined by \( g(X,\xi) = T(X) \), \( \xi \) is a unit vector field. For the scalar curvature \( \gamma \) we suppose that \( \gamma \neq 0 \) for each point of...
M. Under the assumption above we know that $M$ is not Einstein. Hence we consider the case of $a \neq 0$ (See §3). We shall prove the following:

**Theorem 1.** A quasi-conformally flat manifold satisfying the condition (1.4) under the assumption of $\gamma \neq 0$ is a manifold of quasi-constant curvature.

**Theorem 2.** In a quasi-conformally flat Riemannian manifold satisfying the condition (1.4) under the same assumption as Theorem 1, the integral curves of the vector field $\xi$ are geodesic.

**Theorem 3.** In a quasi-conformally flat manifold satisfying (1.4) under the same assumption as Theorem 1, the vector field $\xi$ is a proper concircular vector field (See §4).

**Theorem 4.** If a quasi-conformally flat manifold satisfies (1.4) under the same assumption as Theorem 1, then the manifold is a locally product manifold.

**Theorem 5.** A quasi-conformally flat manifold satisfying (1.4) under the same assumption as Theorem 1 can be expressed as a locally warped product $I \times e^{\phi} M^*$ where $M^*$ is an Einstein manifold (See §4).

§2. Preliminaries

From (1.1) we obtain


$$+ g(Y, Z)(\nabla_W Q)(X) - g(X, Z)(\nabla_W Q)(Y)]$$

$$- \frac{d\gamma(W)}{n} \left[ \frac{a}{n - 1} + 2b \right] [g(Y, Z)X - g(X, Z)Y],$$

(2.1)

where $\nabla$ is the covariant differentiation with respect to the Riemannian metric $g$. We know that $(\text{div } R)(X, Y)Z = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$. Hence contracting (2.1) we obtain

$$(\text{div } C^*)(X, Y)Z = (a + b)((\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z))$$

$$+ \frac{1}{n} \left[ \frac{n - 1}{2} - \frac{a}{n - 1} \right] [g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)].$$

(2.2)

Here we consider quasi-conformally flat manifold i.e., $C^* = 0$. Hence div $C^* = 0$, where ‘div’ denotes the divergence. If $a + b \neq 0$, then from (2.2) it follows that

$$\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$$

$$= \frac{1}{n(a + b)} \left[ \frac{a}{n - 1} - \frac{(n - 1)b}{2} \right] [g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)].$$
This can be written as

\[
(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \alpha [g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)], \quad (2.3)
\]

where \( \alpha = \frac{1}{n(a + b)} \left[ \frac{a}{n - 1} - \frac{(n - 4)b}{2} \right] = \text{constant} \).

§3. Quasi-conformally flat manifold satisfying the condition (1.4)

From (1.1) we get

\[
\tilde{C}^*(X, Y, Z, W) = a \tilde{R}(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)
+ S(X, W)g(Y, Z) - S(Y, W)g(X, Z)]
- \gamma \left[ \frac{a}{n(n - 1)} + 2b[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \right]. \quad (3.1)
\]

If the manifold is quasi-conformally flat under the assumption of \( \gamma \neq 0 \), then we get

\[
\gamma(a + (n - 2)b) = 0.
\]

Then we note that \( \left( \frac{(n - 4)b}{2} - \frac{n}{n - 1} \right) = \frac{3na}{2(n - 1)(n - 2)} \). Since \( a \neq 0 \) under the assumption of \( \gamma \neq 0 \), we know that \( a + b \neq 0 \) and \( \alpha \neq 0 \). Moreover, from (1.4) we have

\[
\tilde{R}(X, Y, Z, W)
= \frac{b}{a} [S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + S(Y, W)g(X, Z) - S(X, W)g(Y, Z)]
+ \frac{\gamma}{na} \left[ \frac{a}{n(n - 1)} + 2b[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \right]. \quad (3.2)
\]

Using (1.4) in (3.2), we obtain

\[
\tilde{R}(X, Y, Z, W)
= \frac{\gamma b}{a} [g(Y, W)T(X)T(Z) - g(X, W)T(Y)T(Z) + g(X, Z)T(Y)T(W)
- g(Y, Z)T(X)T(W)] + \frac{\gamma}{na} \left[ \frac{a}{n(n - 1)} + 2b[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \right],
\]

which implies that the manifold is a manifold of quasi-constant curvature.

Hence we can state that

**Theorem 1.** A quasi-conformally flat manifold satisfying the condition (1.4) under the assumption of \( \gamma \neq 0 \) is a manifold of quasi-constant curvature.
§4. The results concerning the product manifold

From (1.4) we have

\[(\nabla_Z S)(X, Y) = d\gamma(Z)T(X)T(Y) + \gamma[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)].\]  

(4.1)

Substituting (4.1) in (2.3), we get

\[d\gamma(Z)T(X)T(Y) + \gamma[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)] - d\gamma(X)T(Z)T(Y) - \gamma[(\nabla_X T)(Z)T(Y) + T(Z)(\nabla_X T)(Y)] = \alpha[g(X,Y)d\gamma(Z) - g(Z,Y)d\gamma(X)].\]  

(4.2)

Putting \(Y = Z = e_i\) in the above expression where \(\{e_i\}\) is an orthonormal basis of the tangent space at each point of the manifold and taking summation over \(i, 1 \leq i \leq n\), we get

\[\alpha(1 - n)d\gamma(X) = d\gamma(\xi)T(X) + \gamma(\nabla_{\xi} T)(X) + \gamma T(X)(\delta T) - d\gamma(X),\]  

(4.3)

where we put \(\delta T = \sum_{i=1}^{n}(\nabla_{e_i} T)(e_i)\). Again putting \(Y = Z = \xi\) in (4.2), it yields

\[\gamma(\nabla_{\xi} T)(X) = (\alpha - 1)[d\gamma(\xi)T(X) - d\gamma(X)].\]  

(4.4)

Substituting (4.4) in (4.3), we get

\[\alpha(n - 2)d\gamma(X) - \alpha d\gamma(\xi)T(X) + \gamma \delta T = 0.\]  

(4.5)

Now putting \(X = \xi\) in (4.5), it yields

\[\alpha(n - 3)d\gamma(\xi) + \gamma \delta T = 0.\]  

(4.6)

From (4.5) and (4.6) it follows that

\[\alpha d\gamma(X) = \alpha d\gamma(\xi)T(X).\]

Since \(\alpha \neq 0\), we have

\[d\gamma(X) = d\gamma(\xi)T(X).\]  

(4.7)

Putting \(Y = \xi\) in (4.2) and using (4.7), we obtain

\[(\nabla_X T)(Z) - (\nabla_Z T)(X) = 0,\]  

(4.8)

since \(\gamma \neq 0\). This means that the 1-form \(T\) defined by \(g(X,\xi) = T(X)\) is closed, i.e., \(dT(X, Y) = 0\). Hence it follows that

\[g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X)\]  

(4.9)
for all $X, Y$. Now putting $Y = \xi$ in (4.9), we get

$$g(\nabla_X \xi, \xi) = g(\nabla_\xi X).$$

(4.10)

Since $g(\nabla_X \xi, \xi) = 0$, from (4.10) it follows that $g(\nabla_\xi X, X) = 0$ for all $X$. Hence $\nabla_\xi \xi = 0$. This means that the integral curves of the vector field $\xi$ are geodesic. Therefore we can state the following:

**Theorem 2.** In a quasi-conformally flat Riemannian manifold satisfying the condition (1.4) under the assumption of $\gamma \neq 0$, the integral curves of the vector field $\xi$ are geodesic.

From (4.4), by virtue of (4.7) we get

$$\langle \nabla_\xi T \rangle(Z) = 0,$$

(4.11)

since $\gamma \neq 0$. Now we consider the scalar function

$$f = \alpha \frac{d\gamma(\xi)}{\gamma}.$$

We have

$$\nabla_X f = \frac{\alpha}{\gamma^2} [d\gamma(\xi) T(\nabla_X \xi - d\gamma(X)d\gamma(\xi)) + \frac{\alpha}{\gamma} d^2\gamma(\xi, X),$$

(4.12)

where the Hessian $d^2\gamma$ is defined by $d^2\gamma(X,Y) = X(Y \gamma) - (\nabla_X Y)\gamma$. On the other hand, (4.7) implies that

$$d^2\gamma(Y, X) = d^2\gamma(\xi, Y) T(X) + d\gamma(\xi) T(\nabla_Y \xi) T(X) + d\gamma(\xi)(\nabla_Y T)(X),$$

from which we get

$$d^2\gamma(\xi, Y) T(X) = d^2\gamma(\xi, X) T(Y),$$

(4.13)

since $\langle \nabla_X T \rangle(Y) = \langle \nabla_Y T \rangle(X)$ and $d^2\gamma(Y, X) = d^2\gamma(X, Y)$. Putting $X = \xi$ in (4.13), it follows that

$$d^2\gamma(\xi, Y) = d^2\gamma(\xi, \xi) T(Y),$$

since $T(\xi) = 1$. Thus

$$\nabla_X f = \mu T(X),$$

(4.14)

where $\mu = \frac{\alpha}{\gamma} [d^2\gamma(\xi, \xi) - \frac{d\gamma(\xi)}{\gamma} d\gamma(\xi)]$ and we used (4.7). Using (4.14), it is easy to show that

$$\omega(X) = \frac{\alpha}{\gamma} d\gamma(\xi) T(X) = f T(X)$$
is closed. In fact, 
\[ d\omega(X,Y) = 0. \]

Using (4.7) and (4.8) in (4.2), we get 
\[ \gamma[T(Z)(\nabla_X T)(Y) - T(X)(\nabla_Z T)(Y)] = \alpha d\gamma[\xi](g(Y,Z)T(X) - g(X,Y)T(Z)]. \]

Now putting \( Z = \xi \) in the above expression it yields 
\[ -(\nabla_X T)(Y) = \frac{d\gamma(\xi)}{\gamma} [T(X)T(Y) - g(X,Y)], \]
by (4.11). Thus (4.15) can be rewritten as follows: 
\[ (\nabla_X T)(Y) = -fg(X,Y) + \omega(X)T(Y), \]
where \( \omega \) is closed. But this means that the vector field \( \xi \) defined by 
\[ g(X,\xi) = T(X) \]
is a proper concircular vector field ([7], [9]). Hence we can state the following:

**Theorem 3.** In a quasi-conformally flat manifold satisfying (1.4) under the assumption of \( \gamma \neq 0 \), the vector field \( \xi \) is a proper concircular vector field.

From (4.16) it follows that 
\[ \nabla_X \xi = -fX + \omega(X)\xi. \]

Let \( \xi^\perp \) denote the \((n-1)\)-dimensional distribution in a quasi-conformally flat manifold orthogonal to \( \xi \). If \( X \) and \( Y \) belong to \( \xi^\perp \), then 
\[ g(X,\xi) = 0 \]
and 
\[ g(Y,\xi) = 0. \]
Since \( (\nabla_X g)(Y,\xi) = 0 \), it follows from (4.17) and (4.19) that 
\[ g(\nabla_X Y,\xi) = g(\nabla_X \xi, Y) = -fg(X,Y). \]
Similarly, we get 
\[ g(\nabla_Y X,\xi) = g(\nabla_Y \xi, X) = -fg(X,Y). \]

Hence
\[ g(\nabla_X Y,\xi) = (\nabla_Y X,\xi). \]
Now $[X, Y] = \nabla_X Y - \nabla_Y X$ and therefore by (4.20) we obtain

$$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi) = 0.$$ 

Hence $[X, Y]$ is orthogonal to $\xi$. That is, $[X, Y]$ belongs to $\xi^{\perp}$. Thus the distribution $\xi^{\perp}$ is involutive [2]. Hence from Frobenius’ theorem [2] it follows that $\xi^{\perp}$ is integrable. This implies that if a quasi-conformally flat manifold satisfies (1.4), then it is a product manifold. We can therefore state the following theorem:

**Theorem 4.** If a quasi-conformally flat manifold satisfies (1.4) under the assumption of $\gamma \neq 0$, then the manifold is a locally product manifold.

If a quasi-conformally flat manifold satisfies (1.4) under the assumption of $\gamma \neq 0$, then in view of Theorem 3, $\xi$ is a concircular vector field. Also, $M$ is a quasi-constant curvature manifold and satisfies (1.2) and from Theorem 4 we know that $\xi^{\perp}$ is integrable and it holds

$$g(\nabla_X Y, \xi) = -(\nabla_X T)(Y)$$

for the local vector fields $X, Y$ belonging to $\xi^{\perp}$. Thus from (4.15) the second fundamental form $k$ for each leaf satisfies

$$k(X, Y) = -\alpha \frac{d\gamma(\xi)}{\gamma} g(X, Y) \xi.$$ 

Hence we know that each leaf is totally umbilic. Therefore each leaf is a manifold of constant curvature. Hence it must be a warped product $I \times_{e^\eta} M^*$ where $M^*$ is an Einstein manifold. Thus we can state the following result (See [9], [5]):

**Theorem 5.** A quasi-conformally flat manifold satisfying (1.4) under the assumption of $\gamma \neq 0$ can be expressed as a locally warped product $I \times_{e^\eta} M^*$ where $M^*$ is an Einstein manifold.

**References**


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