SUT Journal of Mathematics
Vol. 42, No. 1 (2006), 59–96

Siotani's modified second approximation for multiple comparisons of mean vectors

Yoshihide Kakizawa

(Received January 10, 2006)

Dedicated to Professor Minoru Siotani on his 80th birthday

Abstract. In the modified second approximation method for multiple comparisons of mean vectors that Professor Minoru Siotani (1959, 1960, 1964) originally proposed, it is crucial to consider the bivariate distribution of two correlated Hotelling's T^2 statistics. We focus on an application of the differential operator developed by Kakizawa and Iwashita (2005a) and then derive asymptotic expansion of the bivariate distribution of (T^2_{ab}, T^2_{cd}) as well as its marginal T^2_{ab}.

AMS 2000 *Mathematics Subject Classification.* 62E20, 62H10, 62J15.

Key words and phrases. Multiple comparisons, simultaneous confidence intervals, Siotani's modified second approximation method, nonnormality, asymptotic expansion, correlated Hotelling's T^2 statistics, differential operator.

§1. Introduction

In the theory of statistical analysis it is basic and important to give the distributions of various statistics. However, in multivariate statistical analysis, we encounter the situation where the exact distributions of a statistic cannot be obtained in a closed form, or even if it is obtained, the exact distribution is sometimes intricate. It is very important to obtain good approximations to various distributions. Professor Minoru Siotani has made an important contribution to many inferential procedures in multivariate statistical analysis, by deriving higher order asymptotic expansions of the distributions of various statistics including Hotelling's generalized statistic. An excellent review of Hotelling's generalized statistic under normality was given by Siotani (1989). Since Kano (1995) and Fujikoshi (1997) independently gave asymptotic expansion for the null distribution of Hotelling's one-sample T^2 statistic under general distributions (see Iwashita (1997), Wakaki (1997) and Iwashita and

Seo (2002) for a class of elliptical distributions), there have been many subsequent works to examine the effect of nonnormality upon some standard multivariate statistics on a general linear hypothesis of GMANOVA model, including multivariate linear regression model and MANOVA model. See, for example, Yanagihara (2001), Fujikoshi (2002a, b), Wakaki et al. (2002), Kakizawa and Iwashita (2005a, b), Kakizawa (2005a–d) and Gupta et al. (2006) for recent developments in asymptotic expansions of the null or nonnull distributions of several multivariate statistics according to situations under consideration. In this paper, we consider the bivariate distribution of correlated Hotelling's T^2 statistics associated with multiple comparisons of mean vectors in the multivariate nonnormal one-way layout model. The derivation of asymptotic expansions is a variant of author's recent works based on the differential operator (see Kakizawa and Iwashita (2005a, b) and Kakizawa (2005a–d)).

The idea of applying the differential operator we have considered was first used by Welch (1947) and James (1954) under normality. They originally dealt with various statistics in connection with the Behrens-Fisher problem. In line with Welch and James, Ito (1956) and Siotani (1956) independently gave asymptotic expansions for the percentile and the null distribution of Hotelling's generalized statistic under normality (see also Siotani (1957, 1971) for the nonnull case). Okamoto (1963), Memon and Okamoto (1971) and McLachlan (1973) used similar method for the W and Z statistics appeared in discriminant problem of two normal populations. Needless to say, there are a number of variants of the differential operator according to situations under consideration (e.g. Bilodeau and Brenner (1999; Chapter 8)). Roughly speaking, the differential operator arises from the moment generating function, so a great success in the literature largely relies upon the assumption of normality. For example, the differential operator for the expectation of a smooth function of the sample covariance matrix under normality (see Siotani et al. (1985; page 172)) is nothing but the moment generating function of the Wishart distribution (see also Iwashita and Siotani (1994) for an elliptical case). But, as in Kakizawa and Iwashita (2005a), the truncation argument (e.g. Bhattacharya and Rao (1976; Section 14)) enables us to develop the differential operator in nonnormal populations, since the moment generating function does not always exist.

§2. Siotani's modified second approximation

Given q (≥ 2) levels, let $\boldsymbol{X}_i^{(a)} = (X_{1i}^{(a)}, \ldots, X_{pi}^{(a)})'$ be the i-th observation on the a-th level and assume the linear model (one-way layout model)

$$\boldsymbol{X}_i^{(a)} = \boldsymbol{\theta}^{(a)} + \boldsymbol{U}_i^{(a)}, \quad a = 1, \ldots, q\,;\ i = 1, \ldots, N_a\,, \tag{1}$$

where $\boldsymbol{U}_i^{(a)}$'s are (unobservable) independent $p \times 1$ random vectors with mean zero vector and positive definite covariance matrix Σ, which is assumed to be unknown. The total number of such vectors is $\sum_{a=1}^{q} N_a = N$ (say). In the model (1), the least squares estimates of the $\boldsymbol{\theta}^{(a)}$'s are given by the sample mean vector $\overline{\boldsymbol{X}}^{(a)} = N_a^{-1} \sum_{i=1}^{N_a} \boldsymbol{X}_i^{(a)}$, $a = 1, \dots, q$. Due to the equality of covariance matrices, a usual unbiased estimate of Σ is given by the pooled sample covariance matrix

$$S_{pool,X} = \frac{1}{N-q} \sum_{a=1}^{q} (N_a - 1) S_X^{(a)} ,$$

where $S_X^{(a)} = (N_a - 1)^{-1} \sum_{i=1}^{N_a} (\boldsymbol{X}_i^{(a)} - \overline{\boldsymbol{X}}^{(a)})(\boldsymbol{X}_i^{(a)} - \overline{\boldsymbol{X}}^{(a)})'$, $a = 1, \dots, q$.

We consider the problem of constructing simultaneous confidence intervals among mean vectors. We focus on the multivariate case $p > 1$ and deal with (I) comparisons with a control when the q-th level is regarded as a control and (II) pairwise comparisons. Roy and Bose (1953; (4.3.1)) and Siotani (1960; (12) and (13)) gave exact $100(1-\alpha)\%$ simultaneous confidence intervals of the form

$$\text{(2.I)} \quad \boldsymbol{\ell}'(\boldsymbol{\theta}^{(a)} - \boldsymbol{\theta}^{(q)}) \in \boldsymbol{\ell}'(\overline{\boldsymbol{X}}^{(a)} - \overline{\boldsymbol{X}}^{(q)}) \pm \{N_{aq}^{-1} t_{\max,\mathrm{I}}^2(\alpha)(\boldsymbol{\ell}' S_{pool,X} \boldsymbol{\ell})\}^{1/2}$$
$$\text{for all } \boldsymbol{\ell} \in \mathbf{R}^p - \{\mathbf{0}\} ,\ a = 1, \dots, q-1$$

and

$$\text{(2.II)} \quad \boldsymbol{\ell}'(\boldsymbol{\theta}^{(a)} - \boldsymbol{\theta}^{(b)}) \in \boldsymbol{\ell}'(\overline{\boldsymbol{X}}^{(a)} - \overline{\boldsymbol{X}}^{(b)}) \pm \{N_{ab}^{-1} t_{\max,\mathrm{II}}^2(\alpha)(\boldsymbol{\ell}' S_{pool,X} \boldsymbol{\ell})\}^{1/2}$$
$$\text{for all } \boldsymbol{\ell} \in \mathbf{R}^p - \{\mathbf{0}\} ,\ a, b = 1, \dots, q \,;\ a < b ,$$

where $N_{ab} = N_a N_b / (N_a + N_b)$. Here, $t_{\max,\mathrm{I}}^2(\alpha)$ and $t_{\max,\mathrm{II}}^2(\alpha)$ are, respectively, the upper $100\alpha\%$ point of maximum of correlated Hotelling's T^2 statistics

$$\text{(3)} \qquad T_{\max,\mathrm{I}}^2 = \max_{a=1,\dots,q-1} (T_{aq}^2) \quad \text{and} \quad T_{\max,\mathrm{II}}^2 = \max_{1 \le a < b \le q} (T_{ab}^2) ,$$

where

$$\text{(4)} \quad T_{ab}^2 = N_{ab} (\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(b)})' S_{pool,U}^{-1} (\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(b)}) , \quad a, b = 1, \dots, q \,;\ a \neq b .$$

Obviously, $T_{ab}^2 = T_{ba}^2$ for $a \neq b$. To implement (2.I) and (2.II) practically, even in the case of normal populations, one encounters the difficulty of computing $t_{\max,\mathrm{I}}^2(\alpha)$ and $t_{\max,\mathrm{II}}^2(\alpha)$. One requires a simple expression for the joint distribution of $(T_{ab}^2)_{a,b=1,\dots,q;a<b}$ or its marginal distribution of $(T_{aq}^2)_{a=1,\dots,q-1}$. For any pair $(a, b) \in \{(a, b) : a, b = 1, \dots, q; a < b\} \equiv J$, let $\boldsymbol{\lambda}_{ab} \in \mathbf{R}^q$ be a column vector with $\{N_b/(N_a + N_b)\}^{1/2}$ at the a-th position, $-\{N_a/(N_a + N_b)\}^{1/2}$ at the b-th position and zero at other position. By virtue of multivariate sampling theory (e.g. Anderson (2003; page 77)), we know that given a subset $\{(a_i, b_i) : i = 1, \ldots, K\} \subset J$ for some $K = 1, \ldots, q(q-1)/2$, the distribution of $(T^2_{a_i b_i}/(N-q))_{i=1,\ldots,K}$ under normality is characterized as the distribution of $(\boldsymbol{Z}'_{a_i b_i} W^{-1} \boldsymbol{Z}_{a_i b_i})_{i=1,\ldots,K}$ with $\boldsymbol{Z}_{ab} = (\boldsymbol{\lambda}'_{ab} \otimes I_p)\boldsymbol{U}$, where W is distributed as Wishart distribution $W_p(I_p, N-q)$, independent of $\boldsymbol{U} \sim N_{pq}(\boldsymbol{0}, I_{pq})$. Letting Λ_K be a $K \times q$ matrix whose i-th row is $\boldsymbol{\lambda}'_{a_i b_i}$, $(\boldsymbol{Z}'_{a_1 b_1}, \ldots, \boldsymbol{Z}'_{a_K b_K})' = (\Lambda_K \otimes I_p)\boldsymbol{U}$ is distributed as a nonsingular or singular normal $N_{Kp}(\boldsymbol{0}, \Lambda_K \Lambda'_K \otimes I_p)$ according as the rank of Λ_K is equal to K or less than K (hence the case $K > q$ is singular). Unfortunately, however, a better characterization for the distribution of $(\boldsymbol{Z}'_{a_i b_i} W^{-1} \boldsymbol{Z}_{a_i b_i})_{i=1,\ldots,K}$ is not available at present for either $p > 1$ or $K > 1$. Hence, the percentiles $t^2_{\max,\mathrm{I}}(\alpha)$ and $t^2_{\max,\mathrm{II}}(\alpha)$, equivalently, the distribution functions of the maximum statistics (3), given by

$$P(T^2_{\max,\mathrm{I}} \le x) = P(T^2_{aq} \le x, a = 1, \ldots, q-1) \tag{5.I}$$

and

$$P(T^2_{\max,\mathrm{II}} \le x) = P(T^2_{ab} \le x, a, b = 1, \ldots, q; a < b)\,, \tag{5.II}$$

are not generally computable. Siotani (1959, 1960, 1964) proposed a creative approximate method for obtaining the percentile of the maximum statistic, which is called the modified second approximation method. His idea is based on Bonferroni's inequalities for (5.I) and (5.II):

$$1 - \sum_{a=1}^{q-1} \overline{P}_{aq}(x) + \sum_{1 \le a < b \le q-1} \overline{P}_{aq:bq}(x) \ge P(T^2_{\max,\mathrm{I}} \le x) \ge 1 - \sum_{a=1}^{q-1} \overline{P}_{aq}(x) \tag{6.I}$$

and

$$1 - \sum_{(a,b)\in J} \overline{P}_{ab}(x) + \sum_{\substack{(a,b),(c,d)\in J \\ (a,b)<(c,d)}} \overline{P}_{ab:cd}(x) \ge P(T^2_{\max,\mathrm{II}} \le x) \ge 1 - \sum_{(a,b)\in J} \overline{P}_{bq}(x) \tag{6.II}$$

(we adopt the lexicographically order $(a, b) < (c, d)$ iff (i) $a < c$ or (ii) $a = c$ and $b < d$), where

$$\overline{P}_{ab}(x) = P(T^2_{ab} > x) \quad \text{and} \quad \overline{P}_{ab:cd}(x) = P(T^2_{ab} > x, T^2_{cd} > x)\,.$$

Notice that equating the lower (upper) bounds in (6.I) and (6.II) to $1 - \alpha$ always yields the overestimate (underestimate) of $t^2_{\max,\mathrm{I}}(\alpha)$ and $t^2_{\max,\mathrm{II}}(\alpha)$. In order to get better approximation than the usual Bonferroni-based critical values $t^2_{\mathrm{I},\alpha}$ and $t^2_{\mathrm{II},\alpha}$ which are, respectively, the solutions of the equation

$$\sum_{a=1}^{q-1} \overline{P}_{aq}(x) = \alpha \quad \text{and} \quad \sum_{(a,b)\in J} \overline{P}_{ab}(x) = \alpha \tag{7}$$

(note that making use of these critical values yields conservative $100(1-\alpha)\%$ simultaneous confidence intervals), Siotani's modified second approximations for $t^2_{\max,\mathrm{I}}(\alpha)$ and $t^2_{\max,\mathrm{II}}(\alpha)$ are, respectively, given by $t^2_{\mathrm{I},\alpha+\beta_\mathrm{I}}$ and $t^2_{\mathrm{II},\alpha+\beta_\mathrm{II}}$, where

$$\beta_\mathrm{I} = \sum_{1\le a<b\le q-1} \overline{P}_{aq:bq}(t^2_{\mathrm{I},\alpha}) \quad \text{and} \quad \beta_\mathrm{II} = \sum_{\substack{(a,b),(c,d)\in J\\(a,b)<(c,d)}} \overline{P}_{ab:cd}(t^2_{\mathrm{II},\alpha})$$

(there will be no theoretically support whether the coverage probability of Siotani's simultaneous confidence intervals with the critical value $t^2_{\mathrm{I},\alpha+\beta_\mathrm{I}}$ (or $t^2_{\mathrm{II},\alpha+\beta_\mathrm{II}}$) is larger than or equal to $1-\alpha$).

If the population distribution is assumed to be normal $\boldsymbol{U}^{(a)} \sim N_p(\mathbf{0},\Sigma)$, $a=1,\ldots,q$, it is well-known that $(n/p)[T^2_{ab}/(N-q)]$, $(a,b)\in J$, are identically distributed as a central F distribution $F_{p,n}$ with (p,n) degrees of freedom (e.g. Anderson (2003; page 176)); $P(T^2_{ab} > x) = 1 - P(F_{p,n} > \overline{x})$, where $n = N-q-p+1$ and $\overline{x} = (n/p)[x/(N-q)]$. It follows from (7) that under normality, $t^2_{\mathrm{I},\alpha}$ and $t^2_{\mathrm{II},\alpha}$ are exactly found to be

$$\frac{(N-q)p}{n} f_{p,n}\Big(\frac{\alpha}{q-1}\Big) \quad \text{and} \quad \frac{(N-q)p}{n} f_{p,n}\Big(\frac{\alpha}{q(q-1)/2}\Big), \tag{8}$$

respectively, where $f_{p,n}(\alpha)$ is the upper $100\alpha\%$ point of $F_{p,n}$. In Siotani's (1959, 1960, 1964) method based on the correction term β_I or β_II, it is crucial to evaluate $\overline{P}_{ab:cd}(x) = P(T^2_{ab} > x, T^2_{cd} > x)$, $(a,b),(c,d)\in J$; $(a,b)<(c,d)$. Siotani (1959) is the first work to obtain its asymptotic expansion formula $\beta_\mathrm{I} = \widetilde{\beta}_\mathrm{I} + \mathrm{o}(\nu^{-2})$ or $\beta_\mathrm{II} = \widetilde{\beta}_\mathrm{II} + \mathrm{o}(\nu^{-2})$ with $\nu = N-q$. See also Fujikoshi and Seo (1999). Seo and Siotani (1992, 1993) reported, via an extensive simulation study, a good performance of the critical value (8) with α replaced by $\alpha+\widetilde{\beta}_\mathrm{I}$ or $\alpha+\widetilde{\beta}_\mathrm{II}$ (see also Seo (1995)).

Under nonnormality, one needs the Cornish-Fisher type expansions for the solutions of (7) as well as asymptotic expansion of the joint probability $\overline{P}_{ab:cd}(x) = P(T^2_{ab} > x, T^2_{cd} > x)$, $(a,b),(c,d)\in J$; $(a,b)<(c,d)$. This problem under elliptical populations, together with a simulation study, has been discussed in Seo (2002), Okamoto and Seo (2004) and Okamoto (2005). We focus on an application of the differential operator developed by Kakizawa and Iwashita (2005a) and then derive asymptotic expansion of the bivariate distribution of (T^2_{ab}, T^2_{cd}) under general distributions.

We set down the following assumptions on the N vectors $\boldsymbol{U}_i^{(a)}$'s:

(A$_1$) $\boldsymbol{U}_i^{(a)}$ and $\boldsymbol{U}_j^{(b)}$ are independent if either $a\ne b$ or $i\ne j$;

($\mathbf{A}_2$) for every fixed a, the random vectors $\boldsymbol{U}_1^{(a)}, \ldots, \boldsymbol{U}_{N_a}^{(a)}$ are identically distributed as a distribution of $\boldsymbol{U}^{(a)} = (U_1^{(a)}, \ldots, U_p^{(a)})'$ with mean $\mathbf{0}$ and positive definite covariance matrix Σ;

($\mathbf{A}_3$) for every fixed a, each component $U_j^{(a)}$ of $\boldsymbol{U}^{(a)} = (U_1^{(a)}, \ldots, U_p^{(a)})'$ has as many cumulants as desired (note that if the population follows p-variate normal distribution $N_p(\mathbf{0}, \Sigma)$, all cumulants higher than second are zero), and we denote s-th order cumulant by $Cum(U_{j_1}^{(a)}, \ldots, U_{j_s}^{(a)}) = \kappa_{j_1,\ldots,j_s}^{(a)}$ $(s \geq 3)$;

($\mathbf{A}_4$) all N_a's are large, in such a way that the total number N of observations goes to infinity, while the ratio $N_a/N = \eta_N^{(a)}$ (say) converges to $\eta_a > 0$, $a = 1, \ldots, q$, where $\sum_{a=1}^q \eta_a = 1$.

In order to assure the validity (e.g. Bhattacharya and Ghosh (1978) and Chandra and Ghosh (1980)) of asymptotic expansions up to $N^{-m/2}$ (m even) for the distributions of smooth functions of $\{(\overline{\boldsymbol{U}}^{(a)\prime}, \{\text{vech}(S_U^{(a)})\}')'\,;\, a = 1, \ldots, q\}$, the following condition on the population distribution $\boldsymbol{U}^{(a)} = (U_1^{(a)}, \ldots, U_p^{(a)})'$ is also assumed (we always set $m = 2$):

($\mathbf{A}_5$)$_m$ for every fixed a, the class of distributions of $\boldsymbol{U}^{(a)}$ is restricted to the distributions such that $\boldsymbol{\mathcal{U}}^{(a)} = (\boldsymbol{U}^{(a)\prime}, \{\text{vech}(\boldsymbol{U}^{(a)}\boldsymbol{U}^{(a)\prime} - \Sigma)\}')'$ satisfies Cramér's condition (e.g. Bhattacharya and Rao (1976; page 207))

$$\limsup_{||\xi|| \to \infty} |E[\exp(\mathrm{i}\boldsymbol{\xi}'\boldsymbol{\mathcal{U}}^{(a)})]| < 1 \quad (\boldsymbol{\xi} \in \mathbf{R}^{p+p(p+1)/2})$$

with a finite $2(m+2)$th absolute moment $E(||\boldsymbol{U}^{(a)}||^{2(m+2)}) < \infty$.

In what follows, we denote $\mathrm{i} = \sqrt{-1}$ and use j, k, without or with suffixes, to denote indices, each such index running from 1 to p unless explicitly stated otherwise. We define

$$K_4^{(a)} = \sum_{j_1 j_2 j_3 j_4 = 1}^{p} \kappa_{j_1,j_2,j_3,j_4}^{(a)} \sigma^{j_1 j_2} \sigma^{j_3 j_4}\,, \tag{9}$$

$$K_{33,1}^{(ab)} = \sum_{j_1 j_2 j_3 j_4 j_5 j_6 = 1}^{p} \kappa_{j_1,j_2,j_3}^{(a)} \kappa_{j_4,j_5,j_6}^{(b)} \sigma^{j_1 j_4} \sigma^{j_2 j_5} \sigma^{j_3 j_6}\,, \tag{10}$$

$$K_{33,2}^{(ab)} = \sum_{j_1 j_2 j_3 j_4 j_5 j_6 = 1}^{p} \kappa_{j_1,j_2,j_3}^{(a)} \kappa_{j_4,j_5,j_6}^{(b)} \sigma^{j_1 j_2} \sigma^{j_3 j_4} \sigma^{j_5 j_6} \tag{11}$$

for $a, b = 1, \ldots, q$; $a \leq b$, with σ^{jk} being the (j,k)-th element of Σ^{-1}.

Throughout this paper, $G_\nu(\cdot)$ denotes the central chi-square distribution χ_ν^2 with ν degrees of freedom, whose density function and upper $100\alpha\%$ point are $g_\nu(\cdot)$ and $\chi_{\nu,\alpha}^2$, respectively, and let $\overline{G}_\nu(x) = 1 - G_\nu(x)$.

§3. Asymptotic expansion for $P(T^2_{ab} \le x)$ and its application

Kano (1995) and Fujikoshi (1997) independently derived an asymptotic expansion up to N_a^{-1} of the null distribution of Hotelling's one-sample T^2 statistic $T^2_{[1],a} = N_a(\overline{\boldsymbol{U}}^{(a)})'(S_U^{(a)})^{-1}\overline{\boldsymbol{U}}^{(a)}$ under (A_1)–(A_3) and $(\mathrm{A}_5)_2$ (see Kakizawa and Iwashita (2005a) for the nonnull case). Kakizawa and Iwashita (2005a) also considered the nonnull distribution of Hotelling's two-sample T^2 statistic

$$T^2_{[2],ab} = N_{ab}(\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(b)})'\Big\{\frac{(N_a-1)S_U^{(a)} + (N_b-1)S_U^{(b)}}{N_a+N_b-2}\Big\}^{-1}(\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(b)})\,.$$

The following result, whose validity is shown by Bhattacharya and Rao (1976; Theorems 20.1 and 20.6), together with Bhattacharya and Ghosh's (1978) transformation (see also Bhattacharya and Denker (1990)), is obtained with a slight modification of Kakizawa and Iwashita (2005a) for the null case. Apart from the validity under (A_1)–$(\mathrm{A}_5)_2$ (that is, the 8th moment condition), the resulting asymptotic expansions (see Theorem 1, (26) and Proposition 10) depend on the cumulants up to the fourth order. Note that the same comment has been made for various statistics on the mean structure (e.g. Kano (1995), Fujikoshi (1997, 2002a, b), Yanagihara (2001), Wakaki et al. (2002), Kakizawa and Iwashita (2005a, b), Kakizawa (2005a–d) and Gupta et al. (2006)), but solving the conjecture for a minimal fourth order moment condition is yet an open problem, except for the univariate case with $p = 1$ (see Hall (1987)).

Theorem 1. *Given integers $1 \le a < b \le q$, an asymptotic expansion for the null distribution of Hotelling's T^2 statistic T^2_{ab}, defined by* (4), *is given by*

$$\begin{aligned}\Pr(T^2_{ab} \le x) &= G_p(x) + \frac{1}{N}\sum_{\ell=0}^{3}\pi_{\ell,ab}\,G_{p+2\ell}(x) + \mathrm{o}(N^{-1})\\ &= G_p(x) + \frac{2x}{Np}\Big\{\widetilde{\pi}_{0,ab} + \frac{\widetilde{\pi}_{1,ab}\,x}{p+2} + \frac{\widetilde{\pi}_{2,ab}\,x^2}{(p+2)(p+4)}\Big\}g_p(x) + \mathrm{o}(N^{-1})\,.\end{aligned}$$

Here, $\widetilde{\pi}_{\ell,ab} = \sum_{i=0}^{\ell}\pi_{i,ab}$, $\ell = 0,1,2$, and each coefficient in the asymptotic expansion formula is given by

$$\begin{aligned}\pi_{0,ab} = -\frac{p^2}{4} &+ \Big[-\frac{K_4^{\mathrm{All}}}{8} + \Big\{-\frac{\eta_b}{4(\eta_a+\eta_b)} + \frac{\eta_b^2}{8(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)}\\ &\qquad + \Big\{-\frac{\eta_a}{4(\eta_a+\eta_b)} + \frac{\eta_a^2}{8(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)}\Big]\\ &+ \Big[\Big\{\frac{\eta_a\eta_b}{4(\eta_a+\eta_b)} - \frac{\eta_b^3}{12(\eta_a+\eta_b)^3\eta_a}\Big\}K_{33,1}^{(aa)}\\ &\quad + \Big\{\frac{\eta_a\eta_b}{4(\eta_a+\eta_b)} - \frac{\eta_a^3}{12(\eta_a+\eta_b)^3\eta_b}\Big\}K_{33,1}^{(bb)}\end{aligned}$$

$$
\begin{aligned}
&+\Big\{-\frac{\eta_a\eta_b}{2(\eta_a+\eta_b)}+\frac{\eta_a\eta_b}{6(\eta_a+\eta_b)^3}\Big\}K_{33,1}^{(ab)}\\
&+\Big\{-\frac{\eta_a\eta_b}{8(\eta_a+\eta_b)}-\frac{\eta_b^3}{8(\eta_a+\eta_b)^3\eta_a}+\frac{\eta_b^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(aa)}\\
&+\Big\{-\frac{\eta_a\eta_b}{8(\eta_a+\eta_b)}-\frac{\eta_a^3}{8(\eta_a+\eta_b)^3\eta_b}+\frac{\eta_a^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(bb)}\\
&+\Big\{\frac{\eta_a\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_a\eta_b}{4(\eta_a+\eta_b)^3}-\frac{\eta_a^2+\eta_b^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(ab)}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\pi_{1,ab}=-\frac{p}{2}&+\Big[-\frac{K_4^{\mathrm{All}}}{4}+\Big\{\frac{\eta_b}{\eta_a+\eta_b}-\frac{\eta_b^2}{4(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)}\\
&\qquad+\Big\{\frac{\eta_a}{\eta_a+\eta_b}-\frac{\eta_a^2}{4(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)}\Big]\\
&+\Big[\Big\{-\frac{\eta_a\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_b^3}{4(\eta_a+\eta_b)^3\eta_a}-\frac{\eta_b^2}{2(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(aa)}\\
&\quad+\Big\{-\frac{\eta_a\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_a^3}{4(\eta_a+\eta_b)^3\eta_b}-\frac{\eta_a^2}{2(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(bb)}\\
&\quad+\Big\{\frac{\eta_a\eta_b}{2(\eta_a+\eta_b)}-\frac{\eta_a\eta_b}{2(\eta_a+\eta_b)^3}+\frac{\eta_a^2+\eta_b^2}{2(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(ab)}\\
&\quad+\Big\{\frac{7\eta_a\eta_b}{8(\eta_a+\eta_b)}+\frac{3\eta_b^3}{8(\eta_a+\eta_b)^3\eta_a}-\frac{5\eta_b^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(aa)}\\
&\quad+\Big\{\frac{7\eta_a\eta_b}{8(\eta_a+\eta_b)}+\frac{3\eta_a^3}{8(\eta_a+\eta_b)^3\eta_b}-\frac{5\eta_a^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(bb)}\\
&\quad+\Big\{-\frac{7\eta_a\eta_b}{4(\eta_a+\eta_b)}-\frac{3\eta_a\eta_b}{4(\eta_a+\eta_b)^3}+\frac{5(\eta_a^2+\eta_b^2)}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(ab)}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\pi_{2,ab}=\frac{p(p+2)}{4}&+\Big[\frac{3K_4^{\mathrm{All}}}{8}+\Big\{-\frac{3\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_b^2}{8(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)}\\
&\qquad+\Big\{-\frac{3\eta_a}{4(\eta_a+\eta_b)}+\frac{\eta_a^2}{8(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)}\Big]\\
&+\Big[\Big\{-\frac{3\eta_a\eta_b}{4(\eta_a+\eta_b)}-\frac{\eta_b^3}{4(\eta_a+\eta_b)^3\eta_a}+\frac{\eta_b^2}{(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(aa)}\\
&\quad+\Big\{-\frac{3\eta_a\eta_b}{4(\eta_a+\eta_b)}-\frac{\eta_a^3}{4(\eta_a+\eta_b)^3\eta_b}+\frac{\eta_a^2}{(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(bb)}\\
&\quad+\Big\{\frac{3\eta_a\eta_b}{2(\eta_a+\eta_b)}+\frac{\eta_a\eta_b}{2(\eta_a+\eta_b)^3}-\frac{\eta_a^2+\eta_b^2}{(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(ab)}\\
&\quad+\Big\{-\frac{15\eta_a\eta_b}{8(\eta_a+\eta_b)}-\frac{3\eta_b^3}{8(\eta_a+\eta_b)^3\eta_a}+\frac{7\eta_b^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(aa)}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{-\frac{15\eta_a\eta_b}{8(\eta_a+\eta_b)}-\frac{3\eta_a^3}{8(\eta_a+\eta_b)^3\eta_b}+\frac{7\eta_a^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(bb)}\\
&+\Big\{\frac{15\eta_a\eta_b}{4(\eta_a+\eta_b)}+\frac{3\eta_a\eta_b}{4(\eta_a+\eta_b)^3}-\frac{7(\eta_a^2+\eta_b^2)}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(ab)}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\pi_{3,ab}=\Big[&\Big\{\frac{3\eta_a\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_b^3}{12(\eta_a+\eta_b)^3\eta_a}-\frac{\eta_b^2}{2(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(aa)}\\
&+\Big\{\frac{3\eta_a\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_a^3}{12(\eta_a+\eta_b)^3\eta_b}-\frac{\eta_a^2}{2(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(bb)}\\
&+\Big\{-\frac{3\eta_a\eta_b}{2(\eta_a+\eta_b)}-\frac{\eta_a\eta_b}{6(\eta_a+\eta_b)^3}+\frac{\eta_a^2+\eta_b^2}{2(\eta_a+\eta_b)^2}\Big\}K_{33,1}^{(ab)}\\
&+\Big\{\frac{9\eta_a\eta_b}{8(\eta_a+\eta_b)}+\frac{\eta_b^3}{8(\eta_a+\eta_b)^3\eta_a}-\frac{3\eta_b^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(aa)}\\
&+\Big\{\frac{9\eta_a\eta_b}{8(\eta_a+\eta_b)}+\frac{\eta_a^3}{8(\eta_a+\eta_b)^3\eta_b}-\frac{3\eta_a^2}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(bb)}\\
&+\Big\{-\frac{9\eta_a\eta_b}{4(\eta_a+\eta_b)}-\frac{3\eta_a\eta_b}{4(\eta_a+\eta_b)^3}+\frac{3(\eta_a^2+\eta_b^2)}{4(\eta_a+\eta_b)^2}\Big\}K_{33,2}^{(ab)}\Big],
\end{aligned}
$$

where $K_4^{\text{All}}=\sum_{i=1}^q \eta_i K_4^{(i)}$.

Proof. We will outline the proof of Theorem 1 in Section 5 (see Remark 4). □

Corollary 2. *If the population distributions* $\boldsymbol{U}^{(a)}$*'s satisfy* $\boldsymbol{U}^{(a)}\overset{d}{=}-\boldsymbol{U}^{(a)}$ *for all* $a=1,\dots,q$*, we then have* $\kappa_{j_1,j_2,j_3}^{(a)}=0$ *for all* $j_1,j_2,j_3=1,\dots,p$ *and* $a=1,\dots,q$*. Hence,*

$$
\begin{aligned}
\pi_{0,ab}&=-\frac{p^2}{4}+\Big[-\frac{K_4^{\text{All}}}{8}+\Big\{-\frac{\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_b^2}{8(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)}\\
&\qquad+\Big\{-\frac{\eta_a}{4(\eta_a+\eta_b)}+\frac{\eta_a^2}{8(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)}\Big],\\
\pi_{1,ab}&=-\frac{p}{2}+\Big[-\frac{K_4^{\text{All}}}{4}+\Big\{\frac{\eta_b}{\eta_a+\eta_b}-\frac{\eta_b^2}{4(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)}\\
&\qquad+\Big\{\frac{\eta_a}{\eta_a+\eta_b}-\frac{\eta_a^2}{4(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)}\Big],\\
\pi_{2,ab}&=\frac{p(p+2)}{4}+\Big[\frac{3K_4^{\text{All}}}{8}+\Big\{-\frac{3\eta_b}{4(\eta_a+\eta_b)}+\frac{\eta_b^2}{8(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)}\\
&\qquad+\Big\{-\frac{3\eta_a}{4(\eta_a+\eta_b)}+\frac{\eta_a^2}{8(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)}\Big]
\end{aligned}
$$

and $\pi_{3,ab}=0$.

Corollary 3. *If the sample sizes satisfy* (i) $\lim_{N\to\infty} N_a/N = 1/q$ *for all* $a = 1,\dots,q$ *and the third order cumulants of the population distributions* $\boldsymbol{U}^{(a)} = (U_1^{(a)},\dots,U_p^{(a)})$*'s satisfy* (ii) $\kappa^{(1)}_{j_1,j_2,j_3} = \cdots = \kappa^{(q)}_{j_1,j_2,j_3}$ *for all* $j_1, j_2, j_3 = 1,\dots,p$, *then we have*

$$\pi_{0,ab} = -\frac{p^2}{4} + \Big\{-\frac{\overline{K}_4}{8} + \Big(-\frac{1}{8} + \frac{q}{32}\Big)(K_4^{(a)} + K_4^{(b)})\Big\},$$
$$\pi_{1,ab} = -\frac{p}{2} + \Big\{-\frac{\overline{K}_4}{4} + \Big(\frac{1}{2} - \frac{q}{16}\Big)(K_4^{(a)} + K_4^{(b)})\Big\},$$
$$\pi_{2,ab} = \frac{p(p+2)}{4} + \Big\{\frac{3\overline{K}_4}{8} + \Big(-\frac{3}{8} + \frac{q}{32}\Big)(K_4^{(a)} + K_4^{(b)})\Big\}$$

and $\pi_{3,ab} = 0$, *where* $\overline{K}_4 = (1/q)\sum_{i=1}^q K_4^{(i)}$.

Now, let us consider asymptotic expansions for the solutions of (7). Since the left hand side of $\sum_{a=1}^{q-1} \overline{P}_{aq}(x) = \alpha$ is given by

$$\sum_{a=1}^{q-1}\Big[\overline{G}_p(x) - \frac{2x}{Np}\Big\{\widetilde{\pi}_{0,aq} + \frac{\widetilde{\pi}_{1,aq}\,x}{p+2} + \frac{\widetilde{\pi}_{2,aq}\,x^2}{(p+2)(p+4)}\Big\}g_p(x) + \mathrm{o}(N^{-1})\Big]$$
$$= (q-1)\overline{G}_p(x) - \frac{2x}{Np}\sum_{a=1}^{q-1}\Big\{\widetilde{\pi}_{0,aq} + \frac{\widetilde{\pi}_{1,aq}\,x}{p+2} + \frac{\widetilde{\pi}_{2,aq}\,x^2}{(p+2)(p+4)}\Big\}g_p(x) + \mathrm{o}(N^{-1}),$$

we have

$$G_p(x) + \frac{2x}{Np}\Big\{\widetilde{\pi}_{0,\bullet q} + \frac{\widetilde{\pi}_{1,\bullet q}\,x}{p+2} + \frac{\widetilde{\pi}_{2,\bullet q}\,x^2}{(p+2)(p+4)}\Big\}g_p(x) + \mathrm{o}(N^{-1}) = \alpha_{\mathrm{I}},$$

where

$$\widetilde{\pi}_{\ell,\bullet q} = \frac{1}{q-1}\sum_{a=1}^{q-1}\widetilde{\pi}_{\ell,aq}, \quad \ell = 0,1,2 \quad \text{and} \quad \alpha_{\mathrm{I}} = \frac{\alpha}{q-1}.$$

Similarly, $\sum_{(a,b)\in J} \overline{P}_{ab}(x) = \alpha$ implies

$$G_p(x) + \frac{2x}{Np}\Big\{\widetilde{\pi}_{0,\bullet\bullet} + \frac{\widetilde{\pi}_{1,\bullet\bullet}\,x}{p+2} + \frac{\widetilde{\pi}_{2,\bullet\bullet}\,x^2}{(p+2)(p+4)}\Big\}g_p(x) + \mathrm{o}(N^{-1}) = \alpha_{\mathrm{II}},$$

where

$$\widetilde{\pi}_{\ell,\bullet\bullet} = \frac{1}{q(q-1)/2}\sum_{q\le a<b\le q}\widetilde{\pi}_{\ell,ab}, \quad \ell = 0,1,2 \quad \text{and} \quad \alpha_{\mathrm{II}} = \frac{\alpha}{q(q-1)/2}.$$

Theorem 4. *Asymptotic expansions for the Bonferroni-based critical values that are the solutions of* (7) *are given by*

$$t^2_{\mathrm{I},\alpha} = \chi^2_{p,\alpha_{\mathrm{I}}}\Big[1 - \frac{2}{Np}\Big\{\widetilde{\pi}_{0,\bullet q} + \frac{\widetilde{\pi}_{1,\bullet q}\,\chi^2_{p,\alpha_{\mathrm{I}}}}{p+2} + \frac{\widetilde{\pi}_{2,\bullet q}\,(\chi^2_{p,\alpha_{\mathrm{I}}})^2}{(p+2)(p+4)}\Big\}\Big] + \mathrm{o}(N^{-1})$$

and

$$t^2_{\mathrm{II},\alpha} = \chi^2_{p,\alpha_{\mathrm{II}}}\Big[1 - \frac{2}{Np}\Big\{\widetilde{\pi}_{0,\bullet\bullet} + \frac{\widetilde{\pi}_{1,\bullet\bullet}\,\chi^2_{p,\alpha_{\mathrm{II}}}}{p+2} + \frac{\widetilde{\pi}_{2,\bullet\bullet}\,(\chi^2_{p,\alpha_{\mathrm{II}}})^2}{(p+2)(p+4)}\Big\}\Big] + \mathrm{o}(N^{-1})\,,$$

respectively.

In view of Theorem 4, asymptotic expansions of $t^2_{\mathrm{I},\alpha}$ and $t^2_{\mathrm{II},\alpha}$ are polynomials of degree at most 3 in one variable $x = \chi^2_{p,\beta} > 0$ with $\beta = \alpha_{\mathrm{I}}$, α_{II}, having the form of

$$C_N(x) = x + \frac{2}{N}\sum_{j=1}^{3} \vartheta_j\, x^j \quad \text{(say)},$$

where ϑ_j's depend on summarized cumulants (9)–(11). Noting $C_N(x) = x + \mathrm{O}(N^{-1})$, one can use the limiting value $x = \chi^2_{p,\beta} > 0$ with $\beta = \alpha_{\mathrm{I}}$, α_{II}, independent of violation of normality. However, the effect of nonnormality via unknown summarized cumulants (9)–(11) will be serious for small N. An estimated version of $C_N(\chi^2_{p,\beta})$ may be preferable to $\chi^2_{p,\beta}$ in such a situation. At the end of this section, we shall estimate (9)–(11) from observations $\boldsymbol{X}_i^{(a)}$'s.

Remark 1. Under elliptical populations with possibly different kurtosis parameters $\kappa^{(1)}, \dots, \kappa^{(q)}$ (e.g. Anderson (2003; Section 2.7)), Seo (2002) and Okamoto and Seo (2004) gave an asymptotic expansion of $t^2_{\mathrm{II},\alpha}$ (in this case, we set $K^{(ab)}_{33,1} = K^{(ab)}_{33,2} = 0$ and $K^{(a)}_4 = p(p+2)\kappa^{(a)}$ for $a, b = 1, \dots, q$; $a \le b$). The Cornish-Fisher type expansion they derived was never of degree 3 (see Corollary 2).

Remark 2. With

$$\beta^*_{\mathrm{I}} = \sum_{1\le a<b\le q-1} \overline{P}^{\mathrm{AE1}}_{aq:bq}\{C_N(\chi^2_{p,\alpha_{\mathrm{I}}})\} \quad \text{and} \quad \beta^*_{\mathrm{II}} = \sum_{\substack{(a,b),(c,d)\in J\\(a,b)<(c,d)}} \overline{P}^{\mathrm{AE1}}_{ab:cd}\{C_N(\chi^2_{p,\alpha_{\mathrm{II}}})\}\,,$$

where $\overline{P}^{\mathrm{AE1}}_{ab:cd}(x)$ denotes two-term asymptotic expansion of $\overline{P}_{ab:cd}(x)$ without remainder term of order $\mathrm{o}(N^{-1})$, Siotani's (1959, 1960, 1964) proposal under general distributions is given by $C_N(\chi^2_{p,\alpha^*_{\mathrm{K}}})$, $\mathrm{K} = \mathrm{I}, \mathrm{II}$, where

$$\alpha^*_{\mathrm{I}} = \frac{\alpha + \beta^*_{\mathrm{I}}}{q-1} \quad \text{and} \quad \alpha^*_{\mathrm{II}} = \frac{\alpha + \beta^*_{\mathrm{II}}}{q(q-1)/2}\,.$$

Seo and Siotani (1992, 1993) gave an extensive simulation study in order to examine the accuracy of their proposal under normality. For a class of elliptical populations, Seo (2002) and Okamoto (2005) recently reported the numerical accuracy for pairwise comparisons.

A potentially serious problem with this approach is that the polynomial $C_N(x)$ or its appropriate estimator can even be negative for some $x > 0$. We address this problem by adding an extra N^{-2} term, for example,

$$C_N^{+}(x) = C_N(x) + \frac{1}{N^2} \sum_{j_1 j_2 = 1}^{3} \vartheta_{j_1} \vartheta_{j_2} x^{j_1+j_2-1} .$$

Cribari-Neto and Ferrari (2001) gave a monotone version

$$C_N^{\uparrow}(x) = C_N(x) + \frac{1}{N^2} \sum_{j_1 j_2 = 1}^{3} \frac{j_1 j_2 \, \vartheta_{j_1} \vartheta_{j_2}}{j_1 + j_2 - 1} \, x^{j_1+j_2-1}$$

by using the same idea as Kakizawa's (1996) monotone adjustment of the Bartlett-type correction for asymptotically chi-square statistics due to Cordeiro and Ferrari (1991) (see also Hall (1992; page 123) for the monotone normalizing transformation of asymptotically normal statistics). This approach is, of course, supported by higher-order asymptotic theory; $\lim_{N\to\infty} N\{C_N^{+}(x) - C_N(x)\} = \lim_{N\to\infty} N\{C_N^{\uparrow}(x) - C_N(x)\} = 0$ for any bounded $x > 0$. However, adding N^{-2} term to the asymptotic expansion formula $C_N(x)$ is ad-hoc. It may be reasonable to use $C_N(x)$ if $C_N(x) > 0$, and $C_N^{+}(x)$ or $C_N^{\uparrow}(x)$ otherwise.

In order to estimate (9)–(11), we use the relations between cumulants and moments to get

$$\begin{aligned}
K_4^{(a)} &= \sum_{j_1 j_2 j_3 j_4 = 1}^{p} E(U_{j_1}^{(a)} U_{j_2}^{(a)} U_{j_3}^{(a)} U_{j_4}^{(a)}) \sigma^{j_1 j_2} \sigma^{j_3 j_4} - p(p+2) , \\
K_{33,1}^{(ab)} &= \sum_{j_1 j_2 j_3 k_1 k_2 k_3 = 1}^{p} E(U_{j_1}^{(a)} U_{j_2}^{(a)} U_{j_3}^{(a)}) E(U_{k_1}^{(b)} U_{k_2}^{(b)} U_{k_3}^{(b)}) \sigma^{j_1 k_1} \sigma^{j_2 k_2} \sigma^{j_3 k_3} , \\
K_{33,2}^{(ab)} &= \sum_{j_1 j_2 j_3 k_1 k_2 k_3 = 1}^{p} E(U_{j_1}^{(a)} U_{j_2}^{(a)} U_{j_3}^{(a)}) E(U_{k_1}^{(b)} U_{k_2}^{(b)} U_{k_3}^{(b)}) \sigma^{j_1 j_2} \sigma^{j_3 k_1} \sigma^{k_2 k_3} .
\end{aligned}$$

Then, we can naturally construct their estimators

$$\widehat{K}_4^{(a)} = \frac{1}{N_a} \sum_{i=1}^{N_a} (M_{ii}^{(aa)})^2 - p(p+2) ,$$

$$\widehat{K}_{33,1}^{(ab)} = \begin{cases} \dfrac{1}{N_a(N_a-1)} \displaystyle\sum_{\substack{ii'=1 \\ i \neq i'}}^{N_a} (M_{ii'}^{(aa)})^3 , & \text{if } a = b \\ \dfrac{1}{N_a N_b} \displaystyle\sum_{i=1}^{N_a} \sum_{i'=1}^{N_b} (M_{ii'}^{(ab)})^3 , & \text{if } a < b, \end{cases}$$

$$\widehat{K}_{33,2}^{(ab)} = \begin{cases} \dfrac{1}{N_a(N_a-1)} \displaystyle\sum_{\substack{ii'=1\\ i\neq i'}}^{N_a} M_{ii}^{(aa)} M_{ii'}^{(aa)} M_{i'i'}^{(aa)}\,, & \text{if } a=b \\ \dfrac{1}{N_a N_b} \displaystyle\sum_{i=1}^{N_a}\sum_{i'=1}^{N_b} M_{ii}^{(ab)} M_{ii'}^{(ab)} M_{i'i'}^{(ab)}\,, & \text{if } a<b, \end{cases}$$

where

$$M_{ii'}^{(ab)} = (\boldsymbol{X}_i^{(a)} - \overline{\boldsymbol{X}}^{(a)})' S_{pool,X}^{-1} (\boldsymbol{X}_{i'}^{(b)} - \overline{\boldsymbol{X}}^{(b)})\,, \quad a,b = 1,\ldots,q\,;\ a \le b\,.$$

See also Mardia (1970), McCullagh (1987; page 107) and Anderson (2003; page 103).

§4. Differential operator approach

We now set down

$$\boldsymbol{S}_{ab} = \Big(I_p - \frac{1}{2}\,\widetilde{\Delta} + \frac{3}{8}\,\widetilde{\Delta}^2\Big)\Sigma^{-1/2} N_{ab}^{1/2} (\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(b)}) \tag{12}$$

(see Kakizawa and Iwashita (2005a)), where $\widetilde{\Delta} = \Sigma^{-1/2}(S_{pool,U} - \Sigma)\Sigma^{-1/2}$ with $\Sigma^{-1/2}$ being the inverse matrix of the symmetric square root matrix $\Sigma^{1/2}$ of Σ. By virtue of Chibisov (1972) and Magdalinos (1992), it suffices to consider

$$\widetilde{T}_{ab}^2 = \boldsymbol{S}_{ab}' \boldsymbol{S}_{ab} \tag{13}$$

in place of T_{ab}^2.

In general, inverting an asymptotic expansion of the characteristic function of a certain statistic is not valid. Fortunately, the existence of a valid expansion of the distribution function and the characteristic function of a smooth function of $\{(\overline{\boldsymbol{U}}^{(a)\prime}, \{\mathrm{vech}(S_U^{(a)})\}')'; a = 1,\ldots,q\}$ is guaranteed by Bhattacharya and Rao (1976; Chapter 4), together with Bhattacharya and Ghosh's (1978) transformation (see also Bhattacharya and Denker (1990)). Then, the unicity property of the Fourier-Stieltjes transform implies that a formal inversion of an asymptotic expansion of the characteristic function of such a statistic must be valid. As pointed out in Fujikoshi (2002b), it is thus crucial to find a convenient device for giving an asymptotic expansion of the characteristic function according to situations under consideration. Our recent works on mean vectors (see Kakizawa and Iwashita (2005a, b) and Kakizawa (2005a–d)) are based on the differential operator, although there have been many derivations.

Recall

$$\widetilde{T}_{ab}^2 = F(N_{ab}^{1/2}(\overline{\boldsymbol{U}}^{(a)} - \overline{\boldsymbol{U}}^{(a)}), S_{pool,U}) = \widetilde{T}_{ba}^2 \tag{14}$$

(see (12) and (13)), where

$$F(\boldsymbol{\gamma},\Gamma)=\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma}+\sum_{\nu=1}^{4}b_\nu\boldsymbol{\gamma}'\Sigma^{-1}\{(\Gamma-\Sigma)\Sigma^{-1}\}^{\nu}\boldsymbol{\gamma}$$

with $(b_1,b_2,b_3,b_4)=(-1,1,-3/8,9/64)$. For simplicity, we write

$$\boldsymbol{Z}_U^{(ab)}=N_{ab}^{1/2}(\overline{\boldsymbol{U}}^{(a)}-\overline{\boldsymbol{U}}^{(a)})=\Big(\frac{\eta_N^{(a)}\eta_N^{(b)}}{\eta_N^{(a)}+\eta_N^{(b)}}\Big)^{1/2}N^{1/2}(\overline{\boldsymbol{U}}^{(a)}-\overline{\boldsymbol{U}}^{(b)}),\quad (a,b)\in J\,.$$

We now want to derive an asymptotic expansion of the characteristic function of $(\widetilde{T}^2_{ab},\widetilde{T}^2_{cd})'$, $(a,b),(c,d)\in J$; $(a,b)<(c,d)$. We have two cases to consider: (i) $a,b,c,d\in\{1,\dots,q\}$ are all different, and (ii) $(a,b)\in J$ and $(c,d)\in J$ have exactly one common index, that is,

$$(a,b,c,d)=(a_1,a_2,a_1,a_3),(a_1,a_2,a_2,a_3),(a_1,a_3,a_2,a_3)\,.$$

For the latter case (ii), since (14) implies $(\widetilde{T}^2_{a_1a_2},\widetilde{T}^2_{a_1a_3})'=(\widetilde{T}^2_{a_2a_1},\widetilde{T}^2_{a_3a_1})'$ and $(\widetilde{T}^2_{a_1a_2},\widetilde{T}^2_{a_2a_3})'=(\widetilde{T}^2_{a_1a_2},\widetilde{T}^2_{a_3a_2})'$, we have only to consider a pattern of $(\widetilde{T}^2_{ac},\widetilde{T}^2_{bc})'$, where $a,b,c\in\{1,\dots,q\}$ are all different.

Unlike Seo (2002), Okamoto and Seo (2004) and Okamoto (2005) for elliptical populations, our approach is based on the differential operator developed by Kakizawa and Iwashita (2005a), as follows:

Notation. Let $\boldsymbol{\gamma}^{(ab)}=(\gamma_j^{(ab)})$, $a,b=1,\dots,q$; $a\neq b$ be $p\times 1$ vectors of variables and $\Gamma=(\gamma_{jk})$ be a $p\times p$ symmetric matrix of variables. We define a vector of differential operators by

$$\boldsymbol{\partial}^{(ab)}=(\partial_j^{(ab)})=\left(\frac{\partial}{\partial\gamma_j^{(ab)}}\right)$$

and a matrix of differential operators by

$$\partial=(\partial_{jk})=\left(\frac{1}{2}\,(1+\delta_{jk})\frac{\partial}{\partial\gamma_{jk}}\right)$$

applied to any analytic function, where δ_{jk} is the Kronecker delta, that is, $\delta_{jk}=1$ iff $j=k$, and 0 otherwise.

For simplicity, we write

$$\kappa^{*I}_{j_1j_2j_3;ab}=\Big(\frac{\eta_N^{(a)}\eta_N^{(b)}}{\eta_N^{(a)}+\eta_N^{(b)}}\Big)^{1/2}(\kappa^{(a)}_{j_1,j_2,j_3}-\kappa^{(b)}_{j_1,j_2,j_3})\,,$$

$$\kappa^{*II}_{j_1j_2j_3;ab}=\frac{1}{(\eta_N^{(a)}+\eta_N^{(b)})^{3/2}}\Big\{\frac{(\eta_N^{(b)})^{3/2}}{(\eta_N^{(a)})^{1/2}}\kappa^{(a)}_{j_1,j_2,j_3}-\frac{(\eta_N^{(a)})^{3/2}}{(\eta_N^{(b)})^{1/2}}\kappa^{(b)}_{j_1,j_2,j_3}\Big\}\,,$$

$$\begin{aligned}
\kappa^{*\mathrm{All}}_{j_1j_2j_3j_4} &= \sum_{i=1}^{q} \eta_N^{(i)} \kappa^{(i)}_{j_1,j_2,j_3,j_4}\,, \\
\kappa^{*I}_{j_1j_2j_3j_4;ab} &= \frac{1}{\eta_N^{(a)}+\eta_N^{(b)}}\,(\eta_N^{(b)}\kappa^{(a)}_{j_1,j_2,j_3,j_4} + \eta_N^{(a)}\kappa^{(b)}_{j_1,j_2,j_3,j_4})\,, \\
\kappa^{*II}_{j_1j_2j_3j_4;ab} &= \frac{1}{(\eta_N^{(a)}+\eta_N^{(b)})^2}\Big\{\frac{(\eta_N^{(b)})^2}{\eta_N^{(a)}}\,\kappa^{(a)}_{j_1,j_2,j_3,j_4} + \frac{(\eta_N^{(a)})^2}{\eta_N^{(b)}}\,\kappa^{(b)}_{j_1,j_2,j_3,j_4}\Big\}\,,
\end{aligned}$$

whose limits as $N \to \infty$ are denoted by

$$\begin{aligned}
\kappa^{I}_{j_1j_2j_3;ab} &= \Big(\frac{\eta_a\eta_b}{\eta_a+\eta_b}\Big)^{1/2}(\kappa^{(a)}_{j_1,j_2,j_3} - \kappa^{(b)}_{j_1,j_2,j_3})\,, \\
\kappa^{II}_{j_1j_2j_3;ab} &= \frac{1}{(\eta_a+\eta_b)^{3/2}}\Big\{\frac{(\eta_b)^{3/2}}{(\eta_a)^{1/2}}\kappa^{(a)}_{j_1,j_2,j_3} - \frac{(\eta_a)^{3/2}}{(\eta_b)^{1/2}}\kappa^{(b)}_{j_1,j_2,j_3}\Big\}\,, \\
\kappa^{\mathrm{All}}_{j_1j_2j_3j_4} &= \sum_{i=1}^{q} \eta_i\kappa^{(i)}_{j_1,j_2,j_3,j_4}\,, \\
\kappa^{I}_{j_1j_2j_3j_4;ab} &= \frac{1}{\eta_a+\eta_b}\,(\eta_b\kappa^{(a)}_{j_1,j_2,j_3,j_4} + \eta_a\kappa^{(b)}_{j_1,j_2,j_3,j_4})\,, \\
\kappa^{II}_{j_1j_2j_3j_4;ab} &= \frac{1}{(\eta_a+\eta_b)^2}\Big\{\frac{(\eta_b)^2}{\eta_a}\,\kappa^{(a)}_{j_1,j_2,j_3,j_4} + \frac{(\eta_a)^2}{\eta_b}\,\kappa^{(b)}_{j_1,j_2,j_3,j_4}\Big\}\,.
\end{aligned}$$

Lemma 5. *Given integers $1 \le a < b^* < c^* < d^* \le q$, we set $(b,c,d) = (b^*,c^*,d^*)$, (c^*,b^*,d^*), (d^*,b^*,c^*). Then,*

$$\begin{aligned}
&E\exp\{\mathrm{i}h(\boldsymbol{Z}_U^{(ab)},\boldsymbol{Z}_U^{(cd)},S_{pool,U})\} \\
&\quad = \Xi\exp\{\mathrm{i}h(\boldsymbol{\gamma}^{(ab)},\boldsymbol{\gamma}^{(cd)},\Gamma)\}\Big|_{\gamma^{(ab)}=\gamma^{(cd)}=0,\Gamma=\Sigma} + \mathrm{o}(N^{-1})
\end{aligned}$$

for any multivariate polynomial $h(\boldsymbol{\gamma}^{(ab)},\boldsymbol{\gamma}^{(cd)},\Gamma)$ of finite degree with coefficients in $\mathbf{R}$, which may depend on N but are of order $\mathrm{O}(1)$, where

$$\Xi = \Xi_0\Big[1 + \frac{1}{N^{1/2}}\,\Xi_1 + \frac{1}{N}\Big\{\mathrm{tr}(\Sigma\partial\Sigma\partial) + \Xi_2 + \frac{1}{2}\,\Xi_1^2\Big\}\Big]$$

with

$$\begin{aligned}
\Xi_0 &= \exp\Big\{\frac{1}{2}\,(\boldsymbol{\partial}^{(ab)})'\Sigma\boldsymbol{\partial}^{(ab)} + \frac{1}{2}\,(\boldsymbol{\partial}^{(cd)})'\Sigma\boldsymbol{\partial}^{(cd)}\Big\}\,, \\
\Xi_1 &= \Xi_1^{*I} + \Xi_1^{*II} \quad \textit{and} \quad \Xi_2 = \Xi_2^{*\mathrm{All}} + \Xi_2^{*I} + \Xi_2^{*II}\,.
\end{aligned}$$

Here,

$$\Xi_2^{*\mathrm{All}} = \frac{1}{2}\sum_{j_1j_2j_3j_4=1}^{p} \kappa^{*\mathrm{All}}_{j_1j_2j_3j_4}\partial_{j_1j_2}\partial_{j_3j_4}$$

and

$$\Xi_\ell^{*J} = \Xi_{\ell,J}^{*(ab)} + \Xi_{\ell,J}^{*(cd)}\,, \quad \ell = 1,2\,;\ J = I, II\,,$$

where

$$\begin{aligned}
\Xi_{1,I}^{*(ab)} &= \sum_{j_1j_2j_3=1}^{p} \kappa^{*I}_{j_1j_2j_3;ab}\partial_{j_1j_2}\partial_{j_3}^{(ab)}\,,\\
\Xi_{1,II}^{*(ab)} &= \frac{1}{6}\sum_{j_1j_2j_3=1}^{p} \kappa^{*II}_{j_1j_2j_3;ab}\partial_{j_1}^{(ab)}\partial_{j_2}^{(ab)}\partial_{j_3}^{(ab)}\,,\\
\Xi_{2,I}^{*(ab)} &= \frac{1}{2}\sum_{j_1j_2j_3j_4=1}^{p} \kappa^{*I}_{j_1j_2j_3j_4;ab}\partial_{j_1j_2}\partial_{j_3}^{(ab)}\partial_{j_4}^{(ab)}\,,\\
\Xi_{2,II}^{*(ab)} &= \frac{1}{24}\sum_{j_1j_2j_3j_4=1}^{p} \kappa^{*II}_{j_1j_2j_3j_4;ab}\partial_{j_1}^{(ab)}\partial_{j_2}^{(ab)}\partial_{j_3}^{(ab)}\partial_{j_4}^{(ab)}\,.
\end{aligned}$$

Lemma 6. *Suppose that $N_a/N = \eta_a$, $a = 1,\dots,q$, where η_a's are fixed positive rational numbers satisfying $\sum_{a=1}^q \eta_a = 1$. Given integers a_1, a_2, c, such that $a_1, a_2, c \in \{1,\dots,q\}$ are all different, let*

$$\boldsymbol{\partial}_{(a_2c)}^{(a_1c)} = \begin{pmatrix} \boldsymbol{\partial}^{(a_1c)} \\ \boldsymbol{\partial}^{(a_2c)} \end{pmatrix} \quad \text{and} \quad R_{a_1a_2,c} = \begin{pmatrix} 1 & \eta_{a_1a_2,c} \\ \eta_{a_1a_2,c} & 1 \end{pmatrix} \otimes \Sigma\,,$$

where

$$\eta_{a_1a_2,c} = \Big(\frac{\eta_{a_1}}{\eta_{a_1}+\eta_c}\Big)^{1/2}\Big(\frac{\eta_{a_2}}{\eta_{a_2}+\eta_c}\Big)^{1/2} \in (0,1)\,.$$

Then,

$$\begin{aligned}
&E\exp\{\mathrm{i}h(\boldsymbol{Z}_U^{(a_1c)}, \boldsymbol{Z}_U^{(a_2c)}, S_{pool,U})\}\\
&\quad = \widetilde{\widetilde{\Xi}}\exp\{\mathrm{i}h(\boldsymbol{\gamma}^{(a_1c)}, \boldsymbol{\gamma}^{(a_2c)}, \Gamma)\}\Big|_{\gamma^{(a_1c)}=\gamma^{(a_2c)}=0,\Gamma=\Sigma} + \mathrm{o}(N^{-1})
\end{aligned}$$

for any multivariate polynomial $h(\boldsymbol{\gamma}^{(a_1c)}, \boldsymbol{\gamma}^{(a_2c)}, \Gamma)$ of finite degree with coefficients in $\mathbf{R}$, which may depend on N but are of order $\mathrm{O}(1)$, where

$$\widetilde{\widetilde{\Xi}} = \widetilde{\widetilde{\Xi}}_0\Big[1 + \frac{1}{N^{1/2}}\,\widetilde{\widetilde{\Xi}}_1 + \frac{1}{N}\Big\{\mathrm{tr}(\Sigma\partial\Sigma\partial) + \widetilde{\widetilde{\Xi}}_2 + \frac{1}{2}\,\widetilde{\widetilde{\Xi}}_1^2\Big\}\Big]$$

with

$$\widetilde{\widetilde{\Xi}}_0 = \exp\Big\{\frac{1}{2}\,(\boldsymbol{\partial}_{(a_2c)}^{(a_1c)})' R_{a_1a_2,c}\boldsymbol{\partial}_{(a_2c)}^{(a_1c)}\Big\}\,, \quad \widetilde{\widetilde{\Xi}}_1 = \Xi_1^I + \Xi_1^{II} + \Xi_1^{III}$$

and

$$\widetilde{\widetilde{\Xi}}_2 = \Xi_2^{\mathrm{All}} + \Xi_2^I + \Xi_2^{II} + \Xi_2^{III(1)} + \Xi_2^{III(2)} + \Xi_2^{III(3)}\,.$$

Here,

$$\begin{aligned}\Xi_1^{III} &= -\frac{\eta_{a_1a_2,c}}{2}\Big\{\frac{\eta_{a_1}}{\eta_c(\eta_{a_1}+\eta_c)}\Big\}^{1/2}\sum_{j_1j_2j_3=1}^{p}\kappa^{(c)}_{j_1,j_2,j_3}\partial^{(a_1c)}_{j_1}\partial^{(a_1c)}_{j_2}\partial^{(a_2c)}_{j_3} \\ &\quad -\frac{\eta_{a_1a_2,c}}{2}\Big\{\frac{\eta_{a_2}}{\eta_c(\eta_{a_2}+\eta_c)}\Big\}^{1/2}\sum_{j_1j_2j_3=1}^{p}\kappa^{(c)}_{j_1,j_2,j_3}\partial^{(a_1c)}_{j_1}\partial^{(a_2c)}_{j_2}\partial^{(a_2c)}_{j_3}\,, \\ \Xi_2^{\mathrm{All}} &= \frac{1}{2}\sum_{j_1j_2j_3j_4=1}^{p}\kappa^{\mathrm{All}}_{j_1j_2j_3j_4}\partial_{j_1j_2}\partial_{j_3j_4}\,, \\ \Xi_2^{III(1)} &= \eta_{a_1a_2,c}\sum_{j_1j_2j_3j_4=1}^{p}\kappa^{(c)}_{j_1,j_2,j_3,j_4}\partial_{j_1j_2}\partial^{(a_1c)}_{j_3}\partial^{(a_2c)}_{j_4}\,, \\ \Xi_2^{III(2)} &= \frac{(\eta_{a_1a_2,c})^2}{4\eta_c}\sum_{j_1j_2j_3j_4=1}^{p}\kappa^{(c)}_{j_1,j_2,j_3,j_4}\partial^{(a_1c)}_{j_1}\partial^{(a_1c)}_{j_2}\partial^{(a_2c)}_{j_3}\partial^{(a_2c)}_{j_4}\,, \\ \Xi_2^{III(3)} &= \frac{\eta_{a_1a_2,c}}{6\eta_c}\Big(\frac{\eta_{a_1}}{\eta_{a_1}+\eta_c}\Big)\sum_{j_1j_2j_3j_4=1}^{p}\kappa^{(c)}_{j_1,j_2,j_3,j_4}\partial^{(a_1c)}_{j_1}\partial^{(a_1c)}_{j_2}\partial^{(a_1c)}_{j_3}\partial^{(a_2c)}_{j_4} \\ &\quad +\frac{\eta_{a_1a_2,c}}{6\eta_c}\Big(\frac{\eta_{a_2}}{\eta_{a_2}+\eta_c}\Big)\sum_{j_1j_2j_3j_4=1}^{p}\kappa^{(c)}_{j_1,j_2,j_3,j_4}\partial^{(a_1c)}_{j_1}\partial^{(a_2c)}_{j_2}\partial^{(a_2c)}_{j_3}\partial^{(a_2c)}_{j_4}\end{aligned}$$

and

$$\Xi_\ell^{J} = \Xi^{(a_1c)}_{\ell,J} + \Xi^{(a_2c)}_{\ell,J}\,, \quad \ell = 1,2\,;\ J = I, II$$

are given by

$$\begin{aligned}\Xi^{(a_1a_2)}_{1,I} &= \sum_{j_1j_2j_3=1}^{p}\kappa^{I}_{j_1j_2j_3;a_1a_2}\partial_{j_1j_2}\partial^{(a_1a_2)}_{j_3}\,, \\ \Xi^{(a_1a_2)}_{1,II} &= \frac{1}{6}\sum_{j_1j_2j_3=1}^{p}\kappa^{II}_{j_1j_2j_3;a_1a_2}\partial^{(a_1a_2)}_{j_1}\partial^{(a_1a_2)}_{j_2}\partial^{(a_1a_2)}_{j_3}\,, \\ \Xi^{(a_1a_2)}_{2,I} &= \frac{1}{2}\sum_{j_1j_2j_3j_4=1}^{p}\kappa^{I}_{j_1j_2j_3j_4;a_1a_2}\partial_{j_1j_2}\partial^{(a_1a_2)}_{j_3}\partial^{(a_1a_2)}_{j_4}\,, \\ \Xi^{(a_1a_2)}_{2,II} &= \frac{1}{24}\sum_{j_1j_2j_3j_4=1}^{p}\kappa^{II}_{j_1j_2j_3j_4;a_1a_2}\partial^{(a_1a_2)}_{j_1}\partial^{(a_1a_2)}_{j_2}\partial^{(a_1a_2)}_{j_3}\partial^{(a_1a_2)}_{j_4}\,.\end{aligned}$$

Proof of Lemmas 5 and 6. In line with Kakizawa and Iwashita (2005a),

$$E\exp\{\mathrm{i}h(\boldsymbol{Z}^{(ab)}_U, \boldsymbol{Z}^{(cd)}_U, S_{pool,U})\} = E\exp\{\mathrm{i}h(\boldsymbol{Z}^{(ab)}_{U^\dagger}, \boldsymbol{Z}^{(cd)}_{U^\dagger}, S_{pool,U^\dagger})\} + \mathrm{o}(N^{-1})\,,$$

with

$$\boldsymbol{U}_i^{\dagger(a)} = \begin{cases} \boldsymbol{U}_i^{(a)}, & ||\boldsymbol{U}_i^{(a)}|| \le N_a^{1/2} \\ \boldsymbol{0}, & ||\boldsymbol{U}_i^{(a)}|| > N_a^{1/2} \end{cases}$$

being truncated random vectors for $\boldsymbol{U}_i^{(a)}$, $a = 1, \ldots, q\,;\ i = 1, 2, \ldots, N_a$. Let

$$\boldsymbol{\partial}_{[\ell]}^{(ab)} = \begin{cases} \{\eta_N^{(b)}/(\eta_N^{(a)} + \eta_N^{(b)})\}^{1/2}\boldsymbol{\partial}^{(ab)}, & \text{if } \ell = a \\ -\{\eta_N^{(a)}/(\eta_N^{(a)} + \eta_N^{(b)})\}^{1/2}\boldsymbol{\partial}^{(ab)}, & \text{if } \ell = b \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

Using the independence of $\boldsymbol{U}_j^{(a)}$ and $\boldsymbol{U}_{j'}^{(a')}$ (hence $\boldsymbol{U}_j^{\dagger(a)}$ and $\boldsymbol{U}_{j'}^{\dagger(a')}$) for $a \ne a'$, we obtain

$$E\exp\{\mathrm{i}h(\boldsymbol{Z}_{U^\dagger}^{(ab)}, \boldsymbol{Z}_{U^\dagger}^{(cd)}, S_{pool,U^\dagger})\} = \Theta\exp\{\mathrm{i}h(\boldsymbol{\gamma}^{(ab)}, \boldsymbol{\gamma}^{(cd)}, \Gamma)\}\Big|_{\gamma^{(ab)}=\gamma^{(cd)}=0,\Gamma=\Sigma},$$

where

$$\begin{aligned} \Theta &= E\exp\Big[(\boldsymbol{Z}_{U^\dagger}^{(ab)})'\boldsymbol{\partial}^{(ab)} + (\boldsymbol{Z}_{U^\dagger}^{(cd)})'\boldsymbol{\partial}^{(cd)} + \mathrm{tr}\{(S_{pool,U^\dagger} - \Sigma)\partial\}\Big] \\ &= \prod_{\ell=1}^{q} E\exp\Big[N_\ell^{1/2}(\overline{\boldsymbol{U}}^{\dagger(\ell)})'(\boldsymbol{\partial}_{[\ell]}^{(ab)} + \boldsymbol{\partial}_{[\ell]}^{(cd)}) + \frac{\eta_N^{(\ell)} - 1/N}{1 - q/N}\,\mathrm{tr}\{(S_{U^\dagger}^{(\ell)} - \Sigma)\partial\}\Big] \\ &= \prod_{\ell=1}^{q} \Theta_\ell(\boldsymbol{\partial}_{[\ell]}^{(ab)} + \boldsymbol{\partial}_{[\ell]}^{(cd)}, \eta_N^{(\ell)}\partial; N_\ell) + \mathrm{o}(N^{-1}), \end{aligned}$$

with $\Theta_\ell(\boldsymbol{\partial}, \partial; N)$ being nothing but the differential operator given by Kakizawa and Iwashita (2005a);

$$\Theta_\ell(\boldsymbol{\partial}, \partial; N) = \exp\Big(\frac{1}{2}\,\boldsymbol{\partial}'\Sigma\boldsymbol{\partial}\Big)\Big[1 + \frac{1}{N^{1/2}}\,\Theta_{\ell,1} + \frac{1}{N}\Big\{\mathrm{tr}(\Sigma\partial\Sigma\partial) + \Theta_{\ell,2} + \frac{1}{2}\,\Theta_{\ell,1}^2\Big\}\Big],$$

where

$$\Theta_{\ell,1} = \sum_{j_1 j_2 j_3 = 1}^{p} \kappa_{j_1,j_2,j_3}^{(\ell)}\Big(\partial_{j_1 j_2}\partial_{j_3} + \frac{1}{6}\,\partial_{j_1}\partial_{j_2}\partial_{j_3}\Big),$$

$$\Theta_{\ell,2} = \frac{1}{2}\sum_{j_1 j_2 j_3 j_4 = 1}^{p} \kappa_{j_1,j_2,j_3,j_4}^{(\ell)}\Big(\partial_{j_1 j_2}\partial_{j_3 j_4} + \partial_{j_1 j_2}\partial_{j_3}\partial_{j_4} + \frac{1}{12}\,\partial_{j_1}\partial_{j_2}\partial_{j_3}\partial_{j_4}\Big). \quad \square$$

The following lemmas of independent interest are useful for our derivation. Lemma 8 can be easily shown with a slight modification of Kakizawa and Iwashita (2005a).

Lemma 7 (Kakizawa and Iwashita (2005a)). *Let $1 \le a < b \le q$ be given integers. For any $t_a \in \mathbf{R}$,*

$$\exp\Big\{\frac{1}{2}\,(\boldsymbol{\partial}^{(ab)})'\Sigma\boldsymbol{\partial}^{(ab)}\Big\}\exp\{\mathrm{i}t_a(\boldsymbol{\gamma}^{(ab)})'\Sigma^{-1}\boldsymbol{\gamma}^{(ab)}\}\Big|_{\gamma^{(ab)}=0} = (\varphi_a)^{p/2}$$

and

$$\exp\Big\{\frac{1}{2}\,(\boldsymbol{\partial}^{(ab)})'\Sigma\boldsymbol{\partial}^{(ab)}\Big\}\Big(\prod_{\ell=1}^{v}\partial_{j_\ell}^{(ab)}\Big)\exp\{\mathrm{i}t_a(\boldsymbol{\gamma}^{(ab)})'\Sigma^{-1}\boldsymbol{\gamma}^{(ab)}\}\Big|_{\gamma^{(ab)}=0} \tag{15}$$
$$= \begin{cases} (\varphi_a)^{p/2}(\varphi_a-1)^m\Big\langle\dfrac{(2m)!}{2^m m!}\Big\rangle\sigma^{j_1j_2}\cdots\sigma^{j_{2m-1}j_{2m}}\,, & \text{if } v=2m(\neq 0)\\ 0\,, & \text{if } v=2m+1\end{cases}$$

($m \in \mathbf{N}_0$; a set of nonnegative integers), where $\varphi_a = (1-2\mathrm{i}t_a)^{-1}$ and $\langle n\rangle$ before terms with indices means a sum of n similar terms obtained by the permutation of $\{j_1,\dots,j_{2m}\}$. Here $(2m)!/(2^m m!)$ is the number of the partitions of $\{1,\dots,2m\}$ into m pairs.

Lemma 8. *Given integers a_1, a_2, c, such that $a_1, a_2, c \in \{1,\dots,q\}$ are all different,*

$$\widetilde{\Xi}_0\Big[\prod_{\ell=1}^{2}\exp\{\mathrm{i}t_\ell(\boldsymbol{\gamma}^{(a_\ell c)})'\Sigma^{-1}\boldsymbol{\gamma}^{(a_\ell c)}\}\Big]\Big|_{\gamma^{(a_1c)}=\gamma^{(a_2c)}=0} = \phi^{-p/2}$$

for any $(t_1,t_2)' \in \mathbf{R}^2$, where

$$\phi = (1-2\mathrm{i}t_1)(1-2\mathrm{i}t_2) - \eta^2_{a_1a_2,c}(2\mathrm{i}t_1)(2\mathrm{i}t_2)\,.$$

Further,

$$\widetilde{\Xi}_0\Big(\prod_{\ell=1}^{v_1}\partial_{j_\ell}^{(a_1c)}\Big)\Big(\prod_{\ell=v_1+1}^{v_1+v_2}\partial_{j_\ell}^{(a_2c)}\Big)\Big[\prod_{\ell=1}^{2}\exp\{\mathrm{i}t_\ell(\boldsymbol{\gamma}^{(a_\ell c)})'\Sigma^{-1}\boldsymbol{\gamma}^{(a_\ell c)}\}\Big]\Big|_{\gamma^{(a_1c)}=\gamma^{(a_2c)}=0} \tag{16}$$
$$= \begin{cases}\phi^{-(p+2m)/2}\Big\langle\dfrac{(2m)!}{2^m m!}\Big\rangle[\Lambda_{a_1a_2,c}]^{\beta_1\beta_2}_{j_1j_2}\cdots[\Lambda_{a_1a_2,c}]^{\beta_{2m-1}\beta_{2m}}_{j_{2m-1}j_{2m}}, & \text{if } v=2m\\ 0, & \text{if } v=2m+1\end{cases}$$

for any nonnegative integers v_1 and v_2 ($v \equiv v_1+v_2 \neq 0$), where we set $\beta_1 = \cdots = \beta_{v_1} = 1$ and $\beta_{v_1+1} = \cdots = \beta_{v_1+v_2} = 2$. Here, $\langle n\rangle$ before terms with indices means a sum of n similar terms obtained by the permutation of $\{\binom{\beta_1}{j_1},\dots,\binom{\beta_{2m}}{j_{2m}}\}$, and $[\Lambda_{a_1a_2,c}]^{\beta_r\beta_{r'}}_{j_rj_{r'}}$ denotes the $(j_r, j_{r'})$-th element of the $(\beta_r,\beta_{r'})$-th block of

$$\Lambda_{a_1a_2,c} = \begin{pmatrix}(2\mathrm{i}t_1)(1-2\mathrm{i}t_2) & \eta_{a_1a_2,c}(2\mathrm{i}t_1)(2\mathrm{i}t_2)\\ \eta_{a_1a_2,c}(2\mathrm{i}t_1)(2\mathrm{i}t_2) & (2\mathrm{i}t_2)(1-2\mathrm{i}t_1)\end{pmatrix}\otimes\Sigma^{-1}\,,$$

that is,

$$[\Lambda_{a_1a_2,c}]_{j_rj_{r'}}^{\beta_r\beta_{r'}} = \begin{cases} \sigma^{j_rj_{r'}}(2\mathrm{i}t_1)(1-2\mathrm{i}t_2)\,, & \textit{if } (\beta_r,\beta_{r'})=(1,1) \\ \sigma^{j_rj_{r'}}\eta_{a_1a_2,c}(2\mathrm{i}t_1)(2\mathrm{i}t_2)\,, & \textit{if } (\beta_r,\beta_{r'})=(1,2),(2,1) \\ \sigma^{j_rj_{r'}}(2\mathrm{i}t_2)(1-2\mathrm{i}t_1)\,, & \textit{if } (\beta_r,\beta_{r'})=(2,2). \end{cases}$$

Remark 3. Let v_1 and v_2 be nonnegative integers and let $n = 1, 2$. To evaluate either

$$\exp\Big\{\frac{1}{2}\,(\boldsymbol{\partial}^{(ab)})'\Sigma\boldsymbol{\partial}^{(ab)} + \frac{1}{2}\,(\boldsymbol{\partial}^{(cd)})'\Sigma\boldsymbol{\partial}^{(cd)}\Big\}\Big(\prod_{\ell=1}^{v_1}\partial_{j_\ell}^{(ab)}\Big)\Big(\prod_{\ell=v_1+1}^{v_1+v_2}\partial_{j_\ell}^{(cd)}\Big) \times\Big(\prod_{\ell=1}^{n}\partial_{k_{2\ell-1}k_{2\ell}}\Big)\exp\{\mathrm{i}t_1F(\boldsymbol{\gamma}^{(ab)},\Gamma)+\mathrm{i}t_2F(\boldsymbol{\gamma}^{(cd)},\Gamma)\}\Big|_{\gamma^{(ab)}=\gamma^{(cd)}=0,\Gamma=\Sigma}$$

for given integers a, b, c, d, such that $a, b, c, d \in \{1, \ldots, q\}$ are all different, or

$$\exp\Big\{\frac{1}{2}\,(\boldsymbol{\partial}_{(a_2c)}^{(a_1c)})'R_{a_1a_2,c}\boldsymbol{\partial}_{(a_2c)}^{(a_1c)}\Big\}\Big(\prod_{\ell=1}^{v_1}\partial_{j_\ell}^{(a_1c)}\Big)\Big(\prod_{\ell=v_1+1}^{v_1+v_2}\partial_{j_\ell}^{(a_2c)}\Big) \times\Big(\prod_{\ell=1}^{n}\partial_{k_{2\ell-1}k_{2\ell}}\Big)\exp\{\mathrm{i}t_1F(\boldsymbol{\gamma}^{(a_1c)},\Gamma)+\mathrm{i}t_2F(\boldsymbol{\gamma}^{(a_2c)},\Gamma)\}\Big|_{\gamma^{(a_1c)}=\gamma^{(a_2c)}=0,\Gamma=\Sigma}$$

for given integers a_1, a_2, c, such that $a_1, a_2, c \in \{1, \ldots, q\}$ are all different, we shall first operate ∂_{jk}'s n times and then put $\Gamma = \Sigma$ before applying the formulae (15) or (16). In that process, we can use the following device (see Kakizawa and Iwashita (2005a)):

$$\partial_{k_1k_2}\exp\{\mathrm{i}t_aF(\boldsymbol{\gamma}^{(ab)},\Gamma)+\mathrm{i}t_cF(\boldsymbol{\gamma}^{(cd)},\Gamma)\}\Big|_{\Gamma=\Sigma}$$
$$= \begin{cases} \frac{1}{2}\Big(2\sigma^{k_1k_2} - \frac{\partial_{k_1}^{(ab)}\partial_{k_2}^{(ab)}}{2\mathrm{i}t_a} - \frac{\partial_{k_1}^{(cd)}\partial_{k_2}^{(cd)}}{2\mathrm{i}t_c}\Big) & \\ \quad\times\exp\{\mathrm{i}t_a(\boldsymbol{\gamma}^{(ab)})'\Sigma^{-1}\boldsymbol{\gamma}^{(ab)}+\mathrm{i}t_c(\boldsymbol{\gamma}^{(cd)})'\Sigma^{-1}\boldsymbol{\gamma}^{(cd)}\}\,, & t_at_c\neq 0 \\ \frac{1}{2}\Big(\sigma^{k_1k_2} - \frac{\partial_{k_1}^{(ab)}\partial_{k_2}^{(ab)}}{2\mathrm{i}t_a}\Big)\exp\{\mathrm{i}t_a\boldsymbol{\gamma}^{(ab)})'\Sigma^{-1}\boldsymbol{\gamma}^{(ab)}\}\,, & t_a\neq 0\,,\ t_c=0 \\ \frac{1}{2}\Big(\sigma^{k_1k_2} - \frac{\partial_{k_1}^{(cd)}\partial_{k_2}^{(cd)}}{2\mathrm{i}t_c}\Big)\exp\{\mathrm{i}t_c(\boldsymbol{\gamma}^{(cd)})'\Sigma^{-1}\boldsymbol{\gamma}^{(cd)}\}\,, & t_a=0\,,\ t_c\neq 0 \\ 0\,, & t_a=t_c=0 \end{cases}$$

and

$$\partial_{k_1k_2}\partial_{k_3k_4}\exp\{\mathrm{i}t_aF(\boldsymbol{\gamma}^{(ab)},\Gamma)+\mathrm{i}t_cF(\boldsymbol{\gamma}^{(cd)},\Gamma)\}\Big|_{\Gamma=\Sigma}$$

$$=\begin{cases}\dfrac{1}{4}\Big\{-2(\sigma^{k_1k_3}\sigma^{k_2k_4}+\sigma^{k_1k_4}\sigma^{k_2k_3}) \\ \quad +\displaystyle\prod_{r=1}^{2}\Big(2\sigma^{k_{2r-1}k_{2r}}-\frac{\partial^{(ab)}_{k_{2r-1}}\partial^{(ab)}_{k_{2r}}}{2\mathrm{i}t_a}-\frac{\partial^{(cd)}_{k_{2r-1}}\partial^{(cd)}_{k_{2r}}}{2\mathrm{i}t_c}\Big)\Big\} \\ \quad \times\exp\{\mathrm{i}t_a(\boldsymbol{\gamma}^{(ab)})'\Sigma^{-1}\boldsymbol{\gamma}^{(ab)}+\mathrm{i}t_c(\boldsymbol{\gamma}^{(cd)})'\Sigma^{-1}\boldsymbol{\gamma}^{(cd)}\}\,, & t_at_c\neq 0 \\ \dfrac{1}{4}\Big\{-(\sigma^{k_1k_3}\sigma^{k_2k_4}+\sigma^{k_1k_4}\sigma^{k_2k_3})+\displaystyle\prod_{r=1}^{2}\Big(\sigma^{k_{2r-1}k_{2r}}-\frac{\partial^{(ab)}_{k_{2r-1}}\partial^{(ab)}_{k_{2r}}}{2\mathrm{i}t_a}\Big)\Big\} \\ \quad \times\exp\{\mathrm{i}t_a(\boldsymbol{\gamma}^{(ab)})'\Sigma^{-1}\boldsymbol{\gamma}^{(ab)}\}\,, & t_a\neq 0\,,\ t_c=0 \\ \dfrac{1}{4}\Big\{-(\sigma^{k_1k_3}\sigma^{k_2k_4}+\sigma^{k_1k_4}\sigma^{k_2k_3})+\displaystyle\prod_{r=1}^{2}\Big(\sigma^{k_{2r-1}k_{2r}}-\frac{\partial^{(cd)}_{k_{2r-1}}\partial^{(cd)}_{k_{2r}}}{2\mathrm{i}t_c}\Big)\Big\} \\ \quad \times\exp\{\mathrm{i}t_c(\boldsymbol{\gamma}^{(cd)})'\Sigma^{-1}\boldsymbol{\gamma}^{(cd)}\}\,, & t_a=0\,,\ t_c\neq 0 \\ 0\,, & t_a=t_c=0\,.\end{cases}$$

We notice that if assumption (ii) in Corollary 3 and the equality $N_1=\cdots=N_q$ of sample sizes are imposed, $\kappa^{*I}_{j_1j_2j_3;ab}=\kappa^{I}_{j_1j_2j_3;ab}=\kappa^{*II}_{j_1j_2j_3;ab}=\kappa^{II}_{j_1j_2j_3;ab}=0$, hence $\Xi_1=0$ (see Lemma 5) and

$$\widetilde{\Xi}_1=-\frac{1}{4}\Big(\frac{q}{2}\Big)^{1/2}\sum_{j_1j_2j_3=1}^{p}\kappa^{(c)}_{j_1,j_2,j_3}(\partial^{(a_1c)}_{j_1}\partial^{(a_1c)}_{j_2}\partial^{(a_2c)}_{j_3}+\partial^{(a_1c)}_{j_1}\partial^{(a_2c)}_{j_2}\partial^{(a_2c)}_{j_3})$$

(see Lemma 6).

§5. Asymptotic expansion for $\overline{P}_{ab:cd}(x)=P(T^2_{ab}>x,T^2_{cd}>x)$

5.1. The case where a,b,c,d are all different

We define

$$C_n(\varphi_a,\varphi_c)$$
$$=\frac{1}{4}\{-2(-p+p^2)-4p(\varphi_a+\varphi_c)+(2p+p^2)(\varphi_a^2+\varphi_c^2)+2p\varphi_a\varphi_c\}\,,$$
$$C_4^{\mathrm{All}}(\varphi_a,\varphi_c)$$
$$=\frac{1}{8}\{-4(\varphi_a+\varphi_c)+3(\varphi_a^2+\varphi_c^2)+2\varphi_a\varphi_c\}K_4^{\mathrm{All}}\,,$$
$$C_4^{I}(\varphi_a,\varphi_c)$$
$$=\frac{1}{4}\{-2(K^{(ab)}_{4,I}+K^{(cd)}_{4,I})+(5K^{(ab)}_{4,I}+K^{(cd)}_{4,I})\varphi_a+(K^{(ab)}_{4,I}+5K^{(cd)}_{4,I})\varphi_c$$

$$
\begin{aligned}
&\qquad - 3(K_{4,I}^{(ab)}\varphi_a^2 + K_{4,I}^{(cd)}\varphi_c^2) - (K_{4,I}^{(ab)} + K_{4,I}^{(cd)})\varphi_a\varphi_c\}\,, \\
&C_4^{II}(\varphi_a, \varphi_c) \\
&\quad = \frac{1}{8}\{(K_{4,II}^{(ab)} + K_{4,II}^{(cd)}) + (-2\varphi_a + \varphi_a^2)K_{4,II}^{(ab)} + (-2\varphi_c + \varphi_c^2)K_{4,II}^{(cd)}\}\,, \\
&C_{33}^{I^2}(\varphi_a, \varphi_c) \\
&\quad = \frac{1}{4}\{2(K_{33,1;I^2}^{(ab)(ab)} + K_{33,1;I^2}^{(cd)(cd)}) - 2K_{33,1;I^2}^{(ab)(ab)}\varphi_a - 2K_{33,1;I^2}^{(cd)(cd)}\varphi_c \\
&\qquad - (3K_{33,1;I^2}^{(ab)(ab)} + K_{33,1;I^2}^{(cd)(cd)})\varphi_a^2 - (K_{33,1;I^2}^{(ab)(ab)} + 3K_{33,1;I^2}^{(cd)(cd)})\varphi_c^2 \\
&\qquad + 3K_{33,1;I^2}^{(ab)(ab)}\varphi_a^3 + 3K_{33,1;I^2}^{(cd)(cd)}\varphi_c^3 + K_{33,1;I^2}^{(cd)(cd)}\varphi_a^2\varphi_c + K_{33,1;I^2}^{(ab)(ab)}\varphi_a\varphi_c^2\} \\
&\qquad + \frac{1}{8}\{-4(K_{33,2;I^2}^{(ab)(ab)} + K_{33,2;I^2}^{(cd)(cd)}) \\
&\qquad + 4(4K_{33,2;I^2}^{(ab)(ab)} + K_{33,2;I^2}^{(cd)(cd)})\varphi_a + 4(K_{33,2;I^2}^{(ab)(ab)} + 4K_{33,2;I^2}^{(cd)(cd)})\varphi_c \\
&\qquad - (21K_{33,2;I^2}^{(ab)(ab)} + K_{33,2;I^2}^{(cd)(cd)})\varphi_a^2 - (K_{33,2;I^2}^{(ab)(ab)} + 21K_{33,2;I^2}^{(cd)(cd)})\varphi_c^2 \\
&\qquad + 9K_{33,2;I^2}^{(ab)(ab)}\varphi_a^3 + 9K_{33,2;I^2}^{(cd)(cd)}\varphi_c^3 - 10(K_{33,2;I^2}^{(ab)(ab)} + K_{33,2;I^2}^{(cd)(cd)})\varphi_a\varphi_c \\
&\qquad + (6K_{33,2;I^2}^{(ab)(ab)} + K_{33,2;I^2}^{(cd)(cd)})\varphi_a^2\varphi_c + (K_{33,2;I^2}^{(ab)(ab)} + 6K_{33,2;I^2}^{(cd)(cd)})\varphi_a\varphi_c^2\}\,, \\
&C_{33}^{II^2}(\varphi_a, \varphi_c) \\
&\quad = \frac{1}{12}\{-(K_{33,1;II^2}^{(ab)(ab)} + K_{33,1;II^2}^{(cd)(cd)}) \\
&\qquad + K_{33,1;II^2}^{(ab)(ab)}(3\varphi_a - 3\varphi_a^2 + \varphi_a^3) + K_{33,1;II^2}^{(cd)(cd)}(3\varphi_c - 3\varphi_c^2 + \varphi_c^3)\} \\
&\qquad + \frac{1}{8}\{-(K_{33,2;II^2}^{(ab)(ab)} + K_{33,2;II^2}^{(cd)(cd)}) \\
&\qquad + K_{33,2;II^2}^{(ab)(ab)}(3\varphi_a - 3\varphi_a^2 + \varphi_a^3) + K_{33,2;II^2}^{(cd)(cd)}(3\varphi_c - 3\varphi_c^2 + \varphi_c^3)\}\,, \\
&C_{33}^{I\cdot II}(\varphi_a, \varphi_c) \\
&\quad = \frac{1}{2}\{(-\varphi_a + 2\varphi_a^2 - \varphi_a^3)K_{33,1;I\cdot II}^{(ab)(ab)} + (-\varphi_c + 2\varphi_c^2 - \varphi_c^3)K_{33,1;I\cdot II}^{(cd)(cd)}\} \\
&\qquad + \frac{1}{4}\{2(K_{33,2;I\cdot II}^{(ab)(ab)} + K_{33,2;I\cdot II}^{(cd)(cd)}) \\
&\qquad - (7K_{33,2;I\cdot II}^{(ab)(ab)} + K_{33,2;I\cdot II}^{(cd)(cd)})\varphi_a - (K_{33,2;I\cdot II}^{(ab)(ab)} + 7K_{33,2;I\cdot II}^{(cd)(cd)})\varphi_c \\
&\qquad + 8K_{33,2;I\cdot II}^{(ab)(ab)}\varphi_a^2 + 8K_{33,2;I\cdot II}^{(cd)(cd)}\varphi_c^2 - 3K_{33,2;I\cdot II}^{(ab)(ab)}\varphi_a^3 - 3K_{33,2;I\cdot II}^{(cd)(cd)}\varphi_c^3 \\
&\qquad + 2(K_{33,2;I\cdot II}^{(ab)(ab)} + K_{33,2;I\cdot II}^{(cd)(cd)})\varphi_a\varphi_c - K_{33,2;I\cdot II}^{(ab)(ab)}\varphi_a^2\varphi_c - K_{33,2;I\cdot II}^{(cd)(cd)}\varphi_a\varphi_c^2\}\,,
\end{aligned}
$$

where

$$
K_4^{\mathrm{All}} = \sum_{j_1j_2j_3j_4=1}^{p} \kappa_{j_1j_2j_3j_4}^{\mathrm{All}}\sigma^{j_1j_2}\sigma^{j_3j_4}\,, \tag{17}
$$

$$K_{4,J}^{(ab)} = \sum_{j_1j_2j_3j_4=1}^{p} \kappa^{J}_{j_1j_2j_3j_4;ab}\sigma^{j_1j_2}\sigma^{j_3j_4}\,, \quad J = I, II\,, \tag{18}$$

$$K_{33,1;I^2}^{(ab)(ab)} = \sum_{j_1j_2j_3j_4j_5j_6=1}^{p} \kappa^{I}_{j_1j_2j_3;ab}\kappa^{I}_{j_4j_5j_6;ab}\sigma^{j_1j_4}\sigma^{j_2j_5}\sigma^{j_3j_6}\,, \tag{19}$$

$$K_{33,2;I^2}^{(ab)(ab)} = \sum_{j_1j_2j_3j_4j_5j_6=1}^{p} \kappa^{I}_{j_1j_2j_3;ab}\kappa^{I}_{j_4j_5j_6;ab}\sigma^{j_1j_2}\sigma^{j_3j_4}\sigma^{j_5j_6}\,, \tag{20}$$

$$K_{33,1;II^2}^{(ab)(ab)} = \sum_{j_1j_2j_3j_4j_5j_6=1}^{p} \kappa^{II}_{j_1j_2j_3;ab}\kappa^{II}_{j_4j_5j_6;ab}\sigma^{j_1j_4}\sigma^{j_2j_5}\sigma^{j_3j_6}\,, \tag{21}$$

$$K_{33,2;II^2}^{(ab)(ab)} = \sum_{j_1j_2j_3j_4j_5j_6=1}^{p} \kappa^{II}_{j_1j_2j_3;ab}\kappa^{II}_{j_4j_5j_6;ab}\sigma^{j_1j_2}\sigma^{j_3j_4}\sigma^{j_5j_6}\,, \tag{22}$$

$$K_{33,1;I\cdot II}^{(ab)(ab)} = \sum_{j_1j_2j_3j_4j_5j_6=1}^{p} \kappa^{I}_{j_1j_2j_3;ab}\kappa^{II}_{j_4j_5j_6;ab}\sigma^{j_1j_4}\sigma^{j_2j_5}\sigma^{j_3j_6}\,, \tag{23}$$

$$K_{33,2;I\cdot II}^{(ab)(ab)} = \sum_{j_1j_2j_3j_4j_5j_6=1}^{p} \kappa^{I}_{j_1j_2j_3;ab}\kappa^{II}_{j_4j_5j_6;ab}\sigma^{j_1j_2}\sigma^{j_3j_4}\sigma^{j_5j_6}\,. \tag{24}$$

We can write (17)–(24) in terms of summarized cumulants (9)–(11), as follows:

$$K_4^{\mathrm{All}} = \sum_{i=1}^{q} \eta_i K_4^{(i)}\,, \quad K_{4,I}^{(ab)} = \frac{\eta_b K_4^{(a)} + \eta_a K_4^{(b)}}{\eta_a + \eta_b}\,, \quad K_{4,II}^{(ab)} = \frac{\eta_b^3 K_4^{(a)} + \eta_a^3 K_4^{(b)}}{(\eta_a+\eta_b)^2\eta_a\eta_b}\,,$$

$$\begin{aligned}
K_{33,J;I^2}^{(ab)(ab)} &= \frac{\eta_a\eta_b}{\eta_a+\eta_b}\,(K_{33,J}^{(aa)} - 2K_{33,J}^{(ab)} + K_{33,J}^{(bb)})\,,\\
K_{33,J;II^2}^{(ab)(ab)} &= \frac{1}{(\eta_a+\eta_b)^3}\left(\frac{\eta_b^3}{\eta_a}\,K_{33,J}^{(aa)} - 2\eta_a\eta_b K_{33,J}^{(ab)} + \frac{\eta_a^3}{\eta_b}\,K_{33,J}^{(aa)}\right),\\
K_{33,J;I\cdot II}^{(ab)(ab)} &= \frac{1}{(\eta_a+\eta_b)^2}\,\{\eta_b^2 K_{33,J}^{(aa)} - (\eta_a^2+\eta_b^2)K_{33,J}^{(ab)} + \eta_a^2 K_{33,J}^{(aa)}\}\,, \quad J = 1, 2\,.
\end{aligned}$$

We also prepare

$$\begin{aligned}
C_n(\varphi_a, 1) &= \frac{1}{4}\,\{-p^2 - 2p\varphi_a + (2p+p^2)\varphi_a^2\}\,,\\
C_4^{\mathrm{All}}(\varphi_a, 1) &= \frac{1}{8}\,(-1 - 2\varphi_a + 3\varphi_a^2)K_4^{\mathrm{All}}\,,\\
C_4^{I}(\varphi_a, 1) &= \frac{1}{4}\,(-1 + 4\varphi_a - 3\varphi_a^2)K_{4,I}^{(ab)}\,,\\
C_4^{II}(\varphi_a, 1) &= \frac{1}{8}\,(1 - 2\varphi_a + \varphi_a^2)K_{4,II}^{(ab)}\,,
\end{aligned}$$

$$
\begin{aligned}
C_{33}^{I^2}(\varphi_a,1) &= \frac{1}{8}\{2(1-\varphi_a-3\varphi_a^2+3\varphi_a^3)K_{33,1;I^2}^{(ab)(ab)} \\
&\quad +(-1+7\varphi_a-15\varphi_a^2+9\varphi_a^3)K_{33,2;I^2}^{(ab)(ab)}\}\,, \\
C_{33}^{II^2}(\varphi_a,1) &= \frac{1}{12}(-1+3\varphi_a-3\varphi_a^2+\varphi_a^3)K_{33,1;II^2}^{(ab)(ab)} \\
&\quad +\frac{1}{8}(-1+3\varphi_a-3\varphi_a^2+\varphi_a^3)K_{33,2;II^2}^{(ab)(ab)}\,, \\
C_{33}^{I\cdot II}(\varphi_a,1) &= \frac{1}{4}\{2(-\varphi_a+2\varphi_a^2-\varphi_a^3)K_{33,1;I\cdot II}^{(ab)(ab)} \\
&\quad +(1-5\varphi_a+7\varphi_a^2-3\varphi_a^3)K_{33,2;I\cdot II}^{(ab)(ab)}\}\,.
\end{aligned}
$$

As a corollary of Lemma 5, it is straightforward to get the following final result (we now used only the formula (15) but left the arrangement of several terms with respect to φ_a and φ_c, as in Remark 5 below):

Proposition 9. *Given integers $1 \le a < b^* < c^* < d^* \le q$, we set $(b,c,d) = (b^*,c^*,d^*)$, (c^*,b^*,d^*), (d^*,b^*,c^*). Then,*

$$
\begin{aligned}
&E[\exp(\mathrm{i}t_a\widetilde{T}_{ab}^2+\mathrm{i}t_c\widetilde{T}_{cd}^2)] \\
&\quad = \Xi\exp\{\mathrm{i}t_aF(\boldsymbol{\gamma}^{(ab)},\Gamma)+\mathrm{i}t_cF(\boldsymbol{\gamma}^{(cd)},\Gamma)\}\Big|_{\gamma^{(ab)}=\gamma^{(cd)}=0,\Gamma=\Sigma}+\mathrm{o}(N^{-1}) \\
&\quad = (\varphi_a\varphi_c)^{p/2}\Big[1+\frac{1}{N}\{C_n(\varphi_a,\varphi_c)+C_4(\varphi_a,\varphi_c)+C_{33}(\varphi_a,\varphi_c)\}\Big]+\mathrm{o}(N^{-1})\,,
\end{aligned}
$$

where

$$C_4(\varphi_a,\varphi_c) = C_4^{\mathrm{All}}(\varphi_a,\varphi_c)+C_4^{I}(\varphi_a,\varphi_c)+C_4^{II}(\varphi_a,\varphi_c)$$

and

$$C_{33}(\varphi_a,\varphi_c) = C_{33}^{I^2}(\varphi_a,\varphi_c)+C_{33}^{II^2}(\varphi_a,\varphi_c)+C_{33}^{I\cdot II}(\varphi_a,\varphi_c)\,.$$

We have $C_{33}(\varphi_a,\varphi_c)\equiv 0$ under the same assumptions as in Corollary 2 or 3.

Remark 4. For any $(a,b)\in J$, we have

$$
\begin{aligned}
(25)\quad & E[\exp(\mathrm{i}t_a\widetilde{T}_{ab}^2)] \\
&\quad = (\varphi_a)^{p/2}\Big[1+\frac{1}{N}\{C_n(\varphi_a,1)+C_4(\varphi_a,1)+C_{33}(\varphi_a,1)\}\Big]+\mathrm{o}(N^{-1}) \\
&\quad = (\varphi_a)^{p/2}\Big[1+\frac{1}{N}\sum_{\ell=0}^{3}\pi_{\ell,ab}(\varphi_a)^{\ell}\Big]+\mathrm{o}(N^{-1})\,,
\end{aligned}
$$

where

$$\pi_{0,ab} = -\frac{p^2}{4}+\Big(-\frac{K_4^{\mathrm{All}}}{8}-\frac{K_{4,I}^{(ab)}}{4}+\frac{K_{4,II}^{(ab)}}{8}\Big)$$

$$
\begin{aligned}
&+\Big(\frac{K^{(ab)(ab)}_{33,1;I^2}}{4}-\frac{K^{(ab)(ab)}_{33,1;II^2}}{12}\Big)\\
&+\Big(-\frac{K^{(ab)(ab)}_{33,2;I^2}}{8}-\frac{K^{(ab)(ab)}_{33,2;II^2}}{8}+\frac{K^{(ab)(ab)}_{33,2;I\cdot II}}{4}\Big)\,,\\
\pi_{1,ab}={}&-\frac{p}{2}+\Big(-\frac{K_4^{\mathrm{All}}}{4}+K^{(ab)}_{4,I}-\frac{K^{(ab)}_{4,II}}{4}\Big)\\
&+\Big(-\frac{K^{(ab)(ab)}_{33,1;I^2}}{4}+\frac{K^{(ab)(ab)}_{33,1;II^2}}{4}-\frac{K^{(ab)(ab)}_{33,1;I\cdot II}}{2}\Big)\\
&+\Big(\frac{7K^{(ab)(ab)}_{33,2;I^2}}{8}+\frac{3K^{(ab)(ab)}_{33,2;II^2}}{8}-\frac{5K^{(ab)(ab)}_{33,2;I\cdot II}}{4}\Big)\,,\\
\pi_{2,ab}={}&\frac{p(p+2)}{4}+\Big(\frac{3K_4^{\mathrm{All}}}{8}-\frac{3K^{(ab)}_{4,I}}{4}+\frac{K^{(ab)}_{4,II}}{8}\Big)\\
&+\Big(-\frac{3K^{(ab)(ab)}_{33,1;I^2}}{4}-\frac{K^{(ab)(ab)}_{33,1;II^2}}{4}+K^{(ab)(ab)}_{33,1;I\cdot II}\Big)\\
&+\Big(-\frac{15K^{(ab)(ab)}_{33,2;I^2}}{8}-\frac{3K^{(ab)(ab)}_{33,2;II^2}}{8}+\frac{7K^{(ab)(ab)}_{33,2;I\cdot II}}{4}\Big)\,,\\
\pi_{3,ab}={}&\Big(\frac{3K^{(ab)(ab)}_{33,1;I^2}}{4}+\frac{K^{(ab)(ab)}_{33,1;II^2}}{12}-\frac{K^{(ab)(ab)}_{33,1;I\cdot II}}{2}\Big)\\
&+\Big(\frac{9K^{(ab)(ab)}_{33,2;I^2}}{8}+\frac{K^{(ab)(ab)}_{33,2;II^2}}{8}-\frac{3K^{(ab)(ab)}_{33,2;I\cdot II}}{4}\Big)\,.
\end{aligned}
$$

A formal inversion of (25), that is,

$$
\Pr(\widetilde{T}^2_{ab}\le x)=G_p(x)+\frac{1}{N}\sum_{\ell=0}^{3}\pi_{\ell,ab}\,G_{p+2\ell}(x)+\mathrm{o}(N^{-1})\,,
$$

is valid under (A_1)–$(\mathrm{A}_5)_2$, as in the second paragraph of Section 4. By virtue of Chibisov (1972) and Magdalinos (1992), Theorem 1 (hence Corollaries 2 and 3) is valid. As a special case of elliptical populations (see Remark 1) with equal sample sizes $N_1=\cdots=N_q$ (hence $\eta_1=\cdots=\eta_q=1/q$), asymptotic expansions of Corollary 2 and Proposition 9 are essentially simplified to those of Seo (2002; Theorem 1 and page 63), since Seo's notation N, which is used for the equal sample size $N_1=\cdots=N_q$, is different from our notation N (the total number of observations).

Remark 5. We note

$$
\begin{aligned}
&C_n(\varphi_a,\varphi_c)+C_4(\varphi_a,\varphi_c)+C_{33}(\varphi_a,\varphi_c)\\
&\quad=d_0+\sum_{\ell=1}^{3}(d'_\ell\varphi_a^\ell+d''_\ell\varphi_c^\ell)+d_4\varphi_a\varphi_c+\varphi_a\varphi_c(d'_5\varphi_a+d''_5\varphi_c)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
d_0 &= -\frac{1}{2}\,(-p+p^2) - \frac{1}{2}\,(K^{(ab)}_{4,I} + K^{(cd)}_{4,I}) + \frac{1}{8}\,(K^{(ab)}_{4,II} + K^{(cd)}_{4,II}) \\
&\quad + \frac{1}{2}\,(K^{(ab)(ab)}_{33,1;I^2} + K^{(cd)(cd)}_{33,1;I^2}) - \frac{1}{2}\,(K^{(ab)(ab)}_{33,2;I^2} + K^{(cd)(cd)}_{33,2;I^2}) \\
&\quad - \frac{1}{12}\,(K^{(ab)(ab)}_{33,1;II^2} + K^{(cd)(cd)}_{33,1;II^2}) - \frac{1}{8}\,(K^{(ab)(ab)}_{33,2;II^2} + K^{(cd)(cd)}_{33,2;II^2}) \\
&\quad + \frac{1}{2}\,(K^{(ab)(ab)}_{33,2;I\cdot II} + K^{(cd)(cd)}_{33,2;I\cdot II})\,, \\
d_1' &= -p - \frac{1}{2}\,K^{\mathrm{All}}_4 + \frac{1}{4}\,(5K^{(ab)}_{4,I} + K^{(cd)}_{4,I}) - \frac{1}{4}\,K^{(ab)}_{4,II} \\
&\quad - \frac{1}{2}\,K^{(ab)(ab)}_{33,1;I^2} + \frac{1}{2}\,(4K^{(ab)(ab)}_{33,2;I^2} + K^{(cd)(cd)}_{33,2;I^2}) \\
&\quad + \frac{1}{4}\,K^{(ab)(ab)}_{33,1;II^2} + \frac{3}{8}\,K^{(ab)(ab)}_{33,2;II^2} - \frac{1}{2}\,K^{(ab)(ab)}_{33,1;I\cdot II} - \frac{1}{4}\,(7K^{(ab)(ab)}_{33,2;I\cdot II} + K^{(cd)(cd)}_{33,2;I\cdot II})\,, \\
d_1'' &= -p - \frac{1}{2}\,K^{\mathrm{All}}_4 + \frac{1}{4}\,(K^{(ab)}_{4,I} + 5K^{(cd)}_{4,I}) - \frac{1}{4}\,K^{(cd)}_{4,II} \\
&\quad - \frac{1}{2}\,K^{(cd)(cd)}_{33,1;I^2} + \frac{1}{2}\,(K^{(ab)(ab)}_{33,2;I^2} + 4K^{(cd)(cd)}_{33,2;I^2}) \\
&\quad + \frac{1}{4}\,K^{(cd)(cd)}_{33,1;II^2} + \frac{3}{8}\,K^{(cd)(cd)}_{33,2;II^2} - \frac{1}{2}\,K^{(cd)(cd)}_{33,1;I\cdot II} - \frac{1}{4}\,(K^{(ab)(ab)}_{33,2;I\cdot II} + 7K^{(cd)(cd)}_{33,2;I\cdot II})\,, \\
d_2' &= \frac{1}{4}\,(2p+p^2) + \frac{3}{8}\,K^{\mathrm{All}}_4 - \frac{3}{4}\,K^{(ab)}_{4,I} + \frac{1}{8}\,K^{(ab)}_{4,II} \\
&\quad - \frac{1}{4}\,(3K^{(ab)(ab)}_{33,1;I^2} + K^{(cd)(cd)}_{33,1;I^2}) - \frac{1}{8}\,(21K^{(ab)(ab)}_{33,2;I^2} + K^{(cd)(cd)}_{33,2;I^2}) \\
&\quad - \frac{1}{4}\,K^{(ab)(ab)}_{33,1;II^2} - \frac{3}{8}\,K^{(ab)(ab)}_{33,2;II^2} + K^{(ab)(ab)}_{33,1;I\cdot II} + 2K^{(ab)(ab)}_{33,2;I\cdot II}\,, \\
d_2'' &= \frac{1}{4}\,(2p+p^2) + \frac{3}{8}\,K^{\mathrm{All}}_4 - \frac{3}{4}\,K^{(cd)}_{4,I} + \frac{1}{8}\,K^{(cd)}_{4,II} \\
&\quad - \frac{1}{4}\,(K^{(ab)(ab)}_{33,1;I^2} + 3K^{(cd)(cd)}_{33,1;I^2}) - \frac{1}{8}\,(K^{(ab)(ab)}_{33,2;I^2} + 21K^{(cd)(cd)}_{33,2;I^2}) \\
&\quad - \frac{1}{4}\,K^{(cd)(cd)}_{33,1;II^2} - \frac{3}{8}\,K^{(cd)(cd)}_{33,2;II^2} + K^{(cd)(cd)}_{33,1;I\cdot II} + 2K^{(cd)(cd)}_{33,2;I\cdot II}\,, \\
d_3' &= \frac{3}{4}\,K^{(ab)(ab)}_{33,1;I^2} + \frac{9}{8}\,K^{(ab)(ab)}_{33,2;I^2} + \frac{1}{12}\,K^{(ab)(ab)}_{33,1;II^2} + \frac{1}{8}\,K^{(ab)(ab)}_{33,2;II^2} \\
&\quad - \frac{1}{2}\,K^{(ab)(ab)}_{33,1;I\cdot II} - \frac{3}{4}\,K^{(ab)(ab)}_{33,2;I\cdot II}\,, \\
d_3'' &= \frac{3}{4}\,K^{(cd)(cd)}_{33,1;I^2} + \frac{9}{8}\,K^{(cd)(cd)}_{33,2;I^2} + \frac{1}{12}\,K^{(cd)(cd)}_{33,1;II^2} + \frac{1}{8}\,K^{(cd)(cd)}_{33,2;II^2} \\
&\quad - \frac{1}{2}\,K^{(cd)(cd)}_{33,1;I\cdot II} - \frac{3}{4}\,K^{(cd)(cd)}_{33,2;I\cdot II}\,, \\
d_4 &= \frac{p}{2} + \frac{1}{4}\,K^{\mathrm{All}}_4 - \frac{1}{4}\,(K^{(ab)}_{4,I} + K^{(cd)}_{4,I})
\end{aligned}
$$

$$-\frac{5}{4}\,(K^{(ab)(ab)}_{33,2;I^2}+K^{(cd)(cd)}_{33,2;I^2})+(K^{(ab)(ab)}_{33,2;I\cdot II}+K^{(cd)(cd)}_{33,2;I\cdot II})\,,$$

$$d'_5=\frac{1}{4}\,K^{(cd)(cd)}_{33,1;I^2}+\frac{1}{8}\,(6K^{(ab)(ab)}_{33,2;I^2}+K^{(cd)(cd)}_{33,2;I^2})-\frac{1}{2}\,K^{(ab)(ab)}_{33,2;I\cdot II}\,,$$

$$d''_5=\frac{1}{4}\,K^{(ab)(ab)}_{33,1;I^2}+\frac{1}{8}\,(K^{(ab)(ab)}_{33,2;I^2}+6K^{(cd)(cd)}_{33,2;I^2})-\frac{1}{2}\,K^{(cd)(cd)}_{33,2;I\cdot II}$$

satisfy $d_0+\sum_{\ell=1}^{3}(d'_\ell+d''_\ell)+d_4+(d'_5+d''_5)=0$. Hence, letting $d_\ell=d'_\ell+d''_\ell$, $\ell=1,2,3,5$, we can show

(26) $\overline{P}_{ab:cd}(x)$

$$=\{\overline{G}_p(x)\}^2+\frac{1}{N}\Big[d_0\{\overline{G}_p(x)\}^2+\sum_{\ell=1}^{3}d_\ell\overline{G}_{p+2\ell}(x)\overline{G}_p(x)$$
$$+d_4\{\overline{G}_{p+2}(x)\}^2+d_5\overline{G}_{p+4}(x)\overline{G}_{p+2}(x)\Big]+\mathrm{o}(N^{-1})$$
$$=\{\overline{G}_p(x)\}^2+\frac{2x}{Np}\left[D_1^{(ab:cd)}(x)g_p(x)\overline{G}_p(x)+\frac{2x}{p}\,D_2^{(ab:cd)}(x)\{g_p(x)\}^2\right]$$
$$+\mathrm{o}(N^{-1})\,,$$

where

$D_1^{(ab:cd)}(x)$

$$=d_1+2d_4+d_5+\Big(1+\frac{x}{p+2}\Big)(d_2+d_5)+\Big\{1+\frac{x}{p+2}+\frac{x^2}{(p+2)(p+4)}\Big\}d_3$$
$$=(-d_0+d_4+d_5)+\frac{x}{p+2}\,(d_2+d_3+d_5)+\frac{x^2}{(p+2)(p+4)}\,d_3$$

and

$$D_2^{(ab:cd)}(x)=d_4+\Big(1+\frac{x}{p+2}\Big)d_5=(d_4+d_5)+\frac{x}{p+2}\,d_5\,,$$

by using the relations

$$\overline{G}_{p+2}(x)=\overline{G}_p(x)+\frac{2x}{p}\,g_p(x)\,,$$
$$\overline{G}_{p+4}(x)=\overline{G}_p(x)+\frac{2x}{p}\Big(1+\frac{x}{p+2}\Big)g_p(x)\,,$$
$$\overline{G}_{p+6}(x)=\overline{G}_p(x)+\frac{2x}{p}\Big\{1+\frac{x}{p+2}+\frac{x^2}{(p+2)(p+4)}\Big\}g_p(x)\,.$$

Writing

$$K^{(ab:cd)}_{4,J}=K^{(ab)}_{4,J}+K^{(cd)}_{4,J}\,,\quad J=I,II$$

and

$$K^{(ab:cd)}_{33,A;J}=K^{(ab)(ab)}_{33,A;J}+K^{(cd)(cd)}_{33,A;J}\,,\quad A=1,2\,;\ J=I^2,II^2,I\cdot II\,,$$

we have

$$\begin{aligned}
d_0 &= -\frac{1}{2}\,(-p+p^2) - \frac{1}{2}\,K_{4,I}^{(ab:cd)} + \frac{1}{8}\,K_{4,II}^{(ab:cd)} \\
&\quad + \frac{1}{2}\,K_{33,1;I^2}^{(ab:cd)} - \frac{1}{2}\,K_{33,2;I^2}^{(ab:cd)} - \frac{1}{12}\,K_{33,1;II^2}^{(ab:cd)} - \frac{1}{8}\,K_{33,2;II^2}^{(ab:cd)} + \frac{1}{2}\,K_{33,2;I\cdot II}^{(ab:cd)}\,, \\
d_1 &= -2p - K_4^{\mathrm{All}} + \frac{3}{2}\,K_{4,I}^{(ab:cd)} - \frac{1}{4}\,K_{4,II}^{(ab:cd)} \\
&\quad - \frac{1}{2}\,K_{33,1;I^2}^{(ab:cd)} + \frac{5}{2}\,K_{33,2;I^2}^{(ab:cd)} + \frac{1}{4}\,K_{33,1;II^2}^{(ab:cd)} + \frac{3}{8}\,K_{33,2;II^2}^{(ab:cd)} \\
&\quad - \frac{1}{2}\,K_{33,1;I\cdot II}^{(ab:cd)} - 2K_{33,2;I\cdot II}^{(ab:cd)}\,, \\
d_2 &= \frac{1}{2}\,(2p+p^2) + \frac{3}{4}\,K_4^{\mathrm{All}} - \frac{3}{4}\,K_{4,I}^{(ab:cd)} + \frac{1}{8}\,K_{4,II}^{(ab:cd)} \\
&\quad - K_{33,1;I^2}^{(ab:cd)} - \frac{11}{4}\,K_{33,2;I^2}^{(ab:cd)} - \frac{1}{4}\,K_{33,1;II^2}^{(ab:cd)} - \frac{3}{8}\,K_{33,2;II^2}^{(ab:cd)} \\
&\quad + K_{33,1;I\cdot II}^{(ab:cd)} + 2K_{33,2;I\cdot II}^{(ab:cd)}\,, \\
d_3 &= \frac{3}{4}\,K_{33,1;I^2}^{(ab:cd)} + \frac{9}{8}\,K_{33,2;I^2}^{(ab:cd)} + \frac{1}{12}\,K_{33,1;II^2}^{(ab:cd)} + \frac{1}{8}\,K_{33,2;II^2}^{(ab:cd)} \\
&\quad - \frac{1}{2}\,K_{33,1;I\cdot II}^{(ab:cd)} - \frac{3}{4}\,K_{33,2;I\cdot II}^{(ab:cd)}\,, \\
d_4 &= \frac{p}{2} + \frac{1}{4}\,K_4^{\mathrm{All}} - \frac{1}{4}\,K_{4,I}^{(ab:cd)} - \frac{5}{4}\,K_{33,2;I^2}^{(ab:cd)} + K_{33,2;I\cdot II}^{(ab:cd)}\,, \\
d_5 &= \frac{1}{4}\,K_{33,1;I^2}^{(ab:cd)} + \frac{7}{8}\,K_{33,2;I^2}^{(ab:cd)} - \frac{1}{2}\,K_{33,2;I\cdot II}^{(ab:cd)}\,.
\end{aligned}$$

Under the same assumptions as in Corollary 2 or 3, we note that

$$D_1^{(ab:cd)}(x) = (-d_0 + d_4) + \frac{x}{p+2}\,d_2 \quad \text{and} \quad D_2^{(ab:cd)}(x) = d_4\,,$$

where

$$\begin{aligned}
-d_0 + d_4 &= \frac{1}{2}\,p^2 + \frac{1}{4}\,K_4^{\mathrm{All}} + \frac{1}{4}\,K_{4,I}^{(ab:cd)} - \frac{1}{8}\,K_{4,II}^{(ab:cd)} \\
&= \frac{1}{2}\,p^2 + \frac{1}{4}\,K_4^{\mathrm{All}} + \Big\{\frac{\eta_b}{4(\eta_a+\eta_b)} - \frac{\eta_b^2}{8(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)} \\
&\qquad + \Big\{\frac{\eta_a}{4(\eta_a+\eta_b)} - \frac{\eta_a^2}{8(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)} \\
&\qquad + \Big\{\frac{\eta_d}{4(\eta_c+\eta_d)} - \frac{\eta_d^2}{8(\eta_c+\eta_d)^2\eta_c}\Big\}K_4^{(c)} \\
&\qquad + \Big\{\frac{\eta_c}{4(\eta_c+\eta_d)} - \frac{\eta_c^2}{8(\eta_c+\eta_d)^2\eta_d}\Big\}K_4^{(d)}\,, \\
d_2 &= \frac{1}{2}\,(2p+p^2) + \frac{3}{4}\,K_4^{\mathrm{All}} - \frac{3}{4}\,K_{4,I}^{(ab:cd)} + \frac{1}{8}\,K_{4,II}^{(ab:cd)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\,p(p+2) + \frac{3}{4}\,K_4^{\rm All} + \Big\{-\frac{3\eta_b}{4(\eta_a+\eta_b)} + \frac{\eta_b^2}{8(\eta_a+\eta_b)^2\eta_a}\Big\}K_4^{(a)} \\
&\qquad + \Big\{-\frac{3\eta_a}{4(\eta_a+\eta_b)} + \frac{\eta_a^2}{8(\eta_a+\eta_b)^2\eta_b}\Big\}K_4^{(b)} \\
&\qquad + \Big\{-\frac{3\eta_d}{4(\eta_c+\eta_d)} + \frac{\eta_d^2}{8(\eta_c+\eta_d)^2\eta_c}\Big\}K_4^{(c)} \\
&\qquad + \Big\{-\frac{3\eta_c}{4(\eta_c+\eta_d)} + \frac{\eta_c^2}{8(\eta_c+\eta_d)^2\eta_d}\Big\}K_4^{(d)}\,, \\
d_4 &= \frac{p}{2} + \frac{1}{4}\,K_4^{\rm All} - \frac{1}{4}\,K_{4,I}^{(ab:cd)} \\
&= \frac{p}{2} + \frac{1}{4}\,K_4^{\rm All} + \Big\{-\frac{\eta_b}{4(\eta_a+\eta_b)}\Big\}K_4^{(a)} + \Big\{-\frac{\eta_a}{4(\eta_a+\eta_b)}\Big\}K_4^{(b)} \\
&\qquad + \Big\{-\frac{\eta_d}{4(\eta_c+\eta_d)}\Big\}K_4^{(c)} + \Big\{-\frac{\eta_c}{4(\eta_c+\eta_d)}\Big\}K_4^{(d)}\,.
\end{aligned}
$$

Especially, letting $\eta_1 = \cdots = \eta_q = 1/q$, we have

$$
\begin{aligned}
-d_0 + d_4 &= \frac{1}{2}\,p^2 + \frac{1}{4}\,\overline{K}_4 + \Big(\frac{1}{8} - \frac{q}{32}\Big)(K_4^{(a)} + K_4^{(b)} + K_4^{(c)} + K_4^{(d)})\,, \\
d_2 &= \frac{1}{2}\,p(p+2) + \frac{3}{4}\,\overline{K}_4 + \Big(-\frac{3}{8} + \frac{q}{32}\Big)(K_4^{(a)} + K_4^{(b)} + K_4^{(c)} + K_4^{(d)})\,, \\
d_4 &= \frac{p}{2} + \frac{1}{4}\,\overline{K}_4 - \frac{1}{8}\,(K_4^{(a)} + K_4^{(b)} + K_4^{(c)} + K_4^{(d)})\,.
\end{aligned}
$$

Under elliptical populations (see Remark 1), our results are essentially simplified to those of Seo (2002; Theorem 3 under the equal sample sizes) or Okamoto (2005; Theorem 1 under the unequal sample sizes), since their notation N (the equal sample size $N_1 = \cdots = N_q$ or the unequal sample size $N = N_1 \geq N_j$, $j \neq 1$) and $g_{p/2}(\eta) = \{\Gamma(p/2)\}^{-1}\eta^{p/2-1}\mathrm{e}^{-\eta}$ and $G_{p/2}(\eta) = \int_\eta^\infty g_{p/2}(t)\,dt$ with $\eta = x/2$ are different from our notation N (the total number of observations) and $g_p(x)$ and $\overline{G}_p(x)$ (the density and survival function of χ_p^2).

5.2. The case where $(a,b),(c,d) \in J$ have exactly one common index

After long but straightforward simplifications of an inversion of

$$
\widetilde{\Xi}\exp\{\mathrm{i}t_1 F(\boldsymbol{\gamma}^{(a_1c)},\Gamma) + \mathrm{i}t_2 F(\boldsymbol{\gamma}^{(a_2c)},\Gamma)\}\Big|_{\gamma^{(a_1c)}=\gamma^{(a_2c)}=0,\Gamma=\Sigma}
$$

(see Lemma 6), together with the relations

$$
\begin{aligned}
\overline{G}_{\nu+2}(x) &= \overline{G}_\nu(x) + \frac{2x}{\nu}\,g_\nu(x)\,, \\
\overline{G}_{\nu+4}(x) &= \overline{G}_\nu(x) + \frac{2x}{\nu}\Big(1 + \frac{x}{\nu+2}\Big)g_\nu(x)\,,
\end{aligned}
$$

we have

Proposition 10. *Suppose that* $N_a/N=\eta_a$, $a=1,\dots,q$, *where* η_a*'s are fixed positive rational numbers satisfying* $\sum_{a=1}^{q}\eta_a=1$. *If the population distributions* $\boldsymbol{U}^{(a)}$*'s satisfy* $\boldsymbol{U}^{(a)}\overset{d}{=}-\boldsymbol{U}^{(a)}$ *for all* $a=1,\dots,q$, *that is,* $\kappa^{(a)}_{j_1,j_2,j_3}=0$ *for all* $j_1,j_2,j_3=1,\dots,p$; $a=1,\dots,q$, *then*

$$\begin{aligned}&P(T^2_{a_1c}>x,T^2_{a_2c}>x)\\&=(1-\eta^2_{a_1a_2,c})^{p/2}\sum_{\ell=0}^{\infty}\frac{\Gamma(p/2+\ell)}{\ell!\Gamma(p/2)}(\eta^2_{a_1a_2,c})^{\ell}\Big[\{\overline{G}_{p+2\ell}(\widetilde{x})\}^2\\&\quad+\frac{\widetilde{x}g_{p+2\ell}(\widetilde{x})}{Np(p+2)}\{D^{(a_1c:a_2c)}_{1,\ell}(\widetilde{x})\overline{G}_{p+2\ell}(\widetilde{x})+\frac{\widetilde{x}}{p+2\ell}D^{(a_1c:a_2c)}_{2,\ell}g_{p+2\ell}(\widetilde{x})\}\Big]\\&\quad+\mathrm{o}(N^{-1})\end{aligned}$$

with

$$\widetilde{x}=\frac{x}{1-\eta^2_{a_1a_2,c}}$$

for given integers a_1,a_2,c, *such that* $a_1,a_2,c\in\{1,\dots,q\}$ *are all different, where*

$$\begin{aligned}&D^{(a_1c:a_2c)}_{1,\ell}(\widetilde{x})\\&=(\widetilde{x}+p-2\ell)p(p+2)+\frac{K_4^{\mathrm{All}}}{2}(3\widetilde{x}+p+2-6\ell)\\&\quad-\frac{K^{(a_1c)}_{4,I}+K^{(a_2c)}_{4,I}}{2(1-\eta^2_{a_1a_2,c})}\{(3\widetilde{x}-p-2-18\ell)+\eta^2_{a_1a_2,c}(3\widetilde{x}+p+2+6\ell)\}\\&\quad+\frac{K^{(a_1c)}_{4,II}+K^{(a_2c)}_{4,II}}{4(1-\eta^2_{a_1a_2,c})^2}\\&\quad\times\{(\widetilde{x}-p-2-6\ell)+2\eta^2_{a_1a_2,c}(p+2+2\ell)+\eta^4_{a_1a_2,c}(\widetilde{x}-p-2-2\ell)\}\\&\quad+\frac{K_4^{(c)}}{1-\eta^2_{a_1a_2,c}}6(\eta^2_{a_1a_2,c}\widetilde{x}-2\ell)\\&\quad+\frac{K_4^{(c)}}{\eta_c(1-\eta^2_{a_1a_2,c})^2}3\eta^2_{a_1a_2,c}(\eta^2_{a_1a_2,c}\widetilde{x}-2\ell)\\&\quad-\frac{K_4^{(c)}}{\eta_c(1-\eta^2_{a_1a_2,c})^2}\Big(\frac{\eta_{a_1}}{\eta_{a_1}+\eta_c}+\frac{\eta_{a_2}}{\eta_{a_2}+\eta_c}\Big)(1+\eta^2_{a_1a_2,c})(\eta^2_{a_1a_2,c}\widetilde{x}-2\ell)\end{aligned}$$

(*this is a linear function*) *and*

$$\begin{aligned}&D^{(a_1c:a_2c)}_{2,\ell}\\&=2(1+2\ell)p(p+2)+K_4^{\mathrm{All}}(p+2+6\ell)\end{aligned}$$

$$
\begin{aligned}
&-\frac{K_{4,I}^{(a_1c)}+K_{4,I}^{(a_2c)}}{1-\eta_{a_1a_2,c}^2}\,(1+\eta_{a_1a_2,c}^2)(p+2+6\ell)\\
&+\frac{K_{4,II}^{(a_1c)}+K_{4,II}^{(a_2c)}}{2(1-\eta_{a_1a_2,c}^2)^2}\,\{2\eta_{a_1a_2,c}^2(p+2+3\ell)-\eta_{a_1a_2,c}^4(p+2+2\ell)\}\\
&+\frac{K_4^{(c)}}{1-\eta_{a_1a_2,c}^2}\,4\eta_{a_1a_2,c}^2(p+2+6\ell)\\
&+\frac{K_4^{(c)}}{\eta_c(1-\eta_{a_1a_2,c}^2)^2}\,\eta_{a_1a_2,c}^2\{(p+2+6\ell)+2\eta_{a_1a_2,c}^2(p+2+3\ell)\}\\
&-\frac{K_4^{(c)}}{\eta_c(1-\eta_{a_1a_2,c}^2)^2}\Big(\frac{\eta_{a_1}}{\eta_{a_1}+\eta_c}+\frac{\eta_{a_2}}{\eta_{a_2}+\eta_c}\Big)2\eta_{a_1a_2,c}^2(p+2+4\ell)
\end{aligned}
$$

(*this is a constant*).

We remark that if the equality $N_1=\cdots=N_q$ of sample sizes is assumed (i.e. $\eta_1=\cdots=\eta_q=1/q$), letting $\widetilde{x}=(4/3)x$, they are simplified to

$$
\begin{aligned}
&D_{1,\ell}^{(a_1c:a_2c)}(\widetilde{x})\\
&\quad=(\widetilde{x}+p-2\ell)p(p+2)+\frac{1}{2}\Big[\overline{K}_4(3\widetilde{x}+p+2-6\ell)\\
&\qquad+\Big\{\frac{K_4^{(a_1)}+K_4^{(a_2)}}{2}\,(-5\widetilde{x}+p+2+22\ell)+K_4^{(c)}(-\widetilde{x}+p+2-10\ell)\Big\}\Big]\\
&\qquad+\frac{q}{72}\Big[\frac{K_4^{(a_1)}+K_4^{(a_2)}}{2}\,\{17\widetilde{x}-9(p+2)-82\ell\}+K_4^{(c)}\{\widetilde{x}-9(p+2)+46\ell\}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&D_{2,\ell}^{(a_1c:a_2c)}\\
&\quad=2(1+2\ell)p(p+2)+\Big[\overline{K}_4-\frac{1}{3}\Big\{\frac{5(K_4^{(a_1)}+K_4^{(a_2)})}{2}+K_4^{(c)}\Big\}\Big](p+2+6\ell)\\
&\qquad+\frac{q}{36}\Big\{\frac{K_4^{(a_1)}+K_4^{(a_2)}}{2}\,(7p+14+22\ell)-K_4^{(c)}(p+2-14\ell)\Big\}.
\end{aligned}
$$

The algebraic structure of $D_{1,\ell}^{(a_1c:a_2c)}(\widetilde{x}), D_{2,\ell}^{(a_1c:a_2c)}$ that we presented above is different from Seo (2002; Theorem 4 under the equal sample sizes, in which $d_2/\eta^2=d_{2,\ell}(\eta)/\eta^2$ is a linear function) or Okamoto (2005; Theorem 2 under the unequal sample sizes, in which $d_2/\eta_2^2=d_{2,\ell}(\eta_2)/\eta_2^2$ is a polynomial of order 2). But, we can validate their results by giving at least two equivalent expressions for the formal asymptotic expansion of the density of $(\widetilde{T}_{a_1c}^2,\widetilde{T}_{a_2c}^2)'$, as follows: Our main task is to rearrange an asymptotic expansion of the characteristic function $E[\exp(\mathrm{i}t_1\widetilde{T}^2_{a_1c}+\mathrm{i}t_2\widetilde{T}^2_{a_2c})]$ in terms of

$$\widetilde{\varphi}_\ell=\{1-2(1-\eta^2_{a_1a_2,c})\mathrm{i}t_\ell\}^{-1}\,,\quad \ell=1,2\,,$$

by noting that $\widetilde{\varphi}_1^{\nu_1}\widetilde{\varphi}_2^{\nu_2}$ is the characteristic function of the bivariate distribution (Y_1,Y_2)', where Y_1 and Y_2 are independent two Gamma distributions with parameters ν_1 (or ν_2) and $2(1-\eta^2_{a_1a_2,c})$. Since

$$2\mathrm{i}t_\ell=\frac{1-\widetilde{\varphi}_\ell^{-1}}{1-\eta^2_{a_1a_2,c}}\quad\text{and}\quad 1-2\mathrm{i}t_\ell=\frac{\widetilde{\varphi}_\ell^{-1}-\eta^2_{a_1a_2,c}}{1-\eta^2_{a_1a_2,c}}\,,$$

we have

$$\phi\equiv(1-2\mathrm{i}t_1)(1-2\mathrm{i}t_2)-\eta^2_{a_1a_2,c}(2\mathrm{i}t_1)(2\mathrm{i}t_2)=\frac{1-\eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2}{(1-\eta^2_{a_1a_2,c})\widetilde{\varphi}_1\widetilde{\varphi}_2}$$

and

$$\begin{aligned}&\phi^{-\nu/2}(1-\eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^{-K}\\&\quad=\{(1-\eta^2_{a_1a_2,c})(\widetilde{\varphi}_1\widetilde{\varphi}_2)\}^{\nu/2}(1-\eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^{-(\nu/2+K)}\\&\quad=\{(1-\eta^2_{a_1a_2,c})(\widetilde{\varphi}_1\widetilde{\varphi}_2)\}^{\nu/2}\Big[1+\sum_{\ell=1}^{\infty}\frac{(\nu/2+K)_\ell}{\ell!}\,(\eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^\ell\Big]\end{aligned}$$

for any $\nu>0$ and $K\geq 0$, where $(x)_\ell=\Gamma(x+\ell)/\Gamma(x)$ for $x>0$ and $\ell\in\mathbf{N}_0$; a set of nonnegative integers. As a corollary of Lemma 6, we have only to derive an asymptotic expansion

$$\begin{aligned}(27)\quad &E[\exp(\mathrm{i}t_1\widetilde{T}^2_{a_1c}+\mathrm{i}t_2\widetilde{T}^2_{a_2c})]\\&\quad=\widetilde{\Xi}\exp\{\mathrm{i}t_1F(\boldsymbol{\gamma}^{(a_1c)},\Gamma)+\mathrm{i}t_2F(\boldsymbol{\gamma}^{(a_2c)},\Gamma)\}\Big|_{\boldsymbol{\gamma}^{(a_1c)}=\boldsymbol{\gamma}^{(a_2c)}=0,\Gamma=\Sigma}+\mathrm{o}(N^{-1})\\&\quad=\phi^{-p/2}\Big[1+\frac{1}{N}\sum_{K=1}^{3}(1-\eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^{-K}h_K(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\Big]+\mathrm{o}(N^{-1})\,,\end{aligned}$$

where h_1,h_2 and h_3 are some polynomials which are not determined uniquely. Especially, even if the population distributions $\boldsymbol{U}^{(a)}$'s satisfy $\boldsymbol{U}^{(a)}\overset{d}{=}-\boldsymbol{U}^{(a)}$ for all $a=1,\ldots,q$, that is, $\kappa^{(a)}_{j_1,j_2,j_3}=0$ for all $j_1,j_2,j_3=1,\ldots,p$; $a=1,\ldots,q$, polynomials h_1 and h_2 (in this case, $h_3\equiv 0$ seems to be the simplest choice) are not determined uniquely. Actually, letting $\lambda_\ell=1-2(1-\eta^2_{a_1a_2,c})\mathrm{i}t_\ell$, $\ell=1,2$, Seo (2002; page 65 under the equal sample sizes) or Okamoto (2005; page 214 under the unequal sample sizes) used Iwashita's (1997) results (the three-term Edgeworth expansion of $\{N_a^{1/2}\overline{\boldsymbol{U}}^{(a)};a=1,\ldots,q\}$ and the two-term

Edgeworth expansion of $\{(N_a^{1/2}\overline{\boldsymbol{U}}^{(a)\prime}, N_a^{1/2}\{\mathrm{vech}(S_U^{(a)} - \Sigma)\}')'; a = 1, \dots, q\}$ under elliptical populations) to get

$$E[\exp(\mathrm{i}t_1\widetilde{T}^2_{a_1c} + \mathrm{i}t_2\widetilde{T}^2_{a_2c})] = \phi^{-p/2}\Big[1 + \frac{1}{N}\sum_{K=1}^{2}\phi^{-K}\Lambda_K(\lambda_1, \lambda_2)\Big] + \mathrm{o}(N^{-1}),$$

where

$$\begin{aligned}\Lambda_1(\lambda_1, \lambda_2) &= c_{11} + c_{12}\lambda_1 + c_{13}\lambda_2 + c_{14}\lambda_1\lambda_2,\\ \Lambda_2(\lambda_1, \lambda_2) &= c_{21}\lambda_1^2 + c_{22}\lambda_2^2 + c_{23}\lambda_1^2\lambda_2 + c_{24}\lambda_1\lambda_2^2\\ &\quad + c_{25}\lambda_1\lambda_2 + c_{26}\lambda_1 + c_{27}\lambda_2 + c_{28}\lambda_1^2\lambda_2^2 + c_{29}\end{aligned}$$

(note that coefficients satisfy $\sum_{i=1}^4 c_{1i} = \sum_{i=1}^9 c_{2i} = 0$; especially, under the equal sample sizes as in Seo (2002), $c_{29} = 0$). It is easy to see that with $c^*_{1i} = c_{1i}(1 - \eta^2_{a_1a_2,c})$, $i = 1, \dots, 4$ and $c^*_{2i} = c_{2i}(1 - \eta^2_{a_1a_2,c})^2$, $i = 1, \dots, 9$,

$$\begin{aligned}&\sum_{K=1}^{2}(1 - \eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^{-K}(1 - \eta^2_{a_1a_2,c})^K(\widetilde{\varphi}_1\widetilde{\varphi}_2)^K\Lambda_K(\widetilde{\varphi}_1^{-1}, \widetilde{\varphi}_2^{-1})\\ &\quad = \sum_{K=1}^{2}(1 - \eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^{-K}h_K^{\mathrm{SO}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2),\end{aligned}$$

where

$$\begin{aligned}h_1^{\mathrm{SO}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2) &= c^*_{11}\widetilde{\varphi}_1\widetilde{\varphi}_2 + c^*_{12}\widetilde{\varphi}_2 + c^*_{13}\widetilde{\varphi}_1 + c^*_{14},\\ h_2^{\mathrm{SO}}(\widetilde{\varphi}_1, \widetilde{\varphi}_2) &= c^*_{21}\widetilde{\varphi}_2^2 + c^*_{22}\widetilde{\varphi}_1^2 + c^*_{23}\widetilde{\varphi}_2 + c^*_{24}\widetilde{\varphi}_1\\ &\quad + c^*_{25}\widetilde{\varphi}_1\widetilde{\varphi}_2 + c^*_{26}\widetilde{\varphi}_1\widetilde{\varphi}_2^2 + c^*_{27}\widetilde{\varphi}_1^2\widetilde{\varphi}_2 + c^*_{28} + c^*_{29}\widetilde{\varphi}_1^2\widetilde{\varphi}_2^2\end{aligned}$$

satisfy $h_1^{\mathrm{SO}}(1,1) = h_2^{\mathrm{SO}}(1,1) = 0$. We can use the identity

$$\begin{aligned}\widetilde{\varphi}_1^2\widetilde{\varphi}_2^2 &= 1 + (\widetilde{\varphi}_1\widetilde{\varphi}_2 + 1)(\widetilde{\varphi}_1\widetilde{\varphi}_2 - 1)\\ &= 1 + (-\eta^{-2}_{a_1a_2,c} + \widetilde{\varphi}_1\widetilde{\varphi}_2)(\widetilde{\varphi}_1\widetilde{\varphi}_2 - 1) + (\eta^{-2}_{a_1a_2,c} + 1)(\widetilde{\varphi}_1\widetilde{\varphi}_2 - 1)\end{aligned}$$

and

$$\begin{aligned}&c^*_{21}\widetilde{\varphi}_2^2 + c^*_{22}\widetilde{\varphi}_1^2 + c^*_{23}\widetilde{\varphi}_2 + c^*_{24}\widetilde{\varphi}_1 + c^*_{25}\widetilde{\varphi}_1\widetilde{\varphi}_2 + c^*_{26}\widetilde{\varphi}_1\widetilde{\varphi}_2^2 + c^*_{27}\widetilde{\varphi}_1^2\widetilde{\varphi}_2 + c^*_{28} + c^*_{29}\widetilde{\varphi}_1^2\widetilde{\varphi}_2^2\\ &\quad = c^*_{21}\widetilde{\varphi}_2^2 + c^*_{22}\widetilde{\varphi}_1^2 + c^*_{23}\widetilde{\varphi}_2 + c^*_{24}\widetilde{\varphi}_1 + (c^*_{25} + c^*_{26} + c^*_{27})\widetilde{\varphi}_1\widetilde{\varphi}_2\\ &\qquad + \widetilde{\varphi}_1\widetilde{\varphi}_2\{c^*_{26}\widetilde{\varphi}_2 + c^*_{27}\widetilde{\varphi}_1 - (c^*_{26} + c^*_{27})\} + (c^*_{28} + c^*_{29})\\ &\qquad + c^*_{29}(-\eta^{-2}_{a_1a_2,c} + \widetilde{\varphi}_1\widetilde{\varphi}_2)(\widetilde{\varphi}_1\widetilde{\varphi}_2 - 1) + c^*_{29}(\eta^{-2}_{a_1a_2,c} + 1)(\widetilde{\varphi}_1\widetilde{\varphi}_2 - 1)\\ &\quad = c^*_{21}\widetilde{\varphi}_2^2 + c^*_{22}\widetilde{\varphi}_1^2 + c^*_{23}\widetilde{\varphi}_2 + c^*_{24}\widetilde{\varphi}_1 + (c^*_{25} + c^*_{26} + c^*_{27})\widetilde{\varphi}_1\widetilde{\varphi}_2\\ &\qquad + (c^*_{28} - \eta^{-2}_{a_1a_2,c}c^*_{29}) + c^*_{29}(\eta^{-2}_{a_1a_2,c} + 1)\widetilde{\varphi}_1\widetilde{\varphi}_2\\ &\qquad + \eta^{-2}_{a_1a_2,c}\{c^*_{26}\widetilde{\varphi}_2 + c^*_{27}\widetilde{\varphi}_1 - (c^*_{26} + c^*_{27})\}\\ &\qquad + (-\eta^{-2}_{a_1a_2,c} + \widetilde{\varphi}_1\widetilde{\varphi}_2)\{c^*_{26}\widetilde{\varphi}_2 + c^*_{27}\widetilde{\varphi}_1 - (c^*_{26} + c^*_{27}) + c^*_{29}(\widetilde{\varphi}_1\widetilde{\varphi}_2 - 1)\}\end{aligned}$$

to obtain at least two different expansions

$$\begin{aligned}
&E[\exp(\mathrm{i}t_1\widetilde{T}^2_{a_1c} + \mathrm{i}t_2\widetilde{T}^2_{a_2c})] \\
&= \phi^{-p/2}\Big[1 + \frac{1}{N}\sum_{K=1}^{2}(1-\eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^{-K}h^{\mathrm{SO}}_K(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\Big] + \mathrm{o}(N^{-1}) \\
&= \phi^{-p/2}\Big[1 + \frac{1}{N}\sum_{K=1}^{2}(1-\eta^2_{a_1a_2,c}\widetilde{\varphi}_1\widetilde{\varphi}_2)^{-K}h_K(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\Big] + \mathrm{o}(N^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
h_1(\widetilde{\varphi}_1,\widetilde{\varphi}_2) &= c^*_{11}\widetilde{\varphi}_1\widetilde{\varphi}_2 + c^*_{12}\widetilde{\varphi}_2 + c^*_{13}\widetilde{\varphi}_1 + c^*_{14} \\
&\quad -\eta^{-2}_{a_1a_2,c}\{c^*_{26}\widetilde{\varphi}_2 + c^*_{27}\widetilde{\varphi}_1 - (c^*_{26}+c^*_{27}+c^*_{29}) + c^*_{29}\widetilde{\varphi}_1\widetilde{\varphi}_2\}, \\
h_2(\widetilde{\varphi}_1,\widetilde{\varphi}_2) &= c^*_{21}\widetilde{\varphi}^2_2 + c^*_{22}\widetilde{\varphi}^2_1 + c^*_{23}\widetilde{\varphi}_2 + c^*_{24}\widetilde{\varphi}_1 + (c^*_{25}+c^*_{26}+c^*_{27})\widetilde{\varphi}_1\widetilde{\varphi}_2 \\
&\quad +(c^*_{28} - \eta^{-2}_{a_1a_2,c}c^*_{29}) + c^*_{29}(\eta^{-2}_{a_1a_2,c}+1)\widetilde{\varphi}_1\widetilde{\varphi}_2 \\
&\quad +\eta^{-2}_{a_1a_2,c}\{c^*_{26}\widetilde{\varphi}_2 + c^*_{27}\widetilde{\varphi}_1 - (c^*_{26}+c^*_{27})\}
\end{aligned}$$

also satisfy $h_1(1,1) = h_2(1,1) = 0$. Unlike h_2, h^{SO}_2 includes higher order terms $\widetilde{\varphi}_1\widetilde{\varphi}^2_2$, $\widetilde{\varphi}^2_1\widetilde{\varphi}_2$ and $\widetilde{\varphi}^2_1\widetilde{\varphi}^2_2$, which may cause an appearance of some terms η in Seo (2002; d_2/η^2 of Theorem 4 under the equal sample sizes) or some terms η_2, η^2_2 in Okamoto (2005; d_2/η^2_2 of Theorem 2 under the unequal sample sizes). On the other hand, our formula $D^{(a_1c:a_2c)}_{2,\ell}$ of Proposition 10 is a constant. This point seems to be important, since Seo's (2002) formula d_2/η^2 or Okamoto's (2005) formula d_2/η^2_2 reduces to a constant under normality, which is a member of distributions with $\kappa^{(a)}_{j_1,j_2,j_3} = 0$ for all $j_1, j_2, j_3 = 1, \ldots, p$; $a = 1, \ldots, q$.

As a concluding remark, we assumed $\kappa^{(a)}_{j_1,j_2,j_3} = 0$ for all $j_1, j_2, j_3 = 1, \ldots, p$; $a = 1, \ldots, q$, only in Proposition 10. Otherwise, in view of the differential operator $\widetilde{\widetilde{\Xi}}$ in Lemma 6, the rearrangement of several terms involving $K^{(ab)}_{33,1}$ and $K^{(ab)}_{33,2}$, $a, b = 1, \ldots, q$; $a \le b$ will be very messy in order to get polynomials h_1, h_2, h_3 in (27). Unlike Corollary 3 and Proposition 9, asymptotic expansion for $P(T^2_{a_1c} > x, T^2_{a_2c} > x)$ turns out to depend on $K^{(cc)}_{33,1}$ and $K^{(cc)}_{33,2}$ even in the case of $N_1 = \cdots = N_q$ and $\kappa^{(1)}_{j_1,j_2,j_3} = \cdots = \kappa^{(q)}_{j_1,j_2,j_3}$ for all $j_1, j_2, j_3 = 1, \ldots, p$.

References

[1] Anderson, T. W. (2003). *An Introduction to Multivariate Statistical Analysis*, 3rd ed. New York: Wiley.

[2] Bhattacharya, R. and Denker, M. (1990). *Asymptotic Statistics*. Basel: Birkhäuser-Verlag.

[3] Bhattacharya, R. N. and Ghosh, J. K. (1978). On the validity of the formal Edgeworth expansion. *Ann. Statist.* **6** 434–451. Correction: (1980). **8** 1399.

[4] Bhattacharya, R. N. and Rao, R. R. (1976). *Normal Approximation and Asymptotic Expansions*. New York: Wiley.

[5] Bilodeau, M. and Brenner, D. (1999). *Theory of Multivariate Statistics*. New York: Springer.

[6] Chandra, T. K. and Ghosh, J. K. (1980). Valid asymptotic expansions for the likelihood ratio and other statistics under contiguous alternatives. *Sankhyā* **42** 170–184.

[7] Chibisov, D. M. (1972). An asymptotic expansion for the distribution of a statistic admitting an asymptotic expansion. *Theory Prob. Appl.* **17** 620–630.

[8] Cordeiro, G. M. and Ferrari, S. L. P. (1991). A modified score test statistic having chi-squared distribution to order n^{-1}. *Biometrika* **78** 573–582.

[9] Cribari-Neto, F. and Ferrari, S. L. P. (2001). Monotonic improved critical values for two χ^2 asymptotic criteria. *Economics Letters* **71** 307–316.

[10] Fujikoshi, Y. (1997). An asymptotic expansion for the distribution of Hotelling's T^2-statistic under nonnormality. *J. Mult. Anal.* **61** 187–193.

[11] Fujikoshi, Y. (2002a). Asymptotic expansions for the distributions of multivariate basic statistics and one-way MANOVA tests under nonnormality. *J. Statist. Plan. Inf.* **108** 263–282.

[12] Fujikoshi, Y. (2002b). Some recent results on asymptotic expansions of multivariate test statistics for mean vectors under nonnormality. *Calcutta Statist. Assoc. Bulletin.* **52** 1–46.

[13] Fujikoshi, Y. and Seo, T. (1999). Asymptotic expansions for the joint distribution of correlated Hotelling's T^2 statistics under normality. *Commun. Statist. Theory Meth.* **28** 773–788.

[14] Gupta, A. K., Xu, J. and Fujikoshi, Y. (2006). An asymptotic expansion of the distribution of Rao's U-statistic under a general condition. *J. Mult. Anal.* **97** 492–513.

[15] Hall, P. (1987). Edgeworth expansion for Student's t statistic under minimal moment conditions. *Ann. Prob.* **15** 920–931.

[16] Hall, P. (1992). *The Bootstrap and Edgeworth Expansion*. New York: Springer.

[17] Ito, K. (1956). Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistic. *Ann. Math. Statist.* **27** 1091–1105.

[18] Iwashita, T. (1997). Asymptotic null and nonnull distribution of Hotelling's T^2-statistic under the elliptical distribution. *J. Statist. Plan. Inf.* **61** 85–104.

[19] Iwashita, T. and Seo, T. (2002). An application of Lévy's inversion formula for higher order asymptotic expansion. *J. Statist. Plan. Inf.* **104** 403–416.

[20] Iwashita, T. and Siotani, M. (1994). Asymptotic distributions of functions of a sample covariance matrix under the elliptical distribution. *Canad. J. Statist.* **22** 273–283.

[21] James, G. S. (1954). Tests of linear hypotheses in univariate and multivariate analysis when the ratios of the population variances are unknown. *Biometrika* **41** 19–43.

[22] Kakizawa, Y. (1996). Higher order monotone Bartlett-type adjustment for some multivariate test statistics. *Biometrika* **83** 923–927.

[23] Kakizawa, Y. (2005a). A comparison of local powers of a class of tests for multivariate linear hypothesis under general distributions. Discussion Paper Series A: No.2005–142 and 2006–162, Hokkaido University.

[24] Kakizawa, Y. (2005b). A test of equality of mean vectors of several heteroscedastic multivariate populations. Discussion Paper Series A: No.2005–144, Hokkaido University.

[25] Kakizawa, Y. (2005c). Multiple comparisons of several heteroscedastic multivariate populations. Discussion Paper Series A: No.2005–146, Hokkaido University.

[26] Kakizawa, Y. (2005d). Asymptotic expansions for the distributions of maximum and sum of quasi-independent Hotelling's T^2 statistics under nonnormality. Discussion Paper Series A: No.2005–148. Hokkaido University.

[27] Kakizawa, Y. and Iwashita, T. (2005a). Hotelling's one-sample and two-sample T^2 tests and the multivariate Behrens-Fisher problem under nonnormality (this paper was accepted for publication in April 30, 2006). To appear in *J. Statist. Plan. Inf.*

[28] Kakizawa, Y. and Iwashita, T. (2005b). A comparison of higher-order local powers of a class of one-way MANOVA tests under general distributions. Discussion Paper Series A: No.2005–140 and 151, Hokkaido University.

[29] Kano, Y. (1995). An asymptotic expansion of the distribution of Hotelling's T^2-statistic under general distributions. *Amer. J. Math. Management Sciences* **15** 317–341.

[30] Magdalinos, M. A. (1992). Stochastic expansions and asymptotic approximations. *Econometric Theory* **8** 343–367.

[31] Mardia, K. V. (1970). Measures of multivariate skewness and kurtosis with applications. *Biometrika* **57** 519–530.

[32] McCullagh, P. (1987). *Tensor Methods in Statistics*. London: Chapman & Hall.

[33] McLachlan, G. J. (1973). Asymptotic expansion of the expectation of the estimated error rate in discriminant analysis. *Aust. J. Statist.* **15** 210–214.

[34] Memon, A. Z. and Okamoto, M. (1971). Asymptotic expansion of the distribution of the Z statistic in discriminant analysis. *J. Mult. Anal.* **1** 294–307.

[35] Okamoto, M. (1963). An asymptotic expansion for the distribution of the linear discriminant function. *Ann. Math. Statist.* **34** 1286–1301.

[36] Okamoto, N. (2005). A modified second order Bonferroni approximation in elliptical populations with unequal sample sizes. *SUT J. Math.* **41** 205–225.

[37] Okamoto, N. and Seo, T. (2004). Pairwise multiple comparisons of mean vectors under elliptical populations with unequal sample sizes. *J. Jpn. Soc. Comp. Statist.* **17** 49–66.

[38] Roy, S. N. and Bose, R. C. (1953). Simultaneous confidence interval estimation. *Ann. Math. Statist.* **24** 513–536.

[39] Seo, T. (1995). Simultaneous confidence procedures for multiple comparisons of mean vectors in multivariate normal populations. *Hiroshima Math. J.* **25** 387-422.

[40] Seo, T. (2002). The effect of nonnormality on the upper percentiles on $T^2_{\max}$ statistic in elliptical distributions. *J. Japan. Statist. Soc.* **32** 57–76.

[41] Seo, T. and Siotani, M. (1992). The multivariate Studentized range and its upper percentiles. *J. Japan. Statist. Soc.* **22** 123–137.

[42] Seo, T. and Siotani, M. (1993). Approximation to the upper percentiles of $T^2_{\max}$-type statistics. In *Statistical Sciences and Data Analysis*. Matusita, K. et al. (Eds), Utrecht: VSP pp265–276.

[43] Siotani, M. (1956). On the distributions of the Hotelling's T^2-statistics. *Ann. Inst. Statist. Math.* **8** 1–14.

[44] Siotani, M. (1957). Note on the utilization of the generalized Student ratio in the analysis of variance or dispersion. *Ann. Inst. Statist. Math.* **9** 157–171.

[45] Siotani, M. (1959). The extreme value of the generalized distances of the individual points in the multivariate normal sample. *Ann. Inst. Statist. Math.* **10** 183–208.

[46] Siotani, M. (1960). Notes on multivariate confidence bounds. *Ann. Inst. Statist. Math.* **11** 167–182.

[47] Siotani, M. (1964). Interval estimation for linear combinations of means. *J. Amer. Statist. Assoc.* **59** 1141–1164.

[48] Siotani, M. (1971). An asymptotic expansion of the non-null distribution of Hotelling's generalized T_0^2-statistic. *Ann. Math. Statist.* **42** 560–571.

[49] Siotani, M. (1989). Distributions of Lawley-Hotelling's T_0^2 and related statistics: A review. *Amer. J. Math. Management Sciences* **9** 5–30.

[50] Siotani, M., Hayakawa, T. and Fujikoshi, Y. (1985). *Modern Multivariate Analysis: A Graduate Course and Handbook*. Ohio: American Sciences Press.

[51] Welch, B. L. (1947). The generalization of 'Student's' problem when several different population variances are involved. *Biometrika* **34** 28–35.

[52] Wakaki, H. (1997). Asymptotic expansion of the joint distribution of sample mean vector and sample covariance matrix from an elliptical population. *Hiroshima Math. J.* **27** 295–305.

[53] Wakaki, H., Yanagihara, H. and Fujikoshi, Y. (2002). Asymptotic expansions of the null distributions of test statistics for multivariate linear hypothesis under nonnormality. *Hiroshima Math. J.* **32** 17–50.

[54] Yanagihara, H. (2001). Asymptotic expansion of the null distribution of three test statistics in a nonnormal GMANOVA model. *Hiroshima Math. J.* **31** 213–262.

Yoshihide Kakizawa
Faculty of Economics, Hokkaido University
Nishi 7, Kita 9, Kita-ku, Sapporo 060–0809, Japan
E-mail: kakizawa@econ.hokudai.ac.jp