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# Vertex disjoint cycles containing specified paths of order 3 in a bipartite graph

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**Abstract.** Let $k$, $n$ be integers with $k \geq 3$ and $n \geq 3k$, and let $G$ be a bipartite graph having partite sets $V_1$, $V_2$ with $|V_1| = |V_2| = n$. We show that if $d_G(u) + d_G(v) \geq n + 2k - 1$ for any $u \in V_1$ and $v \in V_2$ with $uv \notin E(G)$, then for any vertex disjoint paths $P_1, P_2, \ldots, P_k$ of order 3, $G$ contains vertex disjoint cycles $H_1, H_2, \ldots, H_k$ such that $\bigcup_{1 \leq i \leq k} V(H_i) = V(G)$ and $H_i$ passes through $P_i$ for each $i$ with $1 \leq i \leq k$.

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## §1. Introduction

In this paper, all graphs considered are finite, undirected and simple graphs with no loops and no multiple edges. Let $G = (V(G), E(G))$ be a graph. The order $|V(G)|$ of $G$ is often denoted by $|G|$ for short. For $v \in V(G)$, we let $d_G(v)$ denote the degree of $v$ in $G$, and we define $\delta(G)$ by $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$. When $G$ is a bipartite graph with partite sets $V_1$ and $V_2$, we further define

$$\sigma_{1,1}(G) = \min\{d_G(u) + d_G(v) \mid u \in V_1, v \in V_2, uv \notin E(G)\}$$

(if $G$ is a complete bipartite graph, we define $\sigma_{1,1}(G) = \infty$).

When $G$ contains vertex disjoint subgraphs $H_1, H_2, \ldots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$, we say that $G$ is partitioned into $H_1, H_2, \ldots, H_k$. If a path $P$ is contained in a cycle $C$ (resp. a path $Q$) as a subgraph, then we write $P \subset C$ (resp. $P \subset Q$).

In this paper, we are concerned with the existence of a partition of a bipartite graph into cycles. A sufficient condition for the existence of a partition into a specified number of cycles in a bipartite graph was given by Wang.

**Theorem 1.1** (Wang [3])**.** *Let $k$, $n$ be integers with $k \geq 1$ and $n \geq 2k+1$. Let $G$ be a bipartite graph having partite sets with equal cardinality $n$, and suppose that $\delta(G) \geq \frac{n}{2} + 1$. Then $G$ can be partitioned into $k$ cycles.*

Wang and Chen et al. independently considered a partition into cycles each of which contains a specified edge, and proved the following theorem.

**Theorem 1.2** (Chen et al. [1]; Wang [2, 4])**.** *Let $k$, $n$ be integers with $k \geq 2$ and $n \geq 3k$. Let $G$ be a bipartite graph having partite sets with equal cardinality $n$, and suppose that $\sigma_{1,1}(G) \geq n + k$. Then for any independent edges $e_1, e_2, \ldots, e_k$, $G$ can be partitioned into $k$ cycles $H_1, H_2, \ldots, H_k$ such that $e_i \in E(H_i)$ for each $i$ with $1 \leq i \leq k$.*

In this paper, we consider a situation in which vertex disjoint paths of order 3 are specified instead of independent edges. The main result of this paper is the following.

**Theorem 1.3.** *Let $k$, $n$ be integers with $k \geq 3$ and $n \geq 3k$. Let $G$ be a bipartite graph having partite sets with equal cardinality $n$, and suppose that $\sigma_{1,1}(G) \geq n + 2k - 1$. Then for any vertex disjoint paths $P_1, P_2, \ldots, P_k$ of order 3, $G$ can be partitioned into $k$ cycles $H_1, H_2, \ldots, H_k$ such that $P_i \subset H_i$ for each $i$ with $1 \leq i \leq k$.*

Theorem 1.3 does not hold for $k = 2$. Let $n \geq 5$, and let $H$ be a complete bipartite graph of order $2n - 2$ with partite sets $W_1 = \{a_i \,|\, 1 \leq i \leq n-1\}$ and $W_2 = \{b_j \,|\, 1 \leq j \leq n-1\}$. Let $L$ be a complete graph of order 2 with $V(L) \cap V(H) = \emptyset$, and write $V(L) = \{c_1, c_2\}$. Define $G$ by $V(G) = V(H) \cup V(L)$ and $E(G) = E(H) \cup E(L) \cup \{c_1 a_i \,|\, 1 \leq i \leq 3\} \cup \{c_2 b_i \,|\, 1 \leq i \leq 3\}$. Let $P_1 = a_1 b_1 a_2$ and $P_2 = b_2 a_3 b_3$ (see Figure 1). Then $c_1 a_1 b_1 a_2 c_1$ is the only cycle which contains $c_1$, passes through one of $P_1$ and $P_2$ and is disjoint from the other, and similarly for $c_2$. Hence $G$ can not be partitioned into two cycles $H_1$, $H_2$ such that $P_i \subset H_i$ for each $i$ with $1 \leq i \leq 2$, while $\sigma_{1,1}(G) = n + 2k - 1 = n + 3$.

Also the degree sum condition in Theorem 1.3 is best possible in the following sense. Define a bipartite graph $G$ of order $2n$ by letting $V(G) = A_1 \cup A_2 \cup A_3 \cup A_4$ with $|A_1| = 1$, $|A_2| = 2k-1$, $|A_3| = n-1$, $|A_4| = n - 2k + 1$, and $E(G) = \bigcup_{i=1}^{3} \{xy \,|\, x \in A_i, y \in A_{i+1}\}$. Write $V(A_1) = \{a\}$, $V(A_2) = \{b_1, b_2, \ldots, b_{2k-1}\}$, and $V(A_3) = \{c_1, c_2, \ldots, c_{n-1}\}$. Let $P_1 = a b_1 c_1$ and $P_i = b_i c_i b_{i+k-1}$ for each $i$ with $2 \leq i \leq k$ (see Figure 2). Then we can not take a cycle passing through $P_1$ without using vertices of other specified paths. Consequently, $G$ can not be partitioned into $k$ cycles $H_1, H_2, \ldots, H_k$ such that $P_i \subset H_i$ for each $i$ with $1 \leq i \leq k$, while $\sigma_{1,1}(G) = n + 2k - 2$.

The first step in the proof of Theorem 1.3 is to show the existence of vertex disjoint cycles that contain the specified paths of order 3.

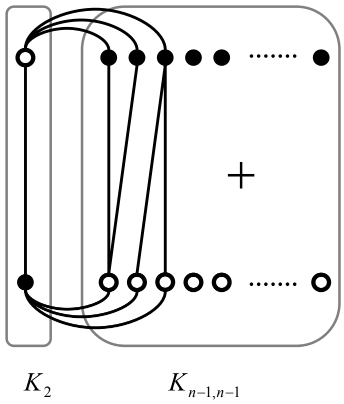


Figure 1.

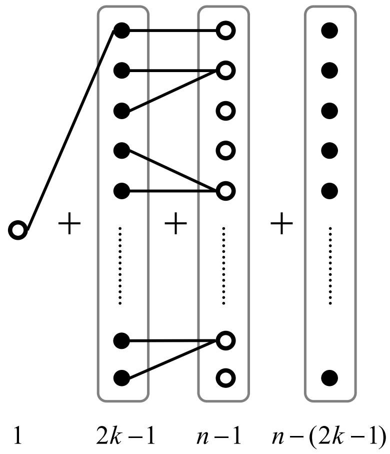


Figure 2.

**Theorem 1.4.** *Let $k$, $n$ be integers with $k \geq 2$ and $n \geq 3k$. Let $G$ be a bipartite graph having partite sets with equal cardinality $n$, and suppose that $\sigma_{1,1}(G) \geq n + 2k - 1$. Then for any vertex disjoint paths $P_1, P_2, \ldots, P_k$ of order 3, $G$ contains $k$ vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that $P_i \subset C_i$ and $|C_i| \leq 6$ for each $i$ with $1 \leq i \leq k$.*

The next step is to show that this collection of cycles can be transformed into a collection of cycles that form a partition of $G$.

**Theorem 1.5.** *Let $k$, $n$ be integers with $k \geq 3$ and $n \geq 2k$. Let $G$ be a bipartite graph having partite sets with equal cardinality $n$, and suppose that $\sigma_{1,1}(G) \geq n + 2k - 1$. Let $P_1, P_2, \ldots, P_k$ be vertex disjoint paths of order 3, and suppose that there exist vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that $P_i \subset C_i$ for each $i$ with $1 \leq i \leq k$. Then there exist vertex disjoint cycles $H_1, H_2, \ldots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$ and $P_i \subset H_i$ for each $i$ with $1 \leq i \leq k$.*

Our notation is standard except possibly for the following. For a vertex $v$ of a graph $G$, the neighborhood of $v$ in $G$ is denoted by $N_G(v)$; thus $d_G(v) = |N_G(v)|$. For a subset $S$ of $V(G)$, we let $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $d_G(S) = \sum_{v \in S} d_G(v)$. For a subgraph $H$ of $G$ and a vertex $v$ of $G$ with $v \in V(G) - V(H)$, let $N_G(v) \cap V(H)$ be denoted by $N_H(v)$, and let $d_H(v) = |N_H(v)|$. For a subgraph $H$ of $G$ and a subset $S$ of $V(G)-V(H)$, we let $N_H(S) = \bigcup_{x\in S} N_H(x)$ and $d_H(S) = \sum_{x\in S} d_H(x)$. For a subset $S$ of $V(G)$, we let $\langle S\rangle_G$ denote the subgraph induced by $S$ in $G$, and let $G - S = \langle V(G) - S\rangle_G$. For a subgraph $H$ of $G$, we write $G - H$ for $G - V(H)$.

A cycle is considered to have a fixed orientation. For a cycle $C = x_1x_2\cdots x_nx_1$ and for two vertices $x_i, x_j \in V(C)$ with $i < j < i + n$, we define segments $C[x_i, x_j]$, $C^-[x_i, x_j]$ and $C(x_i, x_j)$ of $C$ by $C[x_i, x_j] = x_ix_{i+1}\cdots x_{j-1}x_j$, $C^-[x_i, x_j] = x_ix_{i-1}\cdots x_{j+1}x_j$ and $C(x_i, x_j) = C[x_i, x_j]-\{x_i, x_j\}$, respectively (here indices are to be read modulo $n$). We let $v^+$ (resp. $v^-$) denote the successor (resp. the predecessor) of $v$ along $C$, and define $v^{++} = (v^+)^+$ (resp. $v^{--} = (v^-)^-$); thus if $v = x_i$, then $v^+ = x_{i+1}$, $v^- = x_{i-1}$, $v^{++} = x_{i+2}$ and $v^{--} = x_{i-2}$. For a path $P = y_1y_2\cdots y_m$ and for two vertices $y_i, y_j \in V(P)$ with $1 \le i < j \le m$, we define segment $P[y_i, y_j]$ of $P$ by $P[y_i, y_j] = y_iy_{i+1}\cdots y_{j-1}y_j$. When $P_1, P_2, \ldots, P_k$ are vertex disjoint paths of order 3, a cycle $C$ is said to be *admissible* with respect to $\{P_1, P_2, \ldots, P_k\}$ if there exists $i$ with $1 \le i \le k$ such that $P_i \subset C$ and $V(C)\cap V(P_j) = \emptyset$ for every $j$ with $1 \le j \le k$ and $j \ne i$.

## §2. Proof of Theorem 1.4

Throughout the rest of this paper, we let $G$ denote a bipartite graph having partite sets $V_1$, $V_2$ with $|V_1| = |V_2|$. Our proof of Theorem 1.4 requires the following lemmas.

**Lemma 2.1.** *Let $P = xyz$ be a path in $G$ with $x \in V_1$, and let $C$ be a cycle in $G$ such that $P \subset C$. Further let $u \in V(G-C)\cap V_1$ and $v \in V(G-C)\cap V_2$.*

(i) *If $d_C(u) \ge 4$, then $\langle V(C) \cup \{u\}\rangle_G$ contains a cycle which is shorter than $C$ and passes through $P$.*

(ii) *If $d_C(v) \ge 3$, then $\langle V(C)\cup\{v\}\rangle_G$ contains a cycle which is shorter than $C$ and passes through $P$.*

*Proof.* (i) Since $x \in V_1$ and $d_C(u) \ge 4$, there exist $x_1, x_2 \in N_C(u)$ such that $|C[x_1, x_2]| \ge 5$ and $P \subset C(x_2, x_1)$. Then $uC[x_2, x_1]u$ is a cycle shorter than $C$ and $P \subset uC[x_2, x_1]u$.
(ii) Since $x \in V_1$ and $d_C(v) \ge 3$, there exist $x_1, x_2 \in N_C(v)$ such that $|C[x_1, x_2]| \ge 5$ and $P \subset C[x_2, x_1]$. Then $vC[x_2, x_1]v$ is a cycle shorter than $C$ and $P \subset vC[x_2, x_1]v$. □

**Lemma 2.2.** *Let $P$ be a path of order 3, and let $C$ be a cycle in $G$ with $P \subset C$. Let $u \in V(G - C) \cap V_1$, $v \in V(G - C) \cap V_2$, and suppose that $d_C(u) + d_C(v) \ge \frac{|C|}{2} + 3$. Then either $\langle V(C) \cup \{v\}\rangle_G$ contains a cycle which* *is shorter than* $C$ *and passes through* $P$*, or there exists* $w \in N_C(u)$ *such that* $\langle V(C) \cup \{v\} - \{w\}\rangle_G$ *contains a cycle passing through* $P$.

*Proof.* If $d_C(v) \geq 4$, then by Lemma 2.1, $\langle V(C) \cup \{v\}\rangle_G$ contains a cycle which is shorter than $C$ and passes through $P$. Thus we may assume that $d_C(v) \leq 3$. Then $d_C(v) = 3$ and $d_C(u) = \frac{|C|}{2}$, that is, $N_C(u) = V(C) \cap V_2$. Since $d_C(v) = 3$, there exist $a, b \in N_C(v)$ with $P \subset C[b, a]$. Take $w \in V(C(a, b)) \cap V_2$. Then $w \in N_C(u)$, and $vC[b, a]v$ is a cycle in $\langle V(C) \cup \{v\} - \{w\}\rangle_G$ passing through $P$. □

In the rest of this section, we let $G$ be an edge-maximal counterexample to Theorem 1.4, and write $P_i = x_i y_i z_i$ for each $i$ with $1 \leq i \leq k$. The term "*admissible*" means "admissible with respect to $\{P_1, P_2, \ldots, P_k\}$," and a cycle is called *short* if its length is at most 6. Note that $G$ is not a complete bipartite graph, because otherwise there exist $k$ vertex disjoint admissible cycles of length 4. Let $x \in V_1$ and $y \in V_2$ be nonadjacent vertices of $G$. Then the graph obtained from $G$ by adding the edge $xy$ is not a counterexample by the maximality of $G$, which implies that $G$ contains $k - 1$ vertex disjoint admissible short cycles $C_1, C_2, \ldots, C_{k-1}$. We choose admissible short cycles $C_1, C_2, \ldots, C_{k-1}$ so that $|\bigcup_{i=1}^{k-1} V(C_i)|$ is as small as possible. Without loss of generality, we may assume that $P_i \subset C_i$ for each $i$ with $1 \leq i \leq k - 1$, and we may also assume that $x_k \in V_1$. Let $L = \langle \bigcup_{i=1}^{k-1} V(C_i)\rangle_G$, $M = G - L$ and $D_1 = M - V(P_k)$, and write $|M| = 2m$. Since $n \geq 3k$, we have $m \geq 3$. If possible, we choose $C_1, C_2, \ldots, C_{k-1}$ so that $d_{D_1}(z_k) > 0$ and $d_{D_1}(x_k) > 0$.

**Claim 2.1.** *We have* $d_{D_1}(z_k) > 0$ *and* $d_{D_1}(x_k) > 0$.

*Proof.* We first remark that we can choose $C_1, C_2, \ldots, C_{k-1}$ so that $d_{D_1}(z_k) > 0$. To see this, suppose that $d_{D_1}(z_k) = 0$ and take any $x \in V(D_1) \cap V_2$. Then

$$d_M(z_k) + d_M(x) \leq 1 + (m - 1) = m.$$

This implies that

$$\begin{aligned} d_L(z_k) + d_L(x) &\geq n + 2k - 1 - m \\ &= \frac{|L|}{2} + 2k - 1 \\ &> \sum_{i=1}^{k-1} \Big( \frac{|C_i|}{2} + 2 \Big). \end{aligned}$$

Therefore there exists $i$ with $1 \leq i \leq k-1$ such that $d_{C_i}(z_k) + d_{C_i}(x) \geq \frac{|C_i|}{2} + 3$. Hence it follows Lemma 2.2 and the minimality of $|L|$ that there exists $u \in N_{C_i}(z_k)$ such that $\langle V(C_i) \cup \{x\} - \{u\}\rangle_G$ contains a cycle $C_i'$ passing through $P_i$. Consequently, replacing $C_i$ by $C_i'$, we may assume $d_{D_1}(z_k) > 0$.

Now suppose that the claim is false. In view of the remark made at the beginning of the proof, we may assume $d_{D_1}(z_k) > 0$ and $d_{D_1}(x_k) = 0$. Take $y \in N_{D_1}(z_k)$ and $v \in (V(D_1) \cap V_2) - \{y\}$. Arguing as above, we see that there exists $j$ such that $d_{C_j}(x_k) + d_{C_j}(v) \geq \frac{|C_j|}{2} + 3$, and there exists $w \in N_{C_j}(x_k)$ such that $\langle V(C_j) \cup \{v\} - \{w\} \rangle_G$ contains a cycle $C'_j$ passing through $P_j$. Now replacing $C_j$ by $C'_j$, we obtain a contradiction to the choice of $C_1, C_2, \ldots, C_{k-1}$ mentioned immediately before the statement of Claim 2.1. This completes the proof of the claim. □

Take $y \in N_{D_1}(z_k)$ and $z \in N_{D_1}(x_k)$, and let $D_2 = D_1 - \{y, z\}$. Since $G$ is a counterexample, $x_k y, z_k z \notin E(G)$, and hence $y \neq z$.

**Claim 2.2.** $d_{D_2}(y) > 0$.

*Proof.* Suppose that $d_{D_2}(y) = 0$ and take any $u \in V(D_2) \cap V_1$. Then since $yx_k \notin E(G)$,

$$d_M(u) + d_M(y) \leq (m-1) + 1 = m.$$

This implies that

$$\begin{aligned} d_L(u) + d_L(y) &\geq n + 2k - 1 - m. \\ &> \sum_{i=1}^{k-1}\Big(\frac{|C_i|}{2} + 2\Big). \end{aligned}$$

Therefore there exists $i$ with $1 \leq i \leq k-1$ such that $d_{C_i}(u) + d_{C_i}(y) \geq \frac{|C_i|}{2} + 3$. Since $C_i$ is short, this forces $|C_i| = 6$ and $d_{C_i}(u) = d_{C_i}(y) = 3$. Hence by Lemma 2.1(ii), $\langle V(C_i) \cup \{y\} \rangle_G$ or $\langle V(C_i) \cup \{u\} \rangle_G$ contains a cycle of length 4 passing through $P_i$. This contradicts the minimality of $|L|$. □

Now take $z' \in N_{D_2}(y)$. Since $G$ is a counterexample, $zz' \notin E(G)$. Let $D_3 = D_2 - \{z'\}$ and $S = \{z, x_k, y, z'\}$. Then again since $G$ is a counterexample, we have $N_{D_3}(z) \cap N_{D_3}(y) = \emptyset$ and $N_{D_3}(x_k) \cap N_{D_3}(z') = \emptyset$. Hence

$$\begin{aligned} d_M(S) &= d_{\langle S \cup \{y_k, z_k\} \rangle_G}(S) + d_{D_3}(S) \\ &\leq 7 + 2m - 6 \\ &= 2m + 1. \end{aligned}$$

This implies that

$$\begin{aligned} d_L(S) &\geq 2(n + 2k - 1) - (2m + 1) \\ &= 2(n - m) + 4(k - 1) + 1 \\ &> \sum_{i=1}^{k-1}(|C_i| + 4). \end{aligned}$$

Therefore there exists $i$ with $1 \le i \le k-1$ such that $d_{C_i}(S) \ge |C_i| + 5$. This implies that $|C_i| = 6$, and we have

$$11 \le d_{C_i}(S) \le 12. \tag{2.1}$$

Write $C_i = x_i y_i z_i abcx_i$.

**CASE 1.** $x_i \in V_1$.

By (2.1), $5 \le d_{C_i}(\{z, y\}) \le 6$, and hence there exists $u \in \{z, y\}$ such that $d_{C_i}(u) = 3$. Then $C'_i = ux_i y_i z_i u$ is an admissible cycle shorter than $C_i$, which contradicts the minimality of $|L|$.

**CASE 2.** $x_i \in V_2$.

If $\{x_i, z_i\} \subseteq N_{C_i}(z')$, then $z' x_i y_i z_i z'$ is an admissible cycle shorter than $C_i$, a contradiction. Thus $\{x_i, z_i\} \not\subseteq N_{C_i}(z')$, and hence

$$d_{C_i}(z') \le 2. \tag{2.2}$$

By (2.1) and (2.2), we obtain $d_{C_i}(z) = 3$, $d_{C_i}(x_k) = 3$ and $d_{C_i}(z') = 2$, and hence $b \in N_{C_i}(z')$. Now if we let $C'_i = zcx_i y_i z_i az$ and $C'_k = bx_k y_k z_k yz'b$, then $C'_i$ and $C'_k$ together with $\{C_1, C_2, \ldots, C_{k-1}\} - \{C_i\}$ form $k$ vertex disjoint short admissible cycles. But this contradicts the assumption that $G$ is a counterexample.

This completes the proof of Theorem 1.4.

## §3. Proof of Theorem 1.5

Recall that $G$ denotes a bipartite graph having partite sets $V_1$, $V_2$ with $|V_1| = |V_2|$. We prepare the following lemmas before proving Theorem 1.5.

**Lemma 3.1.** *Let $P$ be a path in $G$ having order 3, and let $C$ be a cycle in $G$ such that $P \subset C$. Let $R$ be a path in $G$ with $V(C) \cap V(R) = \emptyset$ such that the endvertices of $R$, $u$ and $v$, belong to different partite sets. Suppose further that $d_C(u) + d_C(v) \ge \frac{|C|}{2} + 2$. Then there exists a cycle $C'$ such that $V(C') = V(C) \cup V(R)$ and $P \subset C'$.*

*Proof.* Write $C = w_1 w_2 \cdots w_r w_1$ with $P = w_1 w_2 w_3$. By the symmetry of the roles of $u$ and $v$, we may assume that $w_1$ and $u$ belong to the same partite set. Then there exists $i$ with $3 \le i \le r-1$ such that $uw_{i+1}, vw_i \in E(G)$. Now the cycle $C'$ obtained by joining $C[w_{i+1}, w_i]$ and $R$ satisfies $V(C') = V(C) \cup V(R)$ and $P \subset C'$. □

**Lemma 3.2.** *Let $P$ be a path in $G$ having order 3, and let $Q$ be a path in $G$ such that $P \subset Q$. Let $R$ be a path with $V(Q) \cap V(R) = \emptyset$ such that the endvertices of $R$, $u$ and $v$, belong to different partite sets. Suppose further that $d_Q(u)+d_Q(v) \geq \frac{|Q|+4}{2}$, Then there exists a path $Q'$ having the same endvertices as $Q$ such that $V(Q') = V(Q) \cup V(R)$ and $P \subset Q'$.*

*Proof.* Write $Q = w_1w_2 \cdots w_r$ with $P = w_jw_{j+1}w_{j+2}$. We may assume that $w_1$ and $u$ belong to the same partite set. Then, regardless of the partite sets which $w_r$ and $w_j$ belong to, there exists $i$ with $1 \leq i \leq j-1$ or $j+2 \leq i \leq r-1$ such that $uw_{i+1}, vw_i \in E(G)$. Now the path $Q'$ obtained by joining $Q[w_1, w_i]$, $R$ and $Q[w_{i+1}, w_r]$ satisfies $V(Q') = V(Q) \cup V(R)$ and $P \subset Q'$. □

**Lemma 3.3.** *Let $P$ be a path of order 3 in $G$. Let $C$ be a cycle in $G$ with $P \subset C$, and suppose that $G$ contains no cycle $D$ satisfying $P \subset D$ and $V(C) \subsetneq V(D)$. Further let $u \in V(G-C) \cap V_1$ and $v \in V(G-C) \cap V_2$. Then $d_C(u) + d_C(v) \leq \frac{|C|}{2} + 2$.*

*Proof.* Write $C = w_1w_2 \cdots w_rw_1$ with $P = w_1w_2w_3$. We may assume that $w_1 \in V_1$. Suppose that $d_C(u) + d_C(v) \geq \frac{|C|}{2} + 3$. Then there exist $i$ and $j$ $(3 \leq i < j \leq r-1)$ with $uw_{i+1}, vw_i, uw_{j+1}, vw_j \in E(G)$. Now if we let $D = uC[w_{i+1}, w_j]vC^-[w_i, w_{j+1}]u$, then we have $P \subset D$ and $V(C) \subsetneq V(D)$. But this contradicts the assumption that there is no such cycle. □

Throughout the rest of this paper, we let $k$, $G$, $P_1, P_2, \ldots, P_k$ be as in Theorem 1.5, and write $P_i = x_iy_iz_i$ for each $i$ with $1 \leq i \leq k$. The term "*admissible*" means "admissible with respect to $\{P_1, P_2, \ldots, P_k\}$". Choose $k$ vertex disjoint cycles $C_1, C_2, \ldots, C_k$ with $P_i \subset C_i$ for each $i$ with $1 \leq i \leq k$ so that $\sum_{i=1}^{k} |C_i|$ is as large as possible, and set $L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle_G$. By way of contradiction, suppose that $L \neq G$, and set $M = G - L$. Let $M_0$ be a connected component of $M$.

**Claim 3.1.** *Let $1 \leq i \leq k$, suppose that $x_i \in V_1$. Then either $N_{C_i}(M_0) \cap V_1 = \emptyset$ or $N_{C_i - \{y_i\}}(M_0) \cap V_2 = \emptyset$.*

*Proof.* Without loss of generality, we may assume that $i = 1$. If $|M_0| = 1$, then the claim clearly holds. We may assume that $|M_0| \geq 2$. Suppose that $N_{C_1}(M_0) \cap V_1 \neq \emptyset$ and $N_{C_1 - \{y_1\}}(M_0) \cap V_2 \neq \emptyset$. Reversing the orientation of $C_1$ if necessary, we may assume that there exist $uw, vz \in E(G)$ with $u \in V(M_0) \cap V_1$, $v \in V(M_0) \cap V_2$ and $w, z \in V(C_1) - \{y_1\}$ satisfying $P_1 \subset C_1[z, w]$ and $N_G(M_0) \cap V(C_1(w, z)) = \emptyset$. If $z = w^+$, then in $\langle V(C_1) \cup V(M_0) \rangle_G$ there exists an admissible cycle longer than $C_1$, a contradiction. Hence we may assume that $|C_1(w, z)| \geq 2$. Let $D$ be the cycle obtained by joining $C_1[z, w]$ and a path $Q$ connecting $u$ and $v$ in $M_0$. Let $S = \{u, v, w^+, z^-\}$. Suppose that $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^-) \geq \frac{|C_1[z,w]|+4}{2}$. Then by Lemma 3.2, there exists a path $Q'$ with endvertices $z$, $w$ such that $V(Q') = V(C_1[z,w]) \cup V(C_1[w^+, z^-])$ and $P_1 \subset Q'$. Since $C_1[z,w]$ is also a segment of $D$, this contradicts the maximality of $|L|$. Thus

$$d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^-) \leq \frac{|C_1[z,w]| + 3}{2}.$$

Similarly it follows from Lemma 3.1 that

$$d_{C_i}(w^+) + d_{C_i}(z^-) \leq \frac{|C_i|}{2} + 1 \text{ for each } i \text{ with } 2 \leq i \leq k.$$

Also

$$d_{C_1[z,w]}(u) + d_{C_1[z,w]}(v) \leq \frac{|C_1[z,w]| + 3}{2}$$

by Lemma 3.2, and we have $d_{C_i}(u) + d_{C_i}(v) \leq \frac{|C_i|}{2} + 1$ for each $i$ with $2 \leq i \leq k$ by Lemma 3.1. Since $d_{C_1(w,z)}(u) + d_{C_1(w,z)}(v) = 0$, $d_{\langle V(C_1(w,z))\rangle_G}(w^+) + d_{\langle V(C_1(w,z))\rangle_G}(z^-) \leq |C_1(w,z)|$, $d_M(u) + d_M(v) \leq |M_0|$ and $d_M(w^+) + d_M(z^-) \leq |M - M_0|$, we now obtain

$$\begin{aligned} d_G(S) &\leq |M| + |C_1(w,z)| + 2\Big(\frac{|C_1[z,w]| + 3}{2}\Big) + \sum_{i=2}^{k} 2\Big(\frac{|C_i|}{2} + 1\Big) \\ &= |M| + \sum_{i=1}^{k}(|C_i| + 2) + 1 \\ &= 2n + 2k + 1. \end{aligned}$$

On the other hand, since $uz^-, vw^+ \notin E(G)$, $d_G(S) \geq 2(n + 2k - 1)$. But this is a contradiction because $k \geq 3$. □

Using an argument similar to the proof of Claim 3.1, we also obtain the following claim.

**Claim 3.2.** *Let $1 \leq i \leq k$, and suppose that $x_i \in V_2$. Then either $N_{C_i - \{y_i\}}(M_0) \cap V_1 = \emptyset$ or $N_{C_i}(M_0) \cap V_2 = \emptyset$.*

Hereafter, we divide the proof into two cases according to the order of $M$.

## 3.1. $|M| = 2$

Write $V(M) = \{u, v\}$. By symmetry, we may assume that $u \in V_1$ and $v \in V_2$.

**Claim 3.3.** *Suppose that $uv \notin E(G)$. Then $d_{C_i}(u) + d_{C_i}(v) = \frac{|C_i|}{2} + 2$ for each $i$ with $1 \leq i \leq k$.*

*Proof.* By Lemma 3.3, we have

$$d_G(u)+d_G(v)=\sum_{i=1}^{k}(d_{C_i}(u)+d_{C_i}(v))\le\sum_{i=1}^{k}\Big(\frac{|C_i|}{2}+2\Big)=n+2k-1.$$

On the other hand, since $uv\notin E(G)$, $d_G(u)+d_G(v)\ge n+2k-1$. Therefore

$$d_{C_i}(u)+d_{C_i}(v)=\frac{|C_i|}{2}+2 \text{ for each } i \text{ with } 1\le i\le k.$$

□

**Claim 3.4.** *Let $1\le i\le k$, and suppose that $|C_i|\ge 6$ and $x_i\in V_1$. Then it is not possible that we have both $d_{C_i}(u)=2$ and $d_{C_i}(v)=\frac{|C_i|}{2}$.*

*Proof.* Without loss of generality, we may assume that $i=1$. Suppose that $d_{C_1}(u)=2$ and $d_{C_1}(v)=\frac{|C_1|}{2}$. By Lemma 3.1, $uv\notin E(G)$. Write $C_1=a_1b_1a_2b_2\cdots a_rb_ra_1$ with $P_1=x_1y_1z_1=a_1b_1a_2$. If $N_{C_1}(u)=\{b_p,b_q\}$ $(2\le p<q\le r)$, then $\{vC_1^-[a_q,b_p]uC_1[b_q,a_p]v,\ C_2,\dots,C_k\}$ is a required partition of $G$, a contradiction. Thus $N_{C_1}(u)=\{b_1,b_h\}$ $(1<h)$. Since $|C_1|\ge 6$, we have $b_{h-1}\ne y_1$ or $b_{h+1}\ne y_1$. By symmetry, we may assume that $b_{h-1}\ne y_1$. Let $C_1'=vC_1[a_h,a_{h-1}]v$ and $C_i'=C_i$ for each $i$ with $2\le i\le k$. Then $C_1',C_2',\dots,C_k'$ are admissible and $|\bigcup_{i=1}^{k}V(C_i')|=|L|$. Hence applying Claim 3.3 with the $C_i$ replaced by the $C_i'$, we obtain

$$d_{C_1'}(u)+d_{C_1'}(b_{h-1})=\frac{|C_1'|}{2}+2.$$

Since we get $N_{C_1'}(u)=\{b_1,b_h\}$ from $N_{C_1}(u)=\{b_1,b_h\}$, this implies $d_{C_1'}(b_{h-1})=\frac{|C_1'|}{2}$; in particular, $b_{h-1}a_{h+1}\in E(G)$. On the other hand, we have $d_{C_2'}(u)+d_{C_2'}(v)=\frac{|C_2'|}{2}+2$ by Claim 3.3. Hence it follows Lemma 3.1 that there exists a cycle $C_2''$ such that $P_2\subset C_2''$ and $V(C_2'')=V(C_2)\cup\{u,b_h,a_h,v\}$. Now if we let $C_1''=C_1[a_{h+1},b_{h-1}]a_{h+1}$ and let $C_i''=C_i$ for each $i$ with $3\le i\le k$, then $\{C_1'',C_2'',\dots,C_k''\}$ is a required partition of $G$, a contradiction. □

**CASE I.** $M$ is connected.

**Claim 3.5.** *Let $1\le i\le k$, and suppose that $x_i\in V_1$. Then $|N_{C_i}(v)|\le 2$.*

*Proof.* Without loss of generality, we may assume that $i=1$. Suppose that $|N_{C_1}(v)|\ge 3$. By Claim 3.1, we have

$$N_{C_1}(u)\subseteq\{y_1\}. \tag{3.1}$$

Since $|N_{C_i}(v)|\ge 3$, we can choose two vertices $w,z\in N_{C_1}(v)$ such that $P_1\subset C_1(z,w]$ and $N_{C_1}(v)\cap V(C_1(w,z))=\emptyset$.

**Case 1.** $|C_1(w,z)| \geq 3$.

Let $S = \{u, v, w^+, z^{--}\}$. Suppose that $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \geq \frac{|C_1[z,w]|+4}{2}$. Then by Lemma 3.2, there exists a path $Q'$ with endvertices $z$, $w$ such that $V(Q') = V(C_1[z,w]) \cup V(C_1[w^+, z^{--}])$ and $P_1 \subset Q'$. Let $D = vQ'v$. Then $|D| \geq 6$ and $|V(D) \cup \bigcup_{i=2}^{k} V(C_i)| = |L|$. Note that $uz^- \notin E(G)$ by (3.1). Hence applying Claim 3.3 to $D, C_2, C_3, \ldots, C_k$, we obtain

$$d_D(u) + d_D(z^-) = \frac{|D|}{2} + 2. \tag{3.2}$$

Note that (3.1) implies $N_D(u) \subseteq \{v, y_1\}$. Hence it follows from (3.2) that $N_D(u) = \{v, y_1\}$ and $d_D(z^-) = \frac{|D|}{2}$. But applying Claim 3.4 to $D, C_2, C_3, \ldots, C_k$, we see that this is impossible. Thus

$$d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \leq \frac{|C_1[z,w]|+3}{2}.$$

Suppose now that there exists $i$ with $2 \leq i \leq k$ such that $d_{C_i}(w^+) + d_{C_i}(z^{--}) \geq \frac{|C_i|}{2} + 2$. Then by Lemma 3.1, there exists a cycle $C_i'$ such that $V(C_i') = V(C_i) \cup V(C_1[w^+, z^{--}])$ and $P_i \subset C_i'$. Let $C_1' = vC_1[z,w]v$ and $C_j' = C_j$ for each $j$ with $2 \leq j \leq k$ and $j \neq i$. Then $|C_1'| \geq 6$ and $|\bigcup_{j=1}^{k} V(C_j')| = |L|$. Hence applying Claim 3.3 to $C_1', C_2', \ldots, C_k'$, we obtain

$$d_{C_1'}(u) + d_{C_1'}(z^-) = \frac{|C_1'|}{2} + 2,$$

which again contradicts Claim 3.4. Thus $d_{C_i}(w^+) + d_{C_i}(z^{--}) \leq \frac{|C_i|}{2} + 1$ for each $i$ with $2 \leq i \leq k$. Also $d_{C_i}(u) + d_{C_i}(v) \leq \frac{|C_i|}{2} + 1$ for each $i$ with $2 \leq i \leq k$ by Lemma 3.1 and, since $N_{C_1(w,z)}(v) = \emptyset$, it follows from (3.1) that $d_{C_1[z,w]}(u) + d_{C_1[z,w]}(v) \leq \frac{|C_1[z,w]|+1}{2} + 1 = \frac{|C_1[z,w]|+3}{2}$. Since $d_{C_1(w,z)}(u) + d_{C_1(w,z)}(v) = 0$, $d_{\langle V(C_1(w,z))\rangle_G}(w^+) + d_{\langle V(C_1(w,z))\rangle_G}(z^{--}) \leq |C_1(w,z)|$, $d_M(u) + d_M(v) = 2$ and $d_M(w^+) + d_M(z^{--}) = 0$, we now obtain

$$\begin{aligned} d_G(S) &\leq 2 + |C_1(w,z)| + 2\Big(\frac{|C_1[z,w]|+3}{2}\Big) + \sum_{i=2}^{k} 2\Big(\frac{|C_i|}{2} + 1\Big) \\ &= 2n + 2k + 1. \end{aligned}$$

On the other hand, since $w^+u, z^{--}v \notin E(G)$, $d_G(S) \geq 2(n + 2k - 1)$. But this is a contradiction because $k \geq 3$.

**Case 2.** $|C_1(w,z)| = 1$.

Write $V(C_1(w,z)) = \{a\}$. Let $C_1' = vC_1[z,w]v$ and $C_i' = C_i$ for each $i$ with $2 \leq i \leq k$. Then $|C_1'| \geq 6$. $C_1', C_2', \ldots, C_k'$ are admissible, and $|\bigcup_{i=1}^{k} V(C_i')| =$

$|L|$. By (3.1), we also have $au \notin E(G)$. Hence by Claim 3.3, we obtain

$$d_{C'_1}(u) + d_{C'_1}(a) = \frac{|C'_1|}{2} + 2$$

In view of (3.1), this forces $N_{C'_1}(u) = \{v, y_1\}$ and $|N_{C'_1}(a)| = \frac{|C'_1|}{2}$. But then we get a contradiction by applying Claim 3.4 to $C'_1, C'_2, \ldots, C'_k$. □

**Claim 3.6.** *Let $1 \leq i \leq k$, and suppose that $x_i \in V_1$. Then $|N_{C_i - \{y_i\}}(u)| \leq 1$.*

*Proof.* Without loss of generality, we may assume that $i = 1$. Suppose that $|N_{C_1 - \{y_1\}}(u)| \geq 2$. We can choose two vertices $w, z \in N_{C_1}(u)$ such that $P_1 \subset C_1(z, w)$ and $N_{C_1}(u) \cap V(C_i(w, z)) = \emptyset$. Since $N_{C_1}(v) = \emptyset$ by Claim 3.1, the rest of the proof is similar to and easier than that of Claim 3.5. □

Using an argument similar to the proof of Claims 3.5 and 3.6, we obtain the following two claims.

**Claim 3.7.** *Let $1 \leq i \leq k$, and suppose that $x_i \in V_2$. Then $|N_{C_i - \{y_i\}}(v)| \leq 1$.*

**Claim 3.8.** *Let $1 \leq i \leq k$, and suppose that $x_i \in V_2$. Then $|N_{C_i}(u)| \leq 2$.*

By Claims 3.1, 3.2, 3.5, 3.6, 3.7 and 3.8, we have $d_L(u) + d_L(v) \leq 3k$. Hence $d_G(u) + d_G(v) \leq 3k + 2$. On the other hand, from the assumption that $\sigma_{1,1}(G) \geq n + 2k - 1$, it follows that $d_G(x) \geq 2k$ for all $x \in V(G)$, and hence $d_G(u) + d_G(v) \geq 4k$. But this is a contradiction because $k \geq 3$.

**CASE II.** $M$ is disconnected.

By Claim 3.3, we have

$$d_{C_i}(u) + d_{C_i}(v) = \frac{|C_i|}{2} + 2 \text{ for each } i \text{ with } 1 \leq i \leq k.$$

Suppose that there exists $i$ with $1 \leq i \leq k$ such that $|C_i| = 4$. Let $C'_i = vx_iy_iz_iv$ or $C'_i = ux_iy_iz_iu$ according as $x_i \in V_1$ or $x_i \in V_2$, and let $C'_j = C_j$ for each $j$ with $1 \leq j \leq k$ and $j \neq i$. Then $G - \bigcup_{i=1}^{k} V(C'_i)$ is connected. Consequently we are reduced to CASE I, which leads to a contradiction. Thus $|C_i| \geq 6$ for each $i$ with $1 \leq i \leq k$. Since $k \geq 3$, we have $|\{i \,|\, x_i \in V_1\}| \geq 2$ or $|\{i \,|\, x_i \in V_2\}| \geq 2$. Without loss of generality, we may assume that $|\{i \,|\, x_i \in V_1\}| \geq 2$. We may also assume that $x_1, x_2 \in V_1$. We write $C_1 = a_1b_1a_2b_2 \cdots a_rb_ra_1$ with $P_1 = x_1y_1z_1 = a_1b_1a_2$.

**Claim 3.9.**

(i) $|\{p \,|\, 2 \leq p \leq r, a_p \in N_G(v), b_p \in N_G(u)\}| \leq 1$.

(ii) $|\{p \mid 2 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\}| \le 1$ (*we take* $a_{r+1} = a_1$).

*Proof.* By way of contradiction, suppose that there exist $p, q$ with $2 \le p < q \le r$ such that $a_p, a_q \in N_G(v)$ and $b_p, b_q \in N_G(u)$. Let $C_1' = uC_1[b_p, a_q]vC_1^-[a_p, b_q]u$. Then $\{C_1', C_2, \ldots, C_k\}$ is a desired partition of $G$, which contradicts the assumption that $G$ is a counterexample. Thus (i) is proved, and (ii) can be verified in a similar way. □

**Claim 3.10.**

(i) (a) $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} \ne \emptyset$ *for each* $p$ *with* $1 \le p \le r$.
(b) $|\{p \mid 1 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\}| = 2$.

(ii) (a) $(N_G(u) \cup N_G(v)) \cap \{b_p, a_{p+1}\} \ne \emptyset$ *for each* $p$ *with* $1 \le p \le r$.
(b) $|\{p \mid 1 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\}| = 2$.

*Proof.* Set $\alpha = |\{p \mid 1 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\}|$ and $\beta = |\{p \mid 1 \le p \le r, (N_G(v) \cup N_G(u)) \cap \{a_p, b_p\} = \emptyset\}|$. Since $d_{C_1}(u) + d_{C_1}(v) = \frac{|C_1|}{2} + 2$ by Claim 3.3, we have $\alpha = \beta + 2$. Since $\alpha \le 2$ by Claim 3.9, this implies $\alpha = 2$ and $\beta = 0$. Thus (i) is proved, and (ii) can be verified in a similar way. □

**Claim 3.11.** *There exist* $s$, $t$ *with* $2 \le s < t \le r$ *such that* $N_{C_1}(u) = \{b_1\} \cup \{b_p \mid s \le p \le t\}$ *and* $N_{C_1}(v) = \{a_p \mid 1 \le p \le s\} \cup \{a_p \mid t+1 \le p \le r\}$.

*Proof.* By Claims 3.9(i) and 3.10(i)(b),

$$a_1 \in N_G(v), b_1 \in N_G(u),$$
$$|\{p \mid 2 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\}| = 1.$$

Similarly by Claims 3.9(ii) and 3.10(ii)(b),

$$a_2 \in N_G(v), b_1 \in N_G(u),$$
$$|\{p \mid 2 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\}| = 1.$$

Write $\{p \mid 2 \le p \le r, a_p \in N_G(v), b_p \in N_G(u)\} = \{s\}$ and $\{p \mid 2 \le p \le r, a_{p+1} \in N_G(v), b_p \in N_G(u)\} = \{t\}$. Suppose that $s > t$. Then since $b_s \in N_G(u)$ and $a_{r+1} = a_1 \in N_G(v)$, it follows from Claim 3.10(i)(a) that there exists $p'$ with $s \le p' \le r$ such that $b_{p'} \in N_G(u)$ and $a_{p'+1} \in N_G(v)$. But since $t < p'$, this contradicts Claim 3.9(ii). Thus $s \le t$. If there exists $p'$ with $2 \le p' \le s-1$ such that $b_{p'} \in N_G(u)$, then by Claim 3.10(i)(a), there exists $q'$ with $p' \le q' \le s-1$ such that $b_{q'} \in N_G(u)$ and $a_{q'+1} \in N_G(v)$, which again contradicts Claim 3.9(ii). Thus $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} = \{a_p\}$ for each $p$ with $2 \le p \le s-1$. Similarly $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} = \{a_p\}$ for each $p$ with $t+1 \le p \le r$. If $s = t$, then it follows that $N_{C_1}(u) = \{b_1, b_s\}$ and $N_{C_1}(v) = \{a_p \mid 1 \le p \le r\}$, which contradicts Claim 3.4. Thus $s < t$. Now since $b_s \in N_{C_1}(u)$, we see from Claims 3.10(i)(a) and 3.9(ii) that $(N_G(u) \cup N_G(v)) \cap \{a_p, b_p\} = \{b_p\}$ for each $p$ with $s+1 \le p \le t-1$. □

We can now complete the proof for the case where $|M| = 2$. We write $C_2 = c_1d_1c_2d_2\cdots c_hd_hc_1$ with $P_2 = x_2y_2z_2 = c_1d_1c_2$. Applying Claim 3.11 to $C_2$, we see that there exists $q$ with $3 \le q \le h$ such that $c_q \notin N_{C_2}(v)$ and $d_{q-1}, d_q \in N_{C_2}(u)$. Let $C_1' = C_1$, $C_2' = uC_2[d_q, d_{q-1}]u$ and $C_i' = C_i$ for each $i$ with $3 \le i \le k$. We apply Claim 3.11 with the $C_i$ replaced by the $C_i'$ (so $u$ is replaced by $c_q$). Note that $C_1 = C_1'$, and hence $N_{C_1}(v) = N_{C_1'}(v)$. Consequently it follows that $N_{C_1}(c_q) = N_{C_1}(u) = \{b_1\} \cup \{b_p \mid s \le p \le t\}$. Now if we let $C_1'' = vC_1[a_{t+1}, a_s]v$, $C_2'' = uC_2[d_q, d_{q-1}]c_qC_1[b_s, b_t]u$ and $C_i'' = C_i$ for each $i$ with $3 \le i \le k$, then $\{C_1'', C_2'', \ldots, C_k''\}$ is a required partition of $G$. This is a contradiction.

This concludes the discussion for the case $|M| = 2$.

### 3.2. $|M| \ge 4$

**Claim 3.12.** *Let $u \in V(M) \cap V_1$ and $v \in V(M) \cap V_2$ with $uv \notin E(G)$. Then $d_M(u) + d_M(v) \ge \frac{|M|}{2} - 1$.*

*Proof.* By Lemma 3.3, we have

$$d_L(u) + d_L(v) \le \sum_{i=1}^{k}\Big(\frac{|C_i|}{2} + 2\Big) = \frac{|L|}{2} + 2k.$$

Hence

$$d_M(u) + d_M(v) \ge n + 2k - 1 - \Big(\frac{|L|}{2} + 2k\Big) = \frac{|M|}{2} - 1.$$

□

**CASE A.** $M$ is disconnected.

Let $M_0$ be a connected component of $M$ which has the smallest order (among all connected components of $M$). Let $M_1 = M - M_0$. We may assume that $|V(M_0) \cap V_1| \ge |V(M_0) \cap V_2|$. Then $|V(M_1) \cap V_1| \le |V(M_1) \cap V_2|$. Take $u \in V(M_0) \cap V_1$ and $v \in V(M_1) \cap V_2$. We divide the proof into the following two subcases according to the value of $|M_0|$.

**Subcase A.1.** $|M_0| \ge 2$.

In this case, the component containing $v$ also has order at least 2. Thus there exists $v' \in V(M_0)$ with $uv' \in E(G)$, and there exists $u' \in V(M_1)$ with

$u'v \in E(G)$. Let $S = \{u, v, u', v'\}$. By Lemma 3.1,

$$\begin{aligned} d_L(u) + d_L(v') &\le \sum_{i=1}^{k}\Big(\frac{|C_i|}{2} + 1\Big) = \frac{|L|}{2} + k, \\ d_L(u') + d_L(v) &\le \sum_{i=1}^{k}\Big(\frac{|C_i|}{2} + 1\Big) = \frac{|L|}{2} + k. \end{aligned}$$

Since we clearly have $d_M(u)+d_M(v') \le |M_0|$ and $d_M(u')+d_M(v) \le |M_1|$, this implies

$$\begin{aligned} d_G(S) &\le |M_0| + |M_1| + 2\Big(\frac{|L|}{2} + k\Big) \\ &= |M| + |L| + 2k \\ &= 2n + 2k. \end{aligned}$$

On the other hand, since $uv, u'v' \notin E(G)$, we have $d_G(S) \ge 2(n + 2k - 1)$, which is a contradiction.

**Subcase A.2.** $|M_0| = 1$.

Since $d_M(u) = 0$, it follows from Claim 3.12 that $d_M(v) = \frac{|M|}{2} - 1$. Since this holds for any $v \in V(M_1) \cap V_2$, $M_1$ is a complete bipartite graph. From the proof of Claim 3.12, we also see that $d_{C_i}(u) + d_{C_i}(v) = \frac{|C_i|}{2} + 2$ for each $i$ with $1 \le i \le k$, which in particular implies that

(3.3) $$d_{C_i}(v) \ge 2 \text{ for each } i \text{ with } 1 \le i \le k.$$

Take $u' \in V(M_1) \cap V_1$. By (3.3) and Claims 3.1 and 3.2,

(3.4) $$N_{C_i}(u') \subseteq \{y_i\} \text{ for each } i \text{ with } 1 \le i \le k,$$

and hence $d_G(u') \le \frac{|M|}{2}+k$. Now take $b \in (V(C_1)\cap V_2)-\{y_1\}$. Since (3.4) holds for any $u' \in V(M_1) \cap V_1$, we have $N_M(b) \subseteq \{u\}$, and hence $d_G(b) \le \frac{|L|}{2} + 1$. Therefore $d_G(u') + d_G(b) \le (\frac{|M|}{2} + k) + (\frac{|L|}{2} + 1) = n + k + 1$. On the other hand, since $u'b \notin E(G)$, we have $d_G(u') + d_G(b) \ge n + 2k - 1$, which is a contradiction.

**CASE B.** $M$ is connected.

**Claim 3.13.** *Let $1 \le i \le k$, and suppose that $x_i \in V_1$. Then $|N_{C_i}(M) \cap V_1| \le 1$.*

*Proof.* Without loss of generality, we may assume that $i = 1$. Suppose that $|N_{C_1}(M) \cap V_1| \ge 2$. Then $N_{C_1}(M) \cap V_2 \subseteq \{y_1\}$ by Claim 3.1. Choose two vertices $w, z \in N_{C_1}(M) \cap V_1$ so that $P_1 \subset C_1[z, w]$ and $N_G(M) \cap V(C_1(w, z)) = \emptyset$.

Take $v \in N_M(w)$ and $v' \in N_M(z)$, and let $Q$ be a path joining $v$ and $v'$ in $M$. Then $\langle V(C_1[z,w]) \cup V(M)\rangle_G$ contains an admissible cycle $D$ such that $V(D) = V(C_1[z,w]) \cup V(Q)$. Suppose that $|C_1(w,z)| = 1$. Write $V(C_1(w,z)) = \{b\}$. If $v \neq v'$, then we get a contradiction to the maximality of $|L|$; if $v = v'$, then since $N_M(b) = \emptyset$, $G - V(D) - \bigcup_{i=2}^{k} V(C_i)$ is disconnected, and hence we are reduced to CASE A, which also leads to a contradiction. Thus $|C_1(w,z)| \geq 3$. Take $u \in V(M) \cap V_1$. Let $S = \{w^+, z^{--}, u, v\}$. Suppose that $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \geq \frac{|C_1[z,w]|+4}{2}$. Then it follows Lemma 3.2 that there exists a cycle $D'$ such that $P_1 \subset D'$ and $V(D') = V(D) \cup V(C_1[w^+, z^{--}]) = (V(C_1) - \{z^-\}) \cup V(Q)$. If $v \neq v'$, then this contradicts the maximality of $|L|$. If $v = v'$, then since $N_M(z^-) = \emptyset$, we are reduced to CASE A. Thus $d_{C_1[z,w]}(w^+) + d_{C_1[z,w]}(z^{--}) \leq \frac{|C_1[z,w]|+3}{2}$. Similarly it follows from Lemma 3.1 that $d_{C_i}(w^+) + d_{C_i}(z^{--}) \leq \frac{|C_i|}{2} + 1$ for each $i$ with $2 \leq i \leq k$. Also $d_{C_i}(u) + d_{C_i}(v) \leq \frac{|C_i|}{2} + 1$ for each $i$ with $2 \leq i \leq k$ by Lemma 3.1, and we have $d_{C_1[z,w]}(u) + d_{C_1[z,w]}(v) \leq \frac{|C_1[z,w]|+3}{2}$ because $N_{C_1}(u) \subseteq \{y_1\}$ by Claim 3.1. Since $d_{C_1(w,z)}(u) + d_{C_1(w,z)}(v) = 0$, $d_{\langle V(C_1(w,z))\rangle_G}(w^+) + d_{\langle V(C_1(w,z))\rangle_G}(z^{--}) \leq |C_1(w,z)|$, $d_M(u) + d_M(v) \leq |M|$ and $d_M(w^+) + d_M(z^{--}) = 0$, we now obtain

$$\begin{aligned} d_G(S) &\leq |M| + |C_1(w,z)| + |C_1[z,w]| + 3 + \sum_{i=2}^{k} 2\Big(\frac{|C_i|}{2} + 1\Big) \\ &= 2n + 2k + 1. \end{aligned}$$

On the other hand, since $w^+u, z^{--}v \notin E(G)$, we have $d_G(S) \geq 2(n + 2k - 1)$, which is a contradiction. □

Using an argument similar to the proof of Claim 3.13, we obtain the following claim.

**Claim 3.14.** *Let $1 \leq i \leq k$, and suppose that $x_i \in V_2$. Then $|N_{C_i - \{y_i\}}(M) \cap V_1| \leq 1$.*

We are now in a position to complete the proof of Theorem 1.5. By symmetry, we may assume that $|\{i \mid x_i \in V_1\}| \geq |\{i \mid x_i \in V_2\}|$. Let $t = |\{i \mid x_i \in V_1\}|$. Since $k \geq 3$, we have $t \geq 2$. At the cost of relabeling, we may assume that $x_1 \in V_1$. Take $a \in (V(C_1) \cap V_1) - N_{C_1}(M)$ and $v \in V(M) \cap V_2$. Then since $d_M(a) = 0$ and $d_L(a) \leq \frac{|L|}{2}$, we have $d_G(a) \leq \frac{|L|}{2}$. By Claims 3.13 and 3.14, $d_L(v) \leq t + 2(k-t) = 2k - t$. Hence $d_G(v) = d_M(v) + d_L(v) \leq \frac{|M|}{2} + 2k - t$. Therefore $d_G(a) + d_G(v) \leq n + 2k - t \leq n + 2k - 2$. But since $av \notin E(G)$, this contradicts the assumption that $\sigma_{1,1}(G) \geq n + 2k - 1$.

This completes the proof of Theorem 1.5.

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