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Quasi-conformally flat manifolds satisfying certain condition on the Ricci tensor

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Abstract. The object of the present paper is to study a non-flat quasi-conformally flat Riemannian manifold whose Ricci tensor S satisfies the condition $S(X,Y) = \gamma T(X)T(Y)$, where γ is the scalar curvature and T is a 1-form defined by $T(X) = g(X,\xi)$, ξ is a unit vector field.

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§1. Introduction

The notion of a quasi-conformal curvature tensor was given by Yano and Sawaki [10]. According to them a quasi-conformal curvature tensor C^* is defined by

$$\begin{aligned} C^*(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ - g(X,Z)QY] - \frac{\gamma}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y], \end{aligned} \quad (1.1)$$

where a and b are constants and R, Q and γ are the Riemannian curvature tensor of type (1, 3), the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and the scalar curvature, respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the

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form

$$\begin{aligned} C^*(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad - g(X,Z)QY] + \frac{\gamma}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z, \end{aligned}$$

where C is the conformal curvature tensor [4]. Thus the conformal curvature tensor C is a particular case of the tensor C^*. For this reason C^* is called the quasi-conformal curvature tensor. A manifold (M^n, g) $(n > 3)$ shall be called quasi-conformally flat if $C^* = 0$. It is known [1] that a quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or Einstein if $a = 0$ and $b \neq 0$. Since they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$.

A Riemannian manifold of quasi-constant curvature was given by B. Y. Chen and K. Yano [3] as a conformally flat manifold with the curvature tensor $\tilde{R}$ of type (0, 4) satisfies the condition

$$\begin{aligned} \tilde{R}(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + q[g(X,W)T(Y)T(Z) + g(Y,Z)T(X)T(W) \\ &\quad - g(X,Z)T(Y)T(W) - g(Y,W)T(X)T(Z)], \end{aligned} \tag{1.2}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type (1, 3), p, q are scalar functions and T is a non-zero 1-form defined by

$$g(X,\tilde{\xi}) = T(X), \tag{1.3}$$

where $\tilde{\xi}$ is a unit vector filed. It can be easily seen that if the curvature tensor $\tilde{R}$ is of the form (1.2), then the manifold is conformally flat. On the other hand, G. Vrănceanu [8] defined the notion of almost constant curvature by the same expression (1.2). Later A. L. Mocanu [6] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Vrănceanu are the same. Hence a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $\tilde{R}$ satisfies the relation (1.2). If $q = 0$, then the manifold reduces to a manifold of constant curvature.

The present paper deals with the quasi-conformally flat manifold (M^n, g) $(n > 3)$ whose Ricci tensor S satisfies

$$S(X,Y) = \gamma T(X)T(Y), \tag{1.4}$$

where T is a non-zero 1-form defined by $g(X,\xi) = T(X)$, ξ is a unit vector field. For the scalar curvature γ we suppose that $\gamma \neq 0$ for each point of

M. Under the assumption above we know that M is not Einstein. Hence we consider the case of $a \neq 0$ (See §3). We shall prove the following:

Theorem 1. *A quasi-conformally flat manifold satisfying the condition* (1.4) *under the assumption of* $\gamma \neq 0$ *is a manifold of quasi-constant curvature.*

Theorem 2. *In a quasi-conformally flat Riemannian manifold satisfying the condition* (1.4) *under the same assumption as Theorem* 1, *the integral curves of the vector field* ξ *are geodesic.*

Theorem 3. *In a quasi-conformally flat manifold satisfying* (1.4) *under the same assumption as Theorem* 1, *the vector field* ξ *is a proper concircular vector field* (*See* §4).

Theorem 4. *If a quasi-conformally flat manifold satisfies* (1.4) *under the same assumption as Theorem* 1, *then the manifold is a locally product manifold.*

Theorem 5. *A quasi-conformally flat manifold satisfying* (1.4) *under the same assumption as Theorem* 1 *can be expressed as a locally warped product* $I \times_{e^q} M^*$ *where* M^* *is an Einstein manifold* (*See* §4).

§2. Preliminaries

From (1.1) we obtain

$$\begin{aligned}(\nabla_W C^*)(X,Y)Z = {} & a(\nabla_W R)(X,Y)Z + b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y \\ & + g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)] \\ & - \frac{d\gamma(W)}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.1}$$

where ∇ is the covariant differentiation with respect to the Riemannian metric g. We know that $(\text{div } R)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$. Hence contracting (2.1) we obtain

$$\begin{aligned}(\text{div } C^*)(X,Y)Z = {} & (a+b)((\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)) \\ & + \frac{1}{n}[\frac{(n-4)b}{2} - \frac{a}{n-1}](g(Y,Z)d\gamma(X) - g(X,Z)d\gamma(Y)). \end{aligned} \tag{2.2}$$

Here we consider quasi-conformally flat manifold i.e., $C^* = 0$. Hence div $C^* = 0$, where 'div' denotes the divergence. If $a + b \neq 0$, then from (2.2) it follows that

$$\begin{aligned}&(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &\quad = \frac{1}{n(a+b)}[\frac{a}{n-1} - \frac{(n-4)b}{2}][g(Y,Z)d\gamma(X) - g(X,Z)d\gamma(Y)].\end{aligned}$$

This can be written as

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \alpha[g(Y,Z)d\gamma(X) - g(X,Z)d\gamma(Y)], \tag{2.3}$$

where $\alpha = \frac{1}{n(a+b)}[\frac{a}{n-1} - \frac{(n-4)b}{2}] = \text{constant}$.

§3. Quasi-conformally flat manifold satisfying the condition (1.4)

From (1.1) we get

$$\begin{aligned}\tilde{C}^*(X,Y,Z,W) = a\tilde{R}(X,Y,Z,W) &+ b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \\ &- \frac{\gamma}{n}[\frac{a}{n-1} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.1}$$

If the manifold is quasi-conformally flat under the assumption of $\gamma \neq 0$, then we get

$$\gamma(a + (n-2)b) = 0.$$

Then we note that $[\frac{(n-4)b}{2} - \frac{a}{n-1}] = \frac{3na}{2(n-1)(n-2)}$. Since $a \neq 0$ under the assumption of $\gamma \neq 0$, we know that $a + b \neq 0$ and $\alpha \neq 0$. Moreover, from (1.4) we have

$$\begin{aligned}&\tilde{R}(X,Y,Z,W) \\ &= \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ &\quad + \frac{\gamma}{na}[\frac{a}{n-1} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{3.2}$$

Using (1.4) in (3.2), we obtain

$$\begin{aligned}&\tilde{R}(X,Y,Z,W) \\ &= \frac{\gamma b}{a}[g(Y,W)T(X)T(Z) - g(X,W)T(Y)T(Z) + g(X,Z)T(Y)T(W) \\ &\quad - g(Y,Z)T(X)T(W)] + \frac{\gamma}{na}[\frac{a}{n-1} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],\end{aligned}$$

which implies that the manifold is a manifold of quasi-constant curvature. Hence we can state that

Theorem 1. *A quasi-conformally flat manifold satisfying the condition* (1.4) *under the assumption of* $\gamma \neq 0$ *is a manifold of quasi-constant curvature.*

§4. The results concerning the product manifold

From (1.4) we have

$$\begin{aligned}(\nabla_Z S)(X,Y)\\ &= d\gamma(Z)T(X)T(Y) + \gamma[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)].\end{aligned} \tag{4.1}$$

Substituting (4.1) in (2.3), we get

$$\begin{aligned}&d\gamma(Z)T(X)T(Y) + \gamma[(\nabla_Z T)(X)T(Y) + T(X)(\nabla_Z T)(Y)]\\ &\quad - d\gamma(X)T(Z)T(Y) - \gamma[(\nabla_X T)(Z)T(Y) + T(Z)(\nabla_X T)(Y)]\\ &\quad\quad = \alpha[g(X,Y)d\gamma(Z) - g(Z,Y)d\gamma(X)].\end{aligned} \tag{4.2}$$

Putting $Y = Z = e_i$ in the above expression where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$\alpha(1-n)d\gamma(X) = d\gamma(\xi)T(X) + \gamma(\nabla_\xi T)(X) + \gamma T(X)(\delta T) - d\gamma(X), \tag{4.3}$$

where we put $\delta T = \sum_{i=1}^{n}(\nabla_{e_i}T)(e_i)$. Again putting $Y = Z = \xi$ in (4.2), it yields

$$\gamma(\nabla_\xi T)(X) = (\alpha - 1)[d\gamma(\xi)T(X) - d\gamma(X)]. \tag{4.4}$$

Substituting (4.4) in (4.3), we get

$$\alpha(n-2)d\gamma(X) - \alpha d\gamma(\xi)T(X) + \gamma\delta T = 0. \tag{4.5}$$

Now putting $X = \xi$ in (4.5), it yields

$$\alpha(n-3)d\gamma(\xi) + \gamma\delta T = 0. \tag{4.6}$$

From (4.5) and (4.6) it follows that

$$\alpha d\gamma(X) = \alpha d\gamma(\xi)T(X).$$

Since $\alpha \neq 0$, we have

$$d\gamma(X) = d\gamma(\xi)T(X). \tag{4.7}$$

Putting $Y = \xi$ in (4.2) and using (4.7), we obtain

$$(\nabla_X T)(Z) - (\nabla_Z T)(X) = 0, \tag{4.8}$$

since $\gamma \neq 0$. This means that the 1-form T defined by $g(X,\xi) = T(X)$ is closed, i.e., $dT(X,Y) = 0$. Hence it follows that

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X) \tag{4.9}$$

for all X, Y. Now putting $Y = \xi$ in (4.9), we get

$$g(\nabla_X \xi, \xi) = g(\nabla_\xi \xi, X). \tag{4.10}$$

Since $g(\nabla_X \xi, \xi) = 0$, from (4.10) it follows that $g(\nabla_\xi \xi, X) = 0$ for all X. Hence $\nabla_\xi \xi = 0$. This means that the integral curves of the vector field ξ are geodesic. Therefore we can state the following:

Theorem 2. *In a quasi-conformally flat Riemannian manifold satisfying the condition* (1.4) *under the assumption of* $\gamma \neq 0$, *the integral curves of the vector field* ξ *are geodesic.*

From (4.4), by virtue of (4.7) we get

$$(\nabla_\xi T)(Z) = 0, \tag{4.11}$$

since $\gamma \neq 0$. Now we consider the scalar function

$$f = \alpha \frac{d\gamma(\xi)}{\gamma}.$$

We have

$$\nabla_X f = \frac{\alpha}{\gamma^2}[d\gamma(\xi)T(\nabla_X \xi)\gamma - d\gamma(X)d\gamma(\xi)] + \frac{\alpha}{\gamma} d^2\gamma(\xi, X), \tag{4.12}$$

where the Hessian $d^2\gamma$ is defined by $d^2\gamma(X, Y) = X(Y\gamma) - (\nabla_X Y)\gamma$. On the other hand, (4.7) implies that

$$d^2\gamma(Y, X) = d^2\gamma(\xi, Y)T(X) + d\gamma(\xi)T(\nabla_Y \xi)T(X) + d\gamma(\xi)(\nabla_Y T)(X),$$

from which we get

$$d^2\gamma(\xi, Y)T(X) = d^2\gamma(\xi, X)T(Y), \tag{4.13}$$

since $(\nabla_X T)(Y) = (\nabla_Y T)(X)$ and $d^2\gamma(Y, X) = d^2\gamma(X, Y)$. Putting $X = \xi$ in (4.13), it follows that

$$d^2\gamma(\xi, Y) = d^2\gamma(\xi, \xi)T(Y),$$

since $T(\xi) = 1$. Thus

$$\nabla_X f = \mu T(X), \tag{4.14}$$

where $\mu = \dfrac{\alpha}{\gamma}[d^2\gamma(\xi, \xi) - \dfrac{d\gamma(\xi)}{\gamma} d\gamma(\xi)]$ and we used (4.7). Using (4.14), it is easy to show that

$$\omega(X) = \frac{\alpha}{\gamma} d\gamma(\xi)T(X) = fT(X)$$

is closed. In fact,

$$d\omega(X, Y) = 0.$$

Using (4.7) and (4.8) in (4.2), we get

$$\gamma[T(Z)(\nabla_X T)(Y) - T(X)(\nabla_Z T)(Y)] \\ = \alpha d\gamma(\xi)[g(Y, Z)T(X) - g(X, Y)T(Z)].$$

Now putting $Z = \xi$ in the above expression it yields

$$-(\nabla_X T)(Y) = \alpha \frac{d\gamma(\xi)}{\gamma}[T(X)T(Y) - g(X, Y)], \tag{4.15}$$

by (4.11). Thus (4.15) can be rewritten as follows:

$$(\nabla_X T)(Y) = -fg(X, Y) + \omega(X)T(Y), \tag{4.16}$$

where ω is closed. But this means that the vector field ξ defined by $g(X, \xi) = T(X)$ is a proper concircular vector field ([7], [9]). Hence we can state the following:

Theorem 3. *In a quasi-conformally flat manifold satisfying* (1.4) *under the assumption of* $\gamma \neq 0$, *the vector field* ξ *is a proper concircular vector field.*

From (4.16) it follows that

$$\nabla_X \xi = -fX + \omega(X)\xi. \tag{4.17}$$

Let $\xi^{\perp}$ denote the $(n-1)$-dimensional distribution in a quasi-conformally flat manifold orthogonal to ξ. If X and Y belong to $\xi^{\perp}$, then

$$g(X, \xi) = 0 \tag{4.18}$$

and

$$g(Y, \xi) = 0. \tag{4.19}$$

Since $(\nabla_X g)(Y, \xi) = 0$, it follows from (4.17) and (4.19) that

$$g(\nabla_X Y, \xi) = g(\nabla_X \xi, Y) = -fg(X, Y).$$

Similarly, we get

$$g(\nabla_Y X, \xi) = g(\nabla_Y \xi, X) = -fg(X, Y).$$

Hence

$$g(\nabla_X Y, \xi) = (\nabla_Y X, \xi). \tag{4.20}$$

Now $[X, Y] = \nabla_X Y - \nabla_Y X$ and therefore by (4.20) we obtain

$$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi) = 0.$$

Hence $[X, Y]$ is orthogonal to ξ. That is, $[X, Y]$ belongs to $\xi^\perp$. Thus the distribution $\xi^\perp$ is involutive [2]. Hence from Frobenius' theorem [2] it follows that $\xi^\perp$ is integrable. This implies that if a quasi-conformally flat manifold satisfies (1.4), then it is a product manifold. We can therefore state the following theorem:

Theorem 4. *If a quasi-conformally flat manifold satisfies* (1.4) *under the assumption of* $\gamma \neq 0$, *then the manifold is a locally product manifold.*

If a quasi-conformally flat manifold satisfies (1.4) under the assumption of $\gamma \neq 0$, then in view of Theorem 3, ξ is a concircular vector field. Also, M is a quasi-constant curvature manifold and satisfies (1.2) and from Theorem 4 we know that $\xi^\perp$ is integrable and it holds

$$g(\nabla_X Y, \xi) = -(\nabla_X T)(Y)$$

for the local vector fields X, Y belonging to $\xi^\perp$. Thus from (4.15) the second fundamental form k for each leaf satisfies

$$k(X, Y) = -\alpha \frac{d\gamma(\xi)}{\gamma} g(X, Y)\xi.$$

Hence we know that each leaf is totally umbilic. Therefore each leaf is a manifold of constant curvature. Hence it must be a warped product $I \times_{e^q} M^*$ where M^* is an Einstein manifold. Thus we can state the following result (See [9], [5]):

Theorem 5. *A quasi-conformally flat manifold satisfying* (1.4) *under the assumption of* $\gamma \neq 0$ *can be expressed as a locally warped product* $I \times_{e^q} M^*$ where M^* is an Einstein manifold.

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