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Transformations with improved asymptotic approximations and their accuracy

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Dedicated to Professor Minoru Siotani on his 80th birthday

Abstract. Suppose that a statistic S is asymptotically distributed as a distribution function $G(x)$ as some parameter $\epsilon \to 0$. We consider monotone transformations of S in order to improve the asymptotic approximation. The transformations proposed here preserve monotonicity and give transformed statistics $T(S)$ whose distribution function is coincident with $G(x)$ up to the order $O(\epsilon^{r-1})$. It may be observed that the proposed transformations give a significant improvement to approximations. Further, we shall also consider error bounds for the remainder term of an asymptotic expansion for the distribution of $T(S)$. Finally, some applications of the findings are demonstrated for some test statistics.

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§1. Introduction

In statistical inference it is basic to obtain the sampling distribution of a statistic. However, we often encounter the situation where the exact distribution cannot be obtained in a closed form, or even if it is obtained, the exact distribution is of no use because of its complexity. To overcome this situation, various approximations of the quantiles as well as the distribution function have been studied. The one to which we restrict attention is that of using asymptotic approximations, especially asymptotic expansions.

Let $F(x)$ be the distribution function of a statistic S depending on some parameter ϵ, not necessary the inverse of a sample size. In this paper, we consider a statistic S whose limiting distribution is $G(x)$ as $\epsilon \to 0$ and suppose that a statistic S has an asymptotic expansion

$$F(x) \equiv \mathrm{P}(S \le x) = G(x) + \epsilon h(x)g(x) + O(\epsilon^2), \tag{1.1}$$

where $g(x)$ is the density function of $G(x)$, $h(x)$ is a polynomial of degree k and $\lim_{x\to\infty} h(x)g(x) = 0$. The asymptotic expansion (1.1) is called an Edgeworth type expansion. When $G(x)$ is the Standard Normal distribution $N(0,1)$, the formula (1.1), with $\epsilon = n^{-1/2}$ where n is the sample size, is termed an Edgeworth expansion. One may refer to Hall (1992a) and its references. In the case that $G(x)$ is a chi-squared distribution χ_f^2 with f degrees of freedom, some examples of the statistic S with $\epsilon = n^{-1}$ are as follows: For $k = 1$, the likelihood ratio test statistic (see Hayakawa (1977)); for $k = 2$, Lawley-Hotelling trace criterion and Bartlett-Nanda-Pillai trace criterion, which are test statistics for multivariate linear hypothesis under normality (see Anderson (1984) and Siotani, Hayakawa and Fujikoshi (1985)); for $k = 3$, the score test statistic (see Harris (1985)), Hotelling's T^2-statistic under nonnormality (see Kano (1995) and Fujikoshi (1997b)) and test statistics for multivariate linear hypothesis under nonnormality (see Wakaki, Yanagihara and Fujikoshi (2002)).

In order to obtain an approximated quantile of statistic S, we consider a monotone function $T = T(S)$ satisfying

$$\mathrm{P}(T \le x) = G(x) + O(\epsilon^2). \tag{1.2}$$

For such a monotone function T, it holds that

$$\mathrm{P}(S \le b(u_\alpha)) = \mathrm{P}(T(S) \le u_\alpha) = 1 - \alpha + O(\epsilon^2),$$

where u_α is the upper α point of $G(x)$ and $b(\cdot)$ is the inverse function of T. We shall propose methods to use $b(u_\alpha)$ as an approximated upper α point of S.

The transformation $T = T(S)$ satisfying property (1.2) is called the Bartlett correction or a Bartlett type correction and it has been investigated by many researchers in the case that the limiting distribution is χ_f^2 (e.g., Cordeiro and Ferrari (1991), Kakizawa (1996), Fujikoshi (1997a), Fujisawa (1997), Cordeiro and Ferrari (1998), Cordeiro, Ferrari and Cysneiros (1998), Fujikoshi (2000), Aoshima, Enoki and Ito (2003), and Enoki and Aoshima (2004)). In the case that the limiting distribution is $N(0,1)$, such transformations were investigated by Hall (1992b) and Fujioka and Maesono (2000), among others. In this paper, we shall consider new transformations given by a different approach from others. Here, the assumption that the limiting distribution of S is $N(0,1)$ or χ_f^2 is not necessary. It may be observed that new transformations, proposed in this paper, give a significant improvement to approximations. Further, we shall also consider error bounds for the remainder term in (1.2) in order to obtain a positive constant c such that

$$|\mathrm{P}(T \le x) - G(x)| \le \epsilon^2 c. \tag{1.3}$$

These findings would lead a broad application with a wide class of statistics.

This paper is organized as in the following way. In Section 2, we propose new monotone transformations $T = T(S)$, satisfying property (1.2), in a different approach from others. In Section 3, we give a method for obtaining such transformations from the moments of a statistic S when the limiting distribution of S is a chi-squared distribution. In Section 4, a uniform or non-uniform error bound for the remainder term in (1.2) is provided for accuracy of improved asymptotic approximations. Finally, in Section 5, some applications of the findings are demonstrated for some test statistics.

§2. Transformations with improved asymptotic approximations

For a statistic S whose limiting distribution is $G(x)$, we assume that the distribution function can be expanded as in (1.1). Then, we consider a monotone transformation $T = T(S)$ based on the Bartlett correction or a Bartlett type correction. That is, we consider a monotone function $T = T(S)$ satisfying property (1.2). We can easily get the following lemma that is a clue to find a desired function $T = T(S)$.

Lemma 2.1. *Suppose that a statistic S has an asymptotic expansion* (1.1). *If the monotone transformation $T = T(S)$ can be expanded as*

$$T = S + \epsilon h(S) + O_p(\epsilon^2), \tag{2.1}$$

then property (1.2) *is satisfied.*

In the case that the limiting distribution is χ_f^2, some monotone transformations that hold (2.1) have been proposed in many articles (e.g., Fujikoshi (2000) for $k = 2$, Cordeiro, Ferrari and Cysneiros (1998) and Aoshima, Enoki and Ito (2003) for $k = 3$, and Kakizawa (1996) and Enoki and Aoshima (2004) for a general k). In the case that the limiting distribution is $N(0, 1)$, such transformations were given by Hall (1992b) and Fujioka and Maesono (2000). We, however, consider new transformations that have not only (2.1) and the monotoneity but also a theoretical background described below for a general setup in (1.1) where the limiting distribution of S is not always assumed $N(0, 1)$ or χ_f^2.

Let x_α and u_α be the upper α points of F and G, respectively. Then, we note that

$$x_\alpha = F^{-1}(G(u_\alpha)). \tag{2.2}$$

Now we define that

$$T^* = G^{-1}(F(S)).$$

From (2.2), since it follows for all $\alpha \in (0,1)$ that

$$\begin{aligned} \mathrm{P}(T^* \le u_\alpha) &= \mathrm{P}(S \le F^{-1}(G(u_\alpha))) \\ &= \mathrm{P}(S \le x_\alpha) \\ &= 1 - \alpha, \end{aligned}$$

we consider T^* as an exact transformation to a limiting distribution G. If $F(x)$ is completely known, T^* is available. As far as $F(x)$ is available in a form (1.1), we have to replace $F(x)$ with

$$\tilde{F}(x) = G(x) + \epsilon h(x) g(x). \tag{2.3}$$

However, $\tilde{F}(x)$ does not hold the monotoneity. Now, we modify $\tilde{F}(x)$ so as to hold the monotoneity and construct a transformation with its modified function. Though Yanagihara and Tonda (2003) proposed such a modification of $\tilde{F}(x)$ in the case that the limiting distribution is χ^2_f, their idea can be extended to the general case at hand as follows:

$$\hat{F}(x) = d^{-1} \left\{ \tilde{F}(x) + \frac{\epsilon^2}{4} \int_{-\infty}^{x} g(t)\{a(t)\}^2 dt \right\}, \tag{2.4}$$

where

$$d = \lim_{x \to \infty} \left\{ \tilde{F}(x) + \frac{\epsilon^2}{4} \int_{-\infty}^{x} g(t)\{a(t)\}^2 dt \right\}, \tag{2.5}$$

$$a(x) = \frac{1}{g(x)} \left(\frac{d}{dx} h(x) g(x) \right). \tag{2.6}$$

Then, $\hat{F}(x)$ is monotone and $\lim_{x\to\infty} \hat{F}(x) = 1$. With a monotone function $\hat{F}(x)$, we propose a new transformation:

$$T_1 = G^{-1}(\hat{F}(S)). \tag{2.7}$$

Theorem 2.1. *Suppose that a statistic S can be expanded as in* (1.1)*. Then, for a monotone transformation $T_1 = T_1(S)$ defined by* (2.7) *with* (2.4)*, it holds that*

$$\mathrm{P}(T_1 \le x) = G(x) + O(\epsilon^2).$$

Proof. As for a function $\tilde{F}(x)$ defined by (2.3), note that $d = 1 + O(\epsilon^2)$ in (2.5). Hence, we have that $\hat{F}(x) = F(x) + O(\epsilon^2)$. Therefore, since T_1 can be expanded as in (2.1), we get the desired result from Lemma 2.1. ■

Next, we consider another adjustment to $\tilde{F}(x)$ under the assumption that $a(x)$ given by (2.6) is a polynomial of a certain degree. That assumption is correct at least for the case that the limiting distribution of S is a normal distribution or a chi-squared distribution:

$$F_*(x) = \int_{-\infty}^{x} g(t) \exp\left(\epsilon a(t) - \epsilon^2 p(t)\right) dt, \tag{2.8}$$

where $a(x)$ is given by (2.6) and $p(x)$ is a polynomial such that $\deg[a(x)]+1 \leq \deg[p(x)] < \infty$ and $p(x) \to +\infty$ as $x \to -\infty$ or ∞. Then, $F_*(x)$ is monotone and $F_*(x) = F(x) + O(\epsilon^2)$. With a monotone function $F_*(x)$, we propose another new transformation:

$$T_2 = G^{-1}(F_*(S)). \tag{2.9}$$

Theorem 2.2. *Suppose that a statistic S can be expanded as in* (1.1)*. Then, for a monotone transformation $T_2 = T_2(S)$ defined by* (2.9) *with* (2.8)*, it holds that*

$$\mathrm{P}(T_2 \leq x) = G(x) + O(\epsilon^2).$$

Proof. The claim is proved similarly to Theorem 2.1. ■

We can see an advantage of T_2 in the following case. Suppose that a statistic S can be expanded as

$$F(x) = \mathrm{P}(S \leq x) = G(x) + \epsilon h_1(x)g(x) + \cdots + \epsilon^{r-1} h_{r-1}(x)g(x) + O(\epsilon^r), \tag{2.10}$$

where $h_i(x)$ is a polynomial of degree k_i. The form (1.1) is the case that $r = 2$. Then, we consider a monotone transformation $T = T(S)$ satisfying

$$\mathrm{P}(T \leq x) = G(x) + O(\epsilon^r). \tag{2.11}$$

For such a monotone function T, it holds that

$$\mathrm{P}(S \leq b(u_\alpha)) = \mathrm{P}(T(S) \leq u_\alpha) = 1 - \alpha + O(\epsilon^r),$$

where u_α is the upper α point of G and $b(\cdot)$ is the inverse function of T. In the case that the limiting distribution is χ^2_f, Kakizawa (1996) proposed a transformation satisfying (2.11). However, his method requires a quite complex calculation to come to a transformation. Now, let us apply transformation T_2 to this case as follows: Let

$$a_i(x) = \frac{1}{g(x)} \left(\frac{d}{dx} h_i(x) g(x) \right).$$

Here, we assume that $a_i(x)$ is a polynomial of a certain degree. Under mild additional assumptions on the smoothness of the underlying distribution, we have that

$$F'(x) = g(x) \left\{ 1 + \epsilon a_1(x) + \epsilon^2 a_2(x) + \cdots + \epsilon^{r-1} a_{r-1}(x) + O(\epsilon^r) \right\}.$$

Define $F_*(x)$ by

$$F_*(x) = \int_{-\infty}^{x} g(t) \exp\left\{ \epsilon a(t) - \frac{\epsilon^2}{2}\{a(t)\}^2 + \cdots \right. \\ \left. \cdots + \frac{(-1)^r \epsilon^{r-1}}{r-1}\{a(t)\}^{r-1} - \epsilon^r p(t) \right\} dt, \tag{2.12}$$

where

$$a(x) = a_1(x) + \epsilon a_2(x) + \cdots + \epsilon^{r-2} a_{r-1}(x),$$

and $p(x)$ is a polynomial such that $\deg[\{a(x)\}^{r-1}] + 1 \le \deg[p(x)] < \infty$ and $p(x) \to +\infty$ as $x \to -\infty$ or ∞. Then, $F_*(x)$ is monotone. By using the relation $\exp(x - x^2/2 + \cdots + (-1)^r x^{r-1}/(r-1)) = 1 + x + O(x^r)$, we have that $F_*(x) = F(x) + O(\epsilon^r)$.

Theorem 2.3. *Suppose that a statistic S can be expanded as in* (2.10). *Then, for a monotone transformation $T_2 = T_2(S)$ defined by* (2.9) *with* (2.12), *it holds that*

$$\mathrm{P}(T_2 \le x) = G(x) + O(\epsilon^r).$$

In order to prove Theorem 2.3, we give the following lemma. When the limiting distribution is χ^2_f, it was given by Kakizawa (1997).

Lemma 2.2. *Suppose that a statistic S has an asymptotic expansion* (2.10). *Let $b(x)$ be the inverse function of the transformation $T = T(S)$ such that $b(x) = x + O(\epsilon)$. If and only if $b(x)$ is coincident with the Cornish-Fisher type expansion for $F^{-1}(G(x))$ up to the order $O(\epsilon^{r-1})$, then property* (2.11) *is satisfied.*

Proof. Let $\tilde{b}(x)$ be the one formed by the terms up to the order $O(\epsilon^{r-1})$ in the Cornish-Fisher type expansion for $F^{-1}(G(x))$. Then,

$$\begin{aligned} F(\tilde{b}(x)) &= F\left(F^{-1}(G(x)) - (F^{-1}(G(x)) - \tilde{b}(x))\right) \\ &= F\left(F^{-1}(G(x))\right) - F'\left(F^{-1}(G(x))\right)\left(F^{-1}(G(x)) - \tilde{b}(x)\right) + \cdots \\ &= G(x) + O(\epsilon^r). \end{aligned} \tag{2.13}$$

On the other hand,

$$F(b(x)) = F(\tilde{b}(x)) + F'(\tilde{b}(x))(b(x) - \tilde{b}(x)) \\ + \frac{1}{2}F''(\tilde{b}(x))(b(x) - \tilde{b}(x))^2 + \cdots . \tag{2.14}$$

It is easy to see that $\tilde{b}(x) = x + O(\epsilon)$. Noting that $b(x) = x + O(\epsilon)$, we get $b(x) - \tilde{b}(x) = O(\epsilon)$. In the case that $F(b(x)) = G(x) + O(\epsilon^r)$, from (2.13) and (2.14), we obtain $b(x) - \tilde{b}(x) = O(\epsilon^r)$. ■

Proof of Theorem 2.3. Since $F_*(x) = F(x) + O(\epsilon^r)$, we get $F_*(F^{-1}(x)) = x + O(\epsilon^r)$. On the other hand,

$$\begin{aligned} x &= F_*(F^{-1}(x) + F_*^{-1}(x) - F^{-1}(x)) \\ &= F_*(F^{-1}(x)) + F_*'(\xi)(F_*^{-1}(x) - F^{-1}(x)), \end{aligned}$$

where $\xi = F^{-1}(x) + \theta(F_*^{-1}(x) - F^{-1}(x))$ with $0 \le \theta \le 1$. From these facts and that $F_*'(x) = F'(x) + O(\epsilon^r)$, we get $F_*^{-1}(x) = F^{-1}(x) + O(\epsilon^r)$. Therefore, Theorem 2.3 is proved by Lemma 2.2. ■

Remark 2.1. The integrability of $F_*(x)$ in (2.12) is proved as follows:

When $x < 0$ and m is even, we have that

$$\exp(x) \le 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{m!}x^m.$$

Let

$$q(x) = \epsilon a(x) - \frac{\epsilon^2}{2}\{a(x)\}^2 + \cdots + \frac{(-1)^r \epsilon^{r-1}}{r-1}\{a(x)\}^{r-1} - \epsilon^r p(x).$$

From the assumption of $p(x)$, there exists a positive number x_0 such that $q(x) < 0$ for any x satisfying $|x| > x_0$. Therefore, we have for any x satisfying $|x| > x_0$ that

$$\exp\{q(x)\} \le 1 + q(x) + \frac{1}{2!}\{q(x)\}^2 + \frac{1}{3!}\{q(x)\}^3 + \cdots + \frac{1}{m!}\{q(x)\}^m (\equiv Q(x)).$$

By using this fact,

$$\begin{aligned} F_*(x) &= \int_{-\infty}^{x} g(t)\exp\{q(t)\}dt \\ &\le \int_{-\infty}^{\min\{x,-x_0\}} g(t)Q(t)dt + \int_{\min\{x,-x_0\}}^{\min\{x,x_0\}} g(t)\exp\{q(t)\}dt \\ &\qquad + \int_{\min\{x,x_0\}}^{x} g(t)Q(t)dt \\ &< \infty. \end{aligned}$$

In addition, by using the relation $\exp(x - x^2/2 + \cdots + (-1)^r x^{r-1}/(r-1)) = 1 + x + O(x^r)$, we have for even number $m \ge r$ that

$$Q(x) = 1 + \epsilon a(x) + \epsilon^r R(x),$$

where $R(x)$ is a polynomial of finite degree and that $R(x) = O(1)$.

Remark 2.2. In (2.12), we assume that $a_i(x)$ is a polynomial. It should be noted that this assumption is not necessary when $F_*(x)$ in (2.12) is integrable.

§3. The method of moments

In Section 2, we assume that $h_i(x)$'s in (2.10) are available in an explicit form. However, we often encounter the situation that it is difficult to access to the polynomials $h_i(x)$'s, even though its existence is assured. Such a situation appears in treating the distribution of a multivariate test statistic under nonnormality. In order to overcome this difficulty, we give a method for obtaining the transformations, proposed in Section 2, from the moments of a statistic S. In order to illustrate the idea explicitly, we focus on a case that the limiting distribution is a chi-squared distribution. The method used here is an extension of Cordeiro and Ferrari (1998). In general, the problem of deriving moments is more tractable than the one of deriving the asymptotic expansion (2.10). It would lead a broad application with a wide class of statistics, especially under nonnormality, where their asymptotic expansions are quite difficult to access.

Suppose that a nonnegative statistic S can be expanded as

$$\begin{aligned}(3.1)\quad F(x) &= \mathrm{P}(S \leq x)\\ &= G_f(x) + \frac{1}{n}h_1(x)g_f(x) + \cdots + \frac{1}{n^{r-1}}h_{r-1}(x)g_f(x) + O(n^{-r}),\end{aligned}$$

where $h_i(x)$ is a polynomial of degree $i \times k$ without constant terms. Here $G_f(\cdot)$ and $g_f(\cdot)$ denote, respectively, the cumulative distribution and density functions of a central chi-squared random variable with f degrees of freedom, χ_f^2. If x and u are corresponding quantiles of F and G_f respectively, then

$$F(x) = G_f(u)$$

and it is required to solve this equation for x in terms of u.

Lemma 3.1. *Suppose that a nonnegative statistic S has an asymptotic expansion* (3.1). *Let x and u be corresponding quantiles of F and G_f respectively. Then, x and u satisfy the following relation:*

$$(3.2)\quad x = u + \frac{1}{n}\sum_{j=1}^{k}\beta_{1,j}u^j + \frac{1}{n^2}\sum_{j=1}^{2k}\beta_{2,j}u^j + \cdots + \frac{1}{n^{r-1}}\sum_{j=1}^{(r-1)k}\beta_{r-1,j}u^j + O(n^{-r}),$$

where $\beta_{i,j}$'s are constants.

Proof. Let

$$D_u \equiv d/du, \qquad \psi(u) = -g_f'(u)/g_f(u) = \frac{1}{2} - \left(\frac{f}{2} - 1\right)\frac{1}{u},$$

$$z_n(x) = \frac{1}{n}h_1(x) + \frac{1}{n^2}h_1(x) + \cdots + \frac{1}{n^{r-1}}h_{r-1}(x).$$

Then, from Hill and Davis (1968), we have that

$$x = u - \sum_{i=1}^{r-1} D_{(i)}(z_n(u))^i/i! + O(n^{-r}), \tag{3.3}$$

where $D_{(1)}$ denotes the identity operator and

$$D_{(i)} = (\psi(u) - D_u)(2\psi(u) - D_u)\cdots((i-1)\psi(u) - D_u), \quad i = 2, 3, \cdots, r-1.$$

In (3.3), the term of order n^{-i}, $q_i(u)$ (say), can be expressed as

$$q_i(u) = -\sum_{j=1}^{i}\sum_{M_i(j)} \frac{D_{(j)}}{j!}\frac{j!}{m_1!m_2!\cdots m_i!}\{h_1(u)\}^{m_1}\{h_2(u)\}^{m_2}\cdots\{h_i(u)\}^{m_i}, \tag{3.4}$$

where $M_i(j) = \{(m_1, m_2, \cdots, m_i) \in \boldsymbol{Z}_+^i; m_1 + 2m_2 + \cdots + im_i = i, m_1 + m_2 + \cdots + m_i = j\}$. Since $h_i(u)$ is a polynomial of u and $\deg\{h_i(u)\} = ik$, we obtain that

$$\deg\{h_1(u)\}^{m_1}\{h_2(u)\}^{m_2}\cdots\{h_i(u)\}^{m_i} = (m_1 + 2m_2 + \cdots + im_i)k = ik.$$

On the other hand, since $h_i(u)$ does not have constant terms, $\{h_1(u)\}^{m_1}$ $\{h_2(u)\}^{m_2}$ $\cdots$ $\{h_i(u)\}^{m_i}$ is a polynomial of lowest degree i. Therefore, we can see that $q_i(u)$ is a polynomial of degree ik without constant terms. ■

Next, we give a method for obtaining the quantities $\beta_{i,j}$ without using (3.1). It is assumed that the quantity k that appears in (3.1) is known. For instance, if the statistic of interest is a likelihood ratio statistic (a score statistic) in particular, we know that $k = 1$ ($k = 3$). There are situations, however, that the quantity k is not known. Such situations will be addressed at the end of this section.

Let

$$\mathrm{E}[S^i] = \mu_i'\left(1 + \frac{c_{i,1}}{n} + \frac{c_{i,2}}{n^2} + \cdots + \frac{c_{i,r-1}}{n^{r-1}}\right) + O(n^{-r}), \qquad (i = 1, 2, \cdots),$$

where $c_{i,j}$'s are constants and $\mu_i' = \mathrm{E}[(\chi_f^2)^i] = f(f+2)\cdots(f+2(i-1))$. Cordeiro and Ferrari (1998) expressed quantities $\beta_{1,1}, \beta_{1,2}, \cdots, \beta_{1,k}$ with $c_{1,1}$, $c_{2,1}$, $\cdots$, $c_{k,1}$. Here, we consider the case of higher order.

Theorem 3.1. *Suppose that x and u satisfy the relation* (3.2). *Let $\boldsymbol{\beta}_i = (\beta_{i,1}, \beta_{i,2}, \cdots, \beta_{i,ik})'$ and $\boldsymbol{c}_j = (c_{1,j}, c_{2,j}, \cdots, c_{jk,j})'$. Then, for $i = 1, 2, \cdots, r-1$, $\boldsymbol{\beta}_i$ can be expressed with $\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_i$.*

Proof. From (3.2), we have that

$$S = \chi_f^2 + \frac{1}{n}\sum_{j=1}^{k}\beta_{1,j}(\chi_f^2)^j + \frac{1}{n^2}\sum_{j=1}^{2k}\beta_{2,j}(\chi_f^2)^j + \cdots$$
$$\cdots + \frac{1}{n^{r-1}}\sum_{j=1}^{(r-1)k}\beta_{r-1,j}(\chi_f^2)^j + O_p(n^{-r}).$$

From this, we have that

$$S^m = \sum_{m_0+m_1+\cdots+m_{r-1}=m}\frac{m!}{m_0!m_1!\cdots m_{r-1}!}(\chi_f^2)^{m_0}\left\{\frac{1}{n}\sum_{j=1}^{k}\beta_{1,j}(\chi_f^2)^j\right\}^{m_1}\cdots$$
$$\cdots\left\{\frac{1}{n^{r-1}}\sum_{j=1}^{(r-1)k}\beta_{r-1,j}(\chi_f^2)^j\right\}^{m_{r-1}} + O_p(n^{-r})$$
$$= (\chi_f^2)^m + \sum_{i=1}^{r-1}\sum_{M_i'}\frac{1}{n^i}\frac{m!}{m_0!m_1!\cdots m_i!}(\chi_f^2)^{m_0}\left\{\sum_{j=1}^{k}\beta_{1,j}(\chi_f^2)^j\right\}^{m_1}\cdots$$
$$\cdots\left\{\sum_{j=1}^{ik}\beta_{i,j}(\chi_f^2)^j\right\}^{m_i} + O_p(n^{-r}),$$

where $M_i' = \{(m_0, m_1, \cdots, m_i) \in \boldsymbol{Z}_+^{i+1}; m_1 + 2m_2 + \cdots + im_i = i, m_0 + m_1 + \cdots + m_i = m\}$. Therefore, we obtain that

$$c_{m,i} = \frac{1}{\mu_m'}\sum_{M_i'}\frac{m!}{m_0!m_1!\cdots m_i!}\mathrm{E}\Bigg[(\chi_f^2)^{m_0}\left\{\sum_{j=1}^{k}\beta_{1,j}(\chi_f^2)^j\right\}^{m_1}\cdots$$
$$\cdots\left\{\sum_{j=1}^{ik}\beta_{i,j}(\chi_f^2)^j\right\}^{m_i}\Bigg]$$
$$= \frac{m}{\mu_m'}\mathrm{E}\left[(\chi_f^2)^{m-1}\left\{\sum_{j=1}^{ik}\beta_{i,j}(\chi_f^2)^j\right\}\right] + B_{m,i}(f, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_{i-1}),$$

where $B_{m,i}(f, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_{i-1})$ is an appropriate function depending on f, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\cdots$, $\boldsymbol{\beta}_{i-1}$. From this, it holds that

$$\sum_{j=1}^{ik}\beta_{i,j}\mu_{m+j-1}' = \frac{\mu_m'}{m}\left\{c_{m,i} - B_{m,i}(f, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_{i-1})\right\} \tag{3.5}$$
$$= \frac{\mu_m'}{m}\tilde{c}_{m,i} \qquad \text{(say)}.$$

Since $\mu'_{m+j-1} = f(f+2)\cdots(f+2(m+j-2))$, we can write $\mu'_{m+j-1} = \mu'_m(f+2m)^{(j-1)}$, where $(s)^{(t)} = s(s+2)\cdots(s+2(t-1))$ for $t = 1,2,\cdots$ and $(s)^{(0)} = 1$. Then, it follows that the system of equations (3.5) can be written as

$$\sum_{j=1}^{ik} \beta_{i,j}(f+2m)^{(j-1)} = \frac{\tilde{c}_{m,i}}{m}, \qquad m = 1,2,\cdots,ik$$

or in matrix form as

$$\boldsymbol{P}_i\boldsymbol{\beta}_i = \tilde{\boldsymbol{c}}_i, \tag{3.6}$$

where $\boldsymbol{P}_i = (p_{i'j'})$ with $p_{i'j'} = (f+2i')^{(j'-1)}$, $i', j' = 1,2,\cdots,ik$ and $\tilde{\boldsymbol{c}}_i = \left(\tilde{c}_{1,i}, 2^{-1}\tilde{c}_{2,i},\cdots,(ik)^{-1}\tilde{c}_{ik,i}\right)'$. Similarly to a method by Cordeiro and Ferrari (1998), it is proved that the system of linear equations in (3.6) has a unique solution. Since, for $i=1$, the theorem has been proved by Cordeiro and Ferrari (1998), the theorem is proved for all $i = 1,2,\cdots,r-1$ by using induction in (3.6). ■

Theorem 3.2. *Suppose that a nonnegative statistic S has an asymptotic expansion* (3.1). *Let x and u be corresponding quantiles of F and G_f respectively. Then, for $i = 1,2,\cdots,r-1$, $h_i(x)$ can be expressed with $\boldsymbol{c}_1,\boldsymbol{c}_2,\cdots,\boldsymbol{c}_i$.*

Proof. From Lemma 3.1, x and u satisfy the relation (3.2). Then, from Theorem 3.1, for $i = 1,2,\cdots,r-1$, $\boldsymbol{\beta}_i$ can be expressed with $\boldsymbol{c}_1,\boldsymbol{c}_2,\cdots,\boldsymbol{c}_i$. For $q_i(u)$ defined in (3.4), it follows that

$$\sum_{j=1}^{ik} \beta_{i,j}u^j = q_i(u),$$

and we can write it as

$$q_i(u) = -h_i(u) + H_i(h_1,h_2,\cdots,h_{i-1})$$

where $H_i(h_1,h_2,\cdots,h_{i-1})$ is an appropriate function depending on $h_1(u)$, $h_2(u)$, $\cdots$, $h_{i-1}(u)$. For $i=1$, the theorem has been proved by Cordeiro and Ferrari (1998). Therefore, the theorem is proved for all $i = 1,2,\cdots,r-1$ by using induction. ■

As mentioned before, the results described above are proved on the assumption that k is known. When this is not the case, one may suppose a positive integer, k_0 say, instead of unknown k. Let us write equation (3.6) for k and k_0 that

$$\boldsymbol{P}_{ik}\boldsymbol{\beta}_{ik} = \tilde{\boldsymbol{c}}_{ik},$$
$$\boldsymbol{P}_{ik_0}\hat{\boldsymbol{\beta}}_{ik_0} = \hat{\tilde{\boldsymbol{c}}}_{ik_0},$$

respectively. Here, indices express the size of matrix or vector. When $k_0 \leq k$, it has that

$$\boldsymbol{P}_{ik} = \begin{pmatrix} \boldsymbol{P}_{ik_0} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{pmatrix},$$

where $\boldsymbol{Q}_\sharp$'s are appropriate matrices. Note that $\hat{\boldsymbol{\beta}}_{ik_0} = (\beta_{i,1}, \beta_{i,2}, \cdots, \beta_{i,ik_0})'$ for $i = 1$. When $k_0 > k$, it has that

$$\boldsymbol{P}_{ik_0} \begin{pmatrix} \boldsymbol{\beta}_{ik} \\ \mathbf{0} \end{pmatrix} = \tilde{\boldsymbol{c}}_{ik_0},$$

where $\mathbf{0}$ is a zero vector with size $i(k_0 - k)$. Note that

$$\hat{\boldsymbol{\beta}}_{ik_0} = \begin{pmatrix} \boldsymbol{\beta}_{ik} \\ \mathbf{0} \end{pmatrix}$$

for $i = 1$. By using induction for $i = 1, 2, \cdots, r-1$, we can get the same results as in Theorems 3.1 and 3.2.

§4. Accuracy of improved asymptotic approximations

Let $F(x)$ be the distribution function of a statistic S depending on some parameter ϵ, not necessary the inverse of a sample size. A typical form of the asymptotic expansion of $F(x)$ around the limiting distribution $G(x)$ of $F(x)$ as $\epsilon \to 0$ is

$$F(x) = \mathrm{P}(S \leq x) = G(x) + g(x) \sum_{j=1}^{r-1} h_j(x)\epsilon^j + R_r(x), \tag{4.1}$$

where $g(x)$ is the density function of $G(x)$ and $h_j(x)$'s are suitable polynomials. When $F(x)$ is approximated by a function of the form

$$\tilde{F}_r(x) = G(x) + g(x) \sum_{j=1}^{r-1} h_j(x)\epsilon^j,$$

it is well known that the error $R_r(x) = F(x) - \tilde{F}_r(x)$ satisfies

$$R_r(x) = O(\epsilon^r)$$

uniformly in x under suitable regularity conditions (see, e.g., Bhattacharya and Ghosh (1978)). This means that there exists a positive constant c_r such that for small ϵ

$$|R_r(x)| \leq \epsilon^r c_r. \tag{4.2}$$

Such c_r and ϵ have been obtained on some special statistics (see, e.g., Fujikoshi (1993), Shimizu and Fujikoshi (1997), Fujikoshi, Ulyanov and Shimizu (2005a,b) and Fujikoshi and Ulyanov (2006)). In this section, our aim is to obtain an error bound for a monotone transformation $T = T(S)$ satisfying property (2.11).

From Section 2, as for an approximation to $F(x)$, let us start with a monotone function $\hat{F}(x)$ expanded as

$$\hat{F}(x) = \tilde{F}_r(x) + R_r^*(x), \tag{4.3}$$

where $R_r^*(x) = O(\epsilon^r)$. For the remainder term $R_r^*(x)$ in (4.3), we assume that there exists a positive constant c_r^* such that

$$|R_r^*(x)| \leq \epsilon^r c_r^*. \tag{4.4}$$

With a monotone function $\hat{F}(x)$, we consider a monotone transformation $T = T(S)$ defined by

$$T = G^{-1}(\hat{F}(S)), \tag{4.5}$$

which satisfies property (2.11).

Theorem 4.1. *If* $\mathrm{P}(S \leq x)$ *can be written in the form* (4.1) *with* (4.2)*, then*

$$|\mathrm{P}(T(S) \leq x) - G(x)| \leq \epsilon^r \tilde{c}_r, \tag{4.6}$$

where T *is a monotone transformation defined by* (4.5) *and* $\tilde{c}_r$ *is a positive constant depending on* c_r *in* (4.2) *and* c_r^* *in* (4.4).

Proof. Let $b(x)$ be the inverse function to T. Then, $b(x)$ can be written as

$$b(x) = \hat{F}^{-1}(G(x)). \tag{4.7}$$

From (4.1), (4.2), (4.4) and (4.7), we get

$$\begin{aligned}
&|\mathrm{P}(T(S) \leq x) - G(x)| = |\mathrm{P}(S \leq b(x)) - G(x)| = |F(b(x)) - \hat{F}(b(x))| \\
&= |R_r(b(x)) - R_r^*(b(x))| \\
&\leq \epsilon^r (c_r + c_r^*).
\end{aligned}$$

By defining

$$\tilde{c}_r = c_r + c_r^*,$$

we obtain (4.6). ■

Remark 4.1. It should be noted that a monotone function $F_*(x)$ in (2.12) satisfies (4.3). In view of Remark 2.1, by noting that

$$\tilde{F}_r(x) = \int_{-\infty}^{x} g(t)(1 + \epsilon a(t))dt,$$

$F_*(x)$ in (2.12) also satisfies (4.4).

Remark 4.2. The case that the error bound in (4.2) or (4.4) is non-uniform can be treated as well.

Remark 4.3. When $r = 2$ and the limiting distribution of S is a chi-squared distribution in (4.1), Enoki and Aoshima (2004) gave sharper error bounds in a way different from here. They applied an improved chi-squared approximation to the asymptotic distribution given by Siotani (1956) and examined its accuracy along with its uniform or non-uniform error bound.

§5. Applications

Here, we consider transformations (2.7) and (2.9) for some test statistics and examine their accuracy of the approximation to the true percentile point x_α of S. We conducted simulation experiments as follows: For parameters given in advance, the approximate percentile point was calculated for each monotone transformation. By using these percentile points, we conducted the Monte Carlo simulation with 100,000 (= R, say) independent trials for a test statistic. Let s_r ($r = 1, ..., R$) be an observed value of S and $p_r = 1$ (or 0) if s_r is (or is not) larger than the approximate percentile point. On the other hand, let $s_{[1]} \leq s_{[2]} \leq \cdots \leq s_{[R]}$ be the ordered values of s_r and let us define $s_{[(1-\alpha)R]}$ as an observed value of x_α. We briefly write it x_α. Let $\bar{p} = 100 \sum_{r=1}^{R} p_r/R$ which estimates the test size ($100\alpha\%$) with its estimated standard error $s(\bar{p}) = 100\sqrt{(\bar{p}/100)(1 - \bar{p}/100)/R}$. In addition, with respect to transformations (2.7) and (2.9), we conducted numerical integrations with *Mathematica*.

Example 5.1 Let $S = (n - q)s_h^2/s_e^2$ be a test statistic for testing the equality of means of q nonnormal populations Π_i ($i = 1, ..., q$) with common variance. Here, s_h^2 and s_e^2 are the sums of squares due to the hypothesis and the error, respectively, based on the sample of the size n_i from Π_i. Let $\rho_i = \sqrt{n_i/n}$, where n is the total sample size. Assume that $\rho_i = O(1)$ as n_j's tend to infinity. Let κ_3 and κ_4 be the third and the fourth cumulants of the standardized variate. Then, under a general condition, an asymptotic expansion for the null distribution of S was given by Fujikoshi, Ohmae and

Yanagihara (1999) in the form (1.1) with $G = \chi^2_f$, $\epsilon = n^{-1}$, $k = 3$, $f = q - 1$, and $h(x)$ given by

$$h(x) = -2\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\frac{a_j}{\prod_{\ell=0}^{i-1}(f+2\ell)}x^i\right),$$

$$a_1 = -\frac{1}{2}(q-1)^2 + 3d_1\kappa_3^2 - 2d_2\kappa_4, \qquad a_2 = \frac{1}{4}(q^2-1) - 3d_1\kappa_3^2 + d_2\kappa_4,$$

$$a_3 = d_1\kappa_3^2,$$

where

$$d_1 = \frac{5}{24}\left(\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right) + \frac{1}{12}(q-1)(q-2), \quad d_2 = \frac{1}{8}\left(\sum_{j=1}^{q}\frac{n}{n_j} - q^2\right) - \frac{1}{4}(q-1).$$

We examined performance of the proposed transformations under the following two nonnormal models:

(i) χ^2 distribution with 4 degrees of freedom;

(ii) Gamma distribution with shape parameter 3 and scale parameter 1/3.

Table 5.1 gives the true percentile point x_α and the approximate percentile points $t_E(u)$, $t_{AEI}(u)$, $t_K(u)$, $t_1(u)$ and $t_2(u)$ for the case $q = 3$. Here, u denotes the upper 5% point of χ^2_2 and $t_E(u)$ is computed from the Cornish-Fisher type expansion up to the order $O(n^{-1})$. On the other hand, $t_{AEI}(u)$, $t_K(u)$, $t_1(u)$ and $t_2(u)$ are computed from Aoshima, Enoki and Ito (2003), Kakizawa (1996), (2.7) and (2.9), respectively. Since the Cornish-Fisher type expansion and the others yield the same approximation up to the order $O(n^{-1})$, the transformations T aim to find an improvement of approximations to x_α in the terms of $O(n^{-2})$.

Table 5.1 The percentile points when $q = 3$

	Sample sizes			Upper 5% points ($\chi^2_2(0.05) = 5.9915$)					
	n_1	n_2	n_3	x_α	$t_E(u)$	$t_{AEI}(u)$	$t_K(u)$	$t_1(u)$	$t_2(u)$
(i)	5	5	5	7.455	6.823	7.012	6.986	6.924	7.566
	10	10	10	6.501	6.407	6.449	6.443	6.433	6.528
	3	6	6	7.521	6.815	7.116	7.033	7.025	7.411
	5	5	10	6.916	6.609	6.759	6.720	6.723	6.884
(ii)	5	5	5	7.367	6.945	7.177	7.231	7.143	7.378
	10	10	10	6.375	6.468	6.519	6.528	6.517	6.545
	3	6	6	7.460	6.939	7.228	7.278	7.226	7.330
	5	5	10	6.918	6.702	6.849	6.870	6.859	6.887

100,000 replications

Table 5.2 The actual test sizes when $q = 3$

	Sample sizes			Nominal 5% test					
	n_1	n_2	n_3	α_1	α_2	α_3	α_4	α_5	α_6
(i)	5	5	5	8.077	6.084	5.736	5.782	5.895	4.798
				0.086	0.076	0.074	0.074	0.074	0.068
	10	10	10	6.200	5.191	5.100	5.112	5.131	4.941
				0.076	0.070	0.070	0.070	0.070	0.069
	3	6	6	8.260	6.260	5.690	5.835	5.849	5.172
				0.087	0.077	0.073	0.074	0.074	0.070
	5	5	10	7.151	5.621	5.303	5.389	5.383	5.061
				0.081	0.073	0.071	0.071	0.071	0.069
(ii)	5	5	5	7.958	5.719	5.290	5.199	5.370	4.984
				0.086	0.073	0.071	0.070	0.071	0.069
	10	10	10	5.917	4.814	4.696	4.670	4.703	4.640
				0.075	0.068	0.067	0.067	0.067	0.067
	3	6	6	8.119	5.921	5.379	5.290	5.381	5.196
				0.086	0.075	0.071	0.071	0.071	0.070
	5	5	10	7.067	5.426	5.144	5.100	5.124	5.067
				0.081	0.072	0.070	0.070	0.070	0.069

100,000 replications

As for the actual test sizes, Table 5.2 gives values of $\bar{p}$ ($s(\bar{p})$), on the first (second) line in each cell, for each monotone transformation. Here, the actual test sizes are defined by

$$\alpha_1 = \mathrm{P}(T > u), \quad \alpha_2 = \mathrm{P}(T > t_E(u)), \quad \alpha_3 = \mathrm{P}(T > t_{AEI}(u)),$$
$$\alpha_4 = \mathrm{P}(T > t_K(u)), \quad \alpha_5 = \mathrm{P}(T > t_1(u)), \quad \alpha_6 = \mathrm{P}(T > t_2(u))$$

for the case that $q = 3$. For the transformation given by (2.9), we consider the following case:

$$a(x) = \sum_{j=1}^{3} a_j \left\{ \frac{x^j}{\prod_{\ell=0}^{j-1}(f + 2\ell)} - 1 \right\}, \quad p(x) = \begin{cases} 2\{a(x)\}^2 & \text{in (i)} \\ -\frac{4}{5}a(x) + \frac{8}{5}\{a(x)\}^2 & \text{in (ii)} \end{cases}.$$

From Tables 5.1 and 5.2, we can see that the transformation given by (2.9) gives a most significant improvement for the approximate percentile point among the others. Note that transformation given by (2.9) is affected by the function $p(x)$. As for a choice of $p(x)$, it is under investigation. Note that the value of $t_1(u)$ is close to $t_{AEI}(u)$ or $t_K(u)$. It seems that transformation given by (2.7) does not make a significant difference from the predecessors.

Example 5.2 Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be independently and identically distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $\overline{\boldsymbol{X}} = \frac{1}{n}\sum_{j=1}^n \boldsymbol{X}_j$ and $\boldsymbol{S} = \frac{1}{\nu}\sum_{j=1}^n (\boldsymbol{X}_j - \overline{\boldsymbol{X}})(\boldsymbol{X}_j - \overline{\boldsymbol{X}})'$ where $\nu = n-1 \geq p$. Then, Hotelling's T^2-statistic is defined by

$$T^2 = n(\overline{\boldsymbol{X}} - \boldsymbol{\mu})'\boldsymbol{S}^{-1}(\overline{\boldsymbol{X}} - \boldsymbol{\mu}).$$

The statistic is used for testing hypotheses about the mean vector $\boldsymbol{\mu}$ or for estimating confidence regions for the unknown $\boldsymbol{\mu}$. Let us put $S = T^2$. Then, an asymptotic expansion for the distribution of S was given by Siotani (1971) as follows:

$$F(x) \equiv \mathrm{P}(S \leq x) = G_p(x) + \frac{1}{\nu}\sum_{j=0}^{2} a_{1j}G_{p+2j}(x) + \frac{1}{\nu^2}\sum_{j=0}^{4} a_{2j}G_{p+2j}(x) + O(\nu^{-3}),$$

where $G_{p+2j}(\cdot)$ is a distribution function of χ^2_{p+2j} and

$$\begin{aligned}
&a_{10} = -\frac{p^2}{4}, \quad a_{11} = -\frac{p}{2}, \quad a_{12} = \frac{1}{4}p(p+2), \quad a_{20} = \frac{1}{96}p(3p^3 - 8p^2 + 8),\\
&a_{21} = \frac{p^3}{8}, \quad a_{22} = -\frac{1}{16}p(p+2)(p^2-6), \quad a_{23} = -\frac{7}{24}p(p+2)(p+4),\\
&a_{24} = \frac{1}{32}p(p+2)(p+4)(p+6).
\end{aligned}$$

Let

$$a_1(x) = a_{10} + \sum_{j=1}^{2} a_{1j}\frac{x^j}{\prod_{\ell=0}^{j-1}(p+2\ell)} \quad \text{and} \quad a_2(x) = a_{20} + \sum_{j=1}^{4} a_{2j}\frac{x^j}{\prod_{\ell=0}^{j-1}(p+2\ell)}.$$

Let $\tilde{F}_i(x)$ be the one formed by the terms of $F(x)$ up to the order $O(n^{-i})$. We examined performance of the proposed transformation given by (2.9) with the following setup:

$$\begin{aligned}
&(1) \quad a(x) = a_1(x),\ p(x) = \frac{1}{2}\{a(x)\}^2;\\
&(2) \quad a(x) = a_1(x),\ p(x) = \frac{1}{2}\{a(x)\}^2 - \frac{1}{3\nu}\{a(x)\}^3 + \frac{1}{4\nu^2}\{a(x)\}^4;\\
&(3) \quad a(x) = a_1(x) + \frac{1}{\nu}a_2(x),\ p(x) = \frac{1}{2}\{a(x)\}^2;\\
&(4) \quad a(x) = a_1(x) + \frac{1}{\nu}a_2(x),\ p(x) = \frac{1}{2}\{a(x)\}^2 - \frac{1}{3\nu}\{a(x)\}^3 + \frac{1}{4\nu^2}\{a(x)\}^4.
\end{aligned}$$

Under the setup (1), (2), (3) or (4), we note for $F_*(x)$ defined by (2.8) that $F_*(x) = \tilde{F}_1(x) + o(n^{-2})$, $F_*(x) = \tilde{F}_1(x) + o(n^{-4})$, $F_*(x) = \tilde{F}_2(x) + o(n^{-2})$ and $F_*(x) = \tilde{F}_2(x) + o(n^{-4})$, respectively.

Table 5.3 The percentile points when $\alpha = 0.05$

p	ν	x_α	t_E	t_1	t_{21}	t_{22}	$\tilde{t}_E$	$\tilde{t}_{21}$	$\tilde{t}_{22}$
2	20	7.4145	7.1885	7.5335	7.2151	7.1984	7.3134	7.4477	7.4325
		0.95	0.9459	0.9518	0.9464	0.9461	0.9482	0.9506	0.9503
	40	6.6423	6.5900	6.6743	6.5994	6.5985	6.6212	6.6450	6.6442
		0.95	0.9489	0.9507	0.9491	0.9491	0.9496	0.9501	0.9501
	60	6.4131	6.3905	6.4278	6.3954	6.3952	6.4044	6.4138	6.4137
		0.95	0.9495	0.9503	0.9496	0.9496	0.9498	0.9500	0.9500
	80	6.3033	6.2907	6.3116	6.2937	6.2937	6.2985	6.3036	6.3035
		0.95	0.9497	0.9502	0.9498	0.9498	0.9499	0.9500	0.9500
	100	6.2389	6.2309	6.2442	6.2329	6.2329	6.2357	6.2390	6.2390
		0.95	0.9498	0.9501	0.9499	0.9499	0.9499	0.9500	0.9500
4	40	11.3559	11.0873	11.3218	11.0363	10.9901	11.2291	11.4130	11.3553
		0.95	0.9457	0.9495	0.9449	0.9441	0.9480	0.9509	0.9500
	60	10.6677	10.5541	10.6674	10.5251	10.5156	10.6171	10.6771	10.6674
		0.95	0.9480	0.9500	0.9475	0.9474	0.9491	0.9500	0.9500
	80	10.3499	10.2875	10.3534	10.2711	10.2680	10.3230	10.3530	10.3500
		0.95	0.9489	0.9501	0.9486	0.9485	0.9495	0.9501	0.9500
	100	10.1669	10.1276	10.1704	10.1173	10.1160	10.1503	10.1683	10.1670
		0.95	0.9493	0.9501	0.9491	0.9491	0.9497	0.9500	0.9500

When $\alpha = 0.05$, Table 5.3 gives the true percentile point x_α, the approximate percentile points t_E, t_1, t_{21}, t_{22}, $\tilde{t}_E$, $\tilde{t}_{21}$ and $\tilde{t}_{22}$ together with the values of $F(x)$ in each second line. Here, t_E ($\tilde{t}_E$) and t_1 are computed from the Cornish-Fisher type expansion up to the order $O(n^{-1})$ ($O(n^{-2})$) and the transformation given by (2.7), respectively. On the other hand, t_{21}, t_{22}, $\tilde{t}_{21}$ and $\tilde{t}_{22}$ are computed from the transformation given by (2.9) under the setups (1), (2), (3) and (4), respectively. By using the fact $\{(\nu - p + 1)/(\nu p)\}T^2$ is distributed as the F-distribution with parameters $(p, \nu - p + 1)$, we obtained the exact value of x_α and $F(x)$ for each percentile point. When the term of $O(n^{-2})$ is obtained, we can observe that the transformations give a significant improvement. Comparing t_{21} with t_{22}, it seems that t_{22} does not always improve t_{21}. On the other hand, comparing $\tilde{t}_{21}$ with $\tilde{t}_{22}$, we can see an excellent improvement in $\tilde{t}_{22}$. From Theorem 4.1, the distance $|F_*(x) - \tilde{F}(x)|$ links to $\tilde{c}_r$ in (4.6). The approximation might be improved by choosing $p(x)$ such that $|F_*(x) - \tilde{F}(x)|$ becomes small.

Next, we examine Theorem 4.1 numerically. We conducted simulation experiments as follows: For parameters given in advance, error bounds for the remainder term of type (4.2) was calculated. We referred to Fujikoshi (1993) for a calculation of (4.2). In the case that error bounds are non-uniform, we referred to Fujikoshi (1988, 1993) and obtained non-uniform bounds of type $\frac{c}{1+x^\ell}$ with a constant c. Here, we set $\ell = 2$ in Example 5.3 and $\ell = 1$ in Examples 5.4 and 5.5. By using these error bounds with some constants and functions, we calculated error bounds of type (4.6). Tables 5.4, 5.6 and 5.8 present that values of the uniform bound (4.2) on the first line in each cell and that values of the uniform bound (4.6) given by Theorem 4.1 with transformation (2.9) ((2.7)) on the second (third) line in each cell. Tables 5.5, 5.7 and 5.9 give values of the non-uniform bound in the same arrangement as before. As for non-uniform bounds, we consider a case $x = u_\alpha$, the upper α point of the limiting distribution.

Example 5.3 We consider the case when $S = Z/Y$ is a scale mixture of standard normal variable Z with $Y = (\chi_n^2/n)^{-1/2}$ where Z and Y are independent. An asymptotic expansion for the distribution of S is given in the form (4.1) with $G = N(0,1)$ (the Standard Normal distribution), $\epsilon = n^{-1}$ and $h(x)$ given by

$$h(x) = -\frac{1}{4}(x^3 + x)$$

Then, a uniform bound for the remainder term of type (4.2), can be obtained (see, e.g., Fujikoshi (1993) and Shimizu and Fujikoshi (1997)). For a uniform bound of type (4.4), it can be obtained numerically. Let us apply Theorem 4.1 to this case. For the transformation given by (2.9), we consider the following case:

$$a(x) = \frac{1}{4}(x^4 - 2x^2 - 1), \quad p(x) = \frac{1}{2}\{a(x)\}^2.$$

Further we consider non-uniform error bounds at $x = u_\alpha$, the upper α point of Φ. A non-uniform bound for the remainder term of type (4.2) can be obtained (see, e.g., Fujikoshi (1988, 1993) and Ulyanov, Fujikoshi and Shimizu (1999)). For a non-uniform bound of type (4.4), it can be obtained numerically. The results of numerical studies are summarized in Tables 5.4 and 5.5.

Table 5.4 Uniform error bounds

$n = 10$	20	30	40	50	70	100	150
0.70008	0.01601	0.00419	0.00187	0.00105	0.00047	0.00021	0.00009
0.70883	0.01820	0.00517	0.00242	0.00140	0.00065	0.00030	0.00012
0.70571	0.01727	0.00466	0.00209	0.00117	0.00051	0.00022	0.00009

Table 5.5 Non-uniform error bounds at $x = u_\alpha$

$\alpha \backslash n$	20	30	40	50	70	100	150
	0.35490	0.03589	0.01050	0.00465	0.00159	0.00059	0.00021
0.05	0.35603	0.03639	0.01078	0.00483	0.00168	0.00063	0.00023
	0.35616	0.03635	0.01072	0.00476	0.00163	0.00060	0.00022
	0.20510	0.02074	0.00607	0.00268	0.00092	0.00034	0.00012
0.01	0.20631	0.02128	0.00637	0.00288	0.00102	0.00039	0.00014
	0.20638	0.02121	0.00629	0.00280	0.00096	0.00035	0.00013

Example 5.4 We consider the case when $S = \chi_f^2/Y$ with $Y = \chi_n^2/n$ where χ_f^2 and Y are independent. An asymptotic expansion for the distribution of S was given by Siotani (1956) in the form (4.1) with $G = \chi_f^2$, $\epsilon = n^{-1}$, $k = 2$, and $h(x)$ given by

$$h(x) = -2\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\frac{a_j}{\prod_{\ell=0}^{i-1}(f+2\ell)}x^i\right),$$
$$a_1 = -\frac{1}{2}f^2, \quad a_2 = \frac{1}{4}f(f+2).$$

Then, a uniform bound for the remainder term of type (4.2), can be obtained (see, e.g., Fujikoshi (1993) and Shimizu and Fujikoshi (1997)). For a uniform bound of type (4.4), it can be obtained numerically. For the transformation given by (2.9), we consider the following case:

$$a(x) = \sum_{j=1}^{2} a_j\left\{\frac{x^j}{\prod_{\ell=0}^{j-1}(f+2\ell)} - 1\right\}, \quad p(x) = \frac{1}{2}\{a(x)\}^2.$$

Further we consider non-uniform error bounds at $x = u_\alpha$, the upper α point of χ_f^2. A non-uniform bound for the remainder term of type (4.2) can be obtained (see, e.g., Fujikoshi (1988, 1993) and Ulyanov, Fujikoshi and Shimizu (1999)). For a non-uniform bound of type (4.4), it can be obtained numerically. The results of numerical studies are summarized in Tables 5.6 and 5.7.

Table 5.6 Uniform error bounds

$f\backslash n$	10	20	30	40	50	70	100	150
	1.36445	0.03108	0.00812	0.00362	0.00203	0.00090	0.00040	0.00017
2	1.38445	0.03608	0.01034	0.00487	0.00283	0.00131	0.00060	0.00025
	1.36456	0.03110	0.00812	0.00362	0.00203	0.00090	0.00040	0.00017
	1.40106	0.03159	0.00819	0.00363	0.00203	0.00090	0.00040	0.00016
4	1.45106	0.04409	0.01374	0.00675	0.00403	0.00192	0.00090	0.00039
	1.40191	0.03170	0.00822	0.00364	0.00204	0.00090	0.00040	0.00016
	0.37938	0.00840	0.00214	0.00094	0.00052	0.00023	0.00010	0.00004
6	0.46938	0.03090	0.01214	0.00657	0.00412	0.00207	0.00100	0.00044
	0.37994	0.00847	0.00216	0.00095	0.00053	0.00023	0.00010	0.00004
	1.04956	0.02659	0.00749	0.00349	0.00202	0.00093	0.00042	0.00018
8	1.18955	0.06159	0.02304	0.01224	0.00762	0.00378	0.00182	0.00080
	1.05241	0.02694	0.00759	0.00353	0.00204	0.00093	0.00043	0.00018
	0.11407	0.00379	0.00123	0.00062	0.00037	0.00018	0.00008	0.00004
10	0.31407	0.05379	0.02345	0.01312	0.00837	0.00426	0.00208	0.00093
	0.12276	0.00490	0.00155	0.00075	0.00044	0.00020	0.00009	0.00004

Table 5.7 Non-uniform error bounds at $x = u_\alpha$

α	$f\backslash n$	20	30	40	50	70	100	150
		0.12445	0.02119	0.00779	0.00393	0.00155	0.00063	0.00025
	2	0.12481	0.02135	0.00788	0.00398	0.00158	0.00065	0.00026
		0.12445	0.02119	0.00779	0.00393	0.00155	0.00063	0.00025
		0.05297	0.00896	0.00328	0.00164	0.00064	0.00026	0.00010
	4	0.05442	0.00960	0.00364	0.00187	0.00076	0.00032	0.00013
		0.05298	0.00896	0.00328	0.00164	0.00064	0.00026	0.00010
		0.00695	0.00116	0.00042	0.00021	0.00008	0.00003	0.00001
0.05	6	0.01034	0.00267	0.00127	0.00075	0.00036	0.00017	0.00007
		0.00695	0.00117	0.00042	0.00021	0.00008	0.00003	0.00001
		0.02649	0.00513	0.00208	0.00113	0.00049	0.00021	0.00009
	8	0.03281	0.00794	0.00366	0.00214	0.00100	0.00047	0.00020
		0.02665	0.00517	0.00209	0.00113	0.00049	0.00021	0.00009
		0.04974	0.00832	0.00301	0.00150	0.00058	0.00023	0.00009
	10	0.06002	0.01289	0.00558	0.00314	0.00142	0.00064	0.00027
		0.05041	0.00849	0.00308	0.00153	0.00059	0.00024	0.00009
		0.08521	0.01451	0.00533	0.00269	0.00106	0.00043	0.00017
	2	0.08659	0.01512	0.00568	0.00291	0.00117	0.00049	0.00019
		0.08578	0.01468	0.00540	0.00272	0.00107	0.00044	0.00017
		0.03891	0.00658	0.00241	0.00121	0.00047	0.00019	0.00007
	4	0.04330	0.00853	0.00350	0.00191	0.00083	0.00037	0.00015
		0.04163	0.00739	0.00274	0.00138	0.00053	0.00021	0.00008
		0.00530	0.00089	0.00032	0.00016	0.00006	0.00003	0.00001
0.01	6	0.01425	0.00486	0.00256	0.00159	0.00079	0.00038	0.00017
		0.01209	0.00295	0.00119	0.00060	0.00022	0.00008	0.00003
		0.02074	0.00402	0.00163	0.00088	0.00038	0.00017	0.00007
	8	0.03582	0.01072	0.00540	0.00330	0.00161	0.00077	0.00034
		0.03367	0.00809	0.00336	0.00176	0.00070	0.00027	0.00010
		0.03967	0.00663	0.00240	0.00119	0.00046	0.00019	0.00007
	10	0.06253	0.01680	0.00812	0.00485	0.00233	0.00110	0.00048
		0.06085	0.01360	0.00540	0.00273	0.00102	0.00037	0.00013

Example 5.5 Suppose that, for a p-variate normal population $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown, we wish to construct a set of simultaneous confidence intervals on $\boldsymbol{a}'\boldsymbol{\mu}$ with a given length 2ℓ for all $\boldsymbol{a}$, $\boldsymbol{a}'\boldsymbol{a} = 1$. A solution to this problem, given by Hyakutake and Siotani (1987), is as follows: First, take a pilot sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ of a given size m and compute

$$\overline{\boldsymbol{X}} = \frac{1}{m}\sum_{j=1}^{m} \boldsymbol{X}_j, \qquad \boldsymbol{S} = \frac{1}{\nu}\sum_{j=1}^{m}(\boldsymbol{X}_j - \overline{\boldsymbol{X}})(\boldsymbol{X}_j - \overline{\boldsymbol{X}})',$$

where $\nu = m - 1 \geq p$. Then, define the total sample size as

$$N = \max\{m + p^2, [c \cdot \mathrm{tr}(\boldsymbol{TS})] + 1\},$$

where c is a positive constant, $[a]$ stands for the greatest integer less than a real number a, and $\boldsymbol{T}$ is a given positive definite matrix which is assumed to be symmetric. Next, take an additional sample $\boldsymbol{X}_{m+1}, \ldots, \boldsymbol{X}_N$ of size $N-m$ and construct the basic random variate $\boldsymbol{Z}$ in the following way:

Choose p matrices $\boldsymbol{A}_j : p \times N = [\boldsymbol{a}_1^{(j)}, \ldots, \boldsymbol{a}_m^{(j)}, \boldsymbol{a}_{m+1}^{(j)}, \ldots, \boldsymbol{a}_N^{(j)}]$, $j = 1, \ldots, p$, satisfying that

(1) $\boldsymbol{a}_1^{(j)} = \cdots = \boldsymbol{a}_m^{(j)} \equiv \boldsymbol{a}_0^{(j)}$ (say) $(j = 1, \ldots, p)$;

(2) $\boldsymbol{A}_j \boldsymbol{1}_N = \boldsymbol{e}_j$, where $\boldsymbol{1}_N : N \times 1 = (1, 1, \ldots, 1)'$ and $\boldsymbol{e}_j : p \times 1 = (0, \ldots, 0, \overset{(j)}{1}, 0, \ldots, 0)'$;

(3) $\boldsymbol{A}\boldsymbol{A}' = \frac{1}{c}\boldsymbol{T}^{-1} \otimes \boldsymbol{S}^{-1}$, where $\boldsymbol{A} : p^2 \times N = [\boldsymbol{A}_1', \boldsymbol{A}_2', \ldots, \boldsymbol{A}_p']'$ and $\otimes$ denotes the direct product. Then, define $\boldsymbol{Z}$ by

$$\boldsymbol{Z} : p \times 1 = [\mathrm{tr}(A_1\boldsymbol{X}'), \mathrm{tr}(A_2\boldsymbol{X}'), \ldots, \mathrm{tr}(A_p\boldsymbol{X}')]',$$

where $\boldsymbol{X} : p \times N = [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m, \boldsymbol{X}_{m+1}, \ldots, \boldsymbol{X}_N]$.

By using the statistic

$$S = \frac{c}{2p}(\boldsymbol{Z} - \boldsymbol{\mu})'\boldsymbol{T}(\boldsymbol{Z} - \boldsymbol{\mu}),$$

taking $\boldsymbol{T} = \boldsymbol{I}_p$ and choosing c as $c = 2px_\alpha/\ell^2$ with x_α the upper α point of S, the solution is obtained as follows:

$$\boldsymbol{a}'\boldsymbol{\mu} \in [\boldsymbol{a}'\boldsymbol{Z} \pm \ell\ ] \quad \text{for all } \boldsymbol{a} \text{ such that } \boldsymbol{a}'\boldsymbol{a} = 1.$$

When $p = 1, 2$, we can evaluate the distribution of S exactly. For $p \geq 3$, the exact treatment of the distribution of S becomes complicated. Hyakutake and Siotani (1987) gave an asymptotic expansion of S in the form (4.1) with $G = \chi_p^2$, $\epsilon = \nu^{-1}$, $k = 2$, and $h(x)$ given by

$$h(x) = -2\sum_{i=1}^{k}\left(\sum_{j=i}^{k}\frac{a_j}{\prod_{\ell=0}^{i-1}(p+2\ell)}x^i\right),$$
$$a_1 = \frac{1}{2}(p^2 - 2), \qquad a_2 = \frac{1}{4}(p+2).$$

Now, a uniform bound for the remainder term of type (4.2), can be obtained (see, e.g., Fujikoshi (1993), Mukaihata and Fujikoshi (1993) and Shimizu and Fujikoshi (1997)). For a uniform bound of type (4.4), it can be obtained numerically. For the transformation given by (2.9), we consider the following case:

$$a(x) = \sum_{j=1}^{2} a_j\left\{\frac{x^j}{\prod_{\ell=0}^{j-1}(p+2\ell)} - 1\right\}, \quad p(x) = \frac{1}{2}\{a(x)\}^2.$$

Similarly to Example 5.4, we also consider non-uniform error bounds at $x = u_\alpha$. Let $\sigma = p^{-1}\mathrm{tr}(\boldsymbol{\Sigma}\boldsymbol{S}^{-1})$. In order to obtain error bounds for the remainder term, of type (4.2), with the help of a method by Fujikoshi (1993), it is necessary to evaluate the exact moments of $\sigma^{\pm 1}$. It is difficult to obtain those in general p, however, Mukaihata and Fujikoshi (1993) gave the ones in the case $p = 2$. Here, we examine Theorem 4.1 in the case $p = 2$. The results of numerical studies are summarized in Tables 5.8 and 5.9.

Table 5.8 Uniform error bounds when $p = 2$

$\nu = 10$	20	30	40	50	70	100	150
1.59553	0.01323	0.00319	0.00140	0.00079	0.00035	0.00016	0.00007
1.62053	0.01948	0.00597	0.00297	0.00179	0.00086	0.00041	0.00018
1.59629	0.01333	0.00322	0.00142	0.00080	0.00036	0.00016	0.00007

Table 5.9 Non-uniform error bounds at $x = u_\alpha$ when $p = 2$

$\alpha \backslash \nu$	20	30	40	50	70	100	150
	0.04098	0.00666	0.00248	0.00127	0.00052	0.00022	0.00009
0.05	0.04265	0.00740	0.00289	0.00154	0.00065	0.00028	0.00012
	0.04108	0.00668	0.00249	0.00128	0.00052	0.00022	0.00009
	0.02806	0.00456	0.00169	0.00087	0.00035	0.00015	0.00006
0.01	0.03149	0.00608	0.00255	0.00142	0.00063	0.00029	0.00012
	0.02932	0.00493	0.00185	0.00095	0.00038	0.00016	0.00006

It should be noted that non-uniform bounds (4.2) in Examples 5.4–5.6 are not available in the case $\epsilon^{-1} = 10$, because of a restriction relative to ϵ^{-1}. From Examples 5.4–5.6, we can see non-uniform bounds at $x = u_\alpha$ have a tendency to be small as α is small. That is, in the case of non-uniform error bounds, Theorem 4.1 gives sharp error bounds more successfully in the tail part of the distribution of S. On the other hand, a non-uniform bound proposed by Fujikoshi (1988) does not necessarily improve the uniform bound of type (4.2) (see Fujikoshi (1988)). In fact, we observe the phenomenon.

Comparing error bounds based on two transformations we proposed, we can see a significant superiority of transformation (2.9). This is due to the fact that the bound (4.4) can be controlled by the relation $\exp(x - x^2/2 + \cdots + (-1)^r x^{r-1}/(r-1)) = 1 + x + O(x^r)$. Examples 5.4 and 5.5 were discussed in Enoki and Aoshima (2004), however, Theorem 4.1 gives sharp error bounds more successfully than Enoki and Aoshima (2004).

Personal Thoughts

Professor Minoru Siotani has been well-known for his many path-breaking contributions in multivariate analysis and the theory of asymptotic expansions in statistics. This article interfaces with both. A major part of this piece goes back to the heart of some of the problems where Professor Minoru Siotani and his colleagues made their distinguished contributions in 1956 and thereafter. We offer this short piece as our personal homage to Professor Minoru Siotani in honour of his 80th birthday.

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