SUT Journal of Mathematics
Vol. 41, No. 2 (2005), 137–152

On the periodicity of the Auslander-Reiten translation and the Nakayama functor for the enveloping algebra of self-injective Nakayama algebras

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(Received August 1, 2005)

Abstract. In this paper, we describe the structures of the left B^e-modules $\tau^i_{B^e}(B)$ and $\mathcal{N}^i_{B^e}(B)$ for $i \geqslant 0$, where B is a certain finite dimensional self-injective Nakayama algebra, B^e is the enveloping algebra of B, τ_{B^e} is the Auslander-Reiten translation in the category $\mathrm{mod}\,(B^e)$ of finitely generated left B^e-modules and $\mathcal{N}_{B^e}\colon \mathrm{mod}\,(B^e) \to \mathrm{mod}\,(B^e)$ is the Nakayama functor. Moreover, we compute the τ_{B^e}-period and the $\mathcal{N}_{B^e}$-period of B.

AMS 2000 *Mathematics Subject Classification.* 16D50, 16E05.

Key words and phrases. Auslander-Reiten translation, Nakayama functor, self-injective Nakayama algebra.

§1. Introduction

Let A be a finite dimensional self-injective algebra over a field K, and let A° be the opposite algebra of A. We denote the category of finitely generated left A-modules by $\mathrm{mod}\,(A)$ and the Auslander-Reiten translation in $\mathrm{mod}\,(A)$ by τ_A. The *Nakayama functor* $\mathcal{N}_A\colon \mathrm{mod}\,(A) \to \mathrm{mod}\,(A)$ is defined by the composition $D(\ \ )^\vee$, where $(\ \ )^\vee$ is the contravariant functor $\mathrm{Hom}_A(\ \ ,A)\colon \mathrm{mod}\,(A) \to \mathrm{mod}\,(A^\circ)$ and D is the duality $\mathrm{Hom}_K(\ \ ,K)\colon \mathrm{mod}\,(A^\circ) \to \mathrm{mod}\,(A)$. In this paper, we deal with τ_A and $\mathcal{N}_A$ in the case where A is the enveloping algebra $B^e := B \otimes_K B^\circ$ of a certain self-injective Nakayama algebra B.

Let K be a field, s a positive integer and Γ the cyclic quiver with s vertices $e_1, e_2, \ldots, e_s$ and s arrows $a_1, a_2, \ldots, a_s$ such that a_i starts at e_i and ends at e_{i+1}. So $a_i = e_{i+1}a_ie_i$ holds for all $1 \leqslant i \leqslant s$ in the path algebra $K\Gamma$, where we regard the subscripts i of e_i modulo s. Denote the sum of all arrows of

Γ by X: $X = a_1 + a_2 + \cdots + a_s \in K\Gamma$. If K is an algebraically closed field, then it is known that a self-injective Nakayama algebra over K which is basic, indecomposable and nonisomorphic to K is of the form $B := K\Gamma/(X^k)$ where $k \geqslant 2$ (see [EH]). And, in [EH] this algebra is denoted by B_s^k. In [P2], Pogorzały computes the τ_{B^e}-period of the left B^e-module B by means of the Galois covering of B^e. In this paper, we determine the structure of the left B^e-modules $\mathcal{N}^i_{B^e}(B)$ as well as the $\tau^i_{B^e}(B)$ for $i \geqslant 0$ by using the structure of syzygy module $\Omega^2_{B^e}(B)$ given in [EH, F], and hence we compute the τ_{B^e}-period and the $\mathcal{N}_{B^e}$-period of B.

In Section 2, as preliminaries, we describe the definitions and some properties of τ_A and $\mathcal{N}_A$ for any finite dimensional self-injective algebra A. Moreover, for any finite dimensional algebra C, any algebra automorphism $\alpha\colon C \to C$ and $M \in \operatorname{mod}(C^e)$, we give the definition of the left C^e-module ${}_1M_\alpha$. In Section 3, we consider the dual module $D(e_iB \otimes_K Be_j)$ $(1 \leqslant i, j \leqslant s)$ for the indecomposable projective right B^e-module $e_iB \otimes_K Be_j$ (Proposition 3.3). In Section 4, we give a minimal injective B^e-copresentation of $\tau_{B^e}({}_1B_{\beta^n})$ for some algebra automorphism $\beta\colon B \to B$ and any integer n with $n \geqslant 0$, and hence we describe the structures of $\tau^i_{B^e}(B)$ and $\mathcal{N}^i_{B^e}(B)$ $(i \geqslant 0)$ (Theorem). Moreover, we compute the τ_{B^e}-period and the $\mathcal{N}_{B^e}$-period of B (Corollary 4.6). Finally, as Appendix, we give an alternative proof of Theorem in Section 4 by means of the Nakayama automorphism ν of B^e.

For general facts on algebras we refer to [ARS]. Throughout this paper, we will denote $\otimes_K$ by $\otimes$.

§2. Preliminaries

Let A be any finite dimensional self-injective algebra over a field K. We denote the contravariant functor $\operatorname{Hom}_A(\ ,A)\colon \operatorname{mod}(A) \to \operatorname{mod}(A^\circ)$ by $(\ )^\vee$ and the duality $\operatorname{Hom}_K(\ ,K)\colon\ \operatorname{mod}(A^\circ) \to \operatorname{mod}(A)$ by D. Since A is a self-injective algebra, $(\ )^\vee\colon \operatorname{mod}(A) \to \operatorname{mod}(A^\circ)$ is a duality. So the Nakayama functor $\mathcal{N}_A := D(\ )^\vee\colon \operatorname{mod}(A) \to \operatorname{mod}(A)$ is an equivalence of the categories.

Take any $M \in \operatorname{mod}(A)$ and fix a minimal projective A-presentation $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ of M. We define a left A-module $\Omega_A(M) := \operatorname{Ker} f_0$ and we put $\Omega^0_A(M) := M$ and $\Omega^i_A(M) := \Omega_A\left(\Omega^{i-1}_A(M)\right)$ for each $i \geqslant 1$. Then we have the exact sequence

$$0 \longrightarrow \Omega^2(M) \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0.$$

Also, we define a A°-module $\operatorname{Tr}_A(M) := \operatorname{Coker} f_1^\vee$, which is called the *transpose* of M. Then we obtain the following exact sequence of left A°-modules:

$$0 \longrightarrow M^\vee \xrightarrow{f_0^\vee} P_0^\vee \xrightarrow{f_1^\vee} P_1^\vee \longrightarrow \operatorname{Tr}_A(M) \longrightarrow 0,$$

where $P_0^\vee \xrightarrow{f_1^\vee} P_1^\vee \to \mathrm{Tr}_A(M) \to 0$ is a minimal projective A°-presentation of $\mathrm{Tr}_A(M)$. Furthermore, we define a left A-module $\tau_A(M) := D\mathrm{Tr}_A(M)$, which is called the *Auslander-Reiten translation.* Then we get the following exact sequence of left A-modules:

$$0 \longrightarrow \tau_A(M) \longrightarrow \mathcal{N}_A(P_1) \xrightarrow{\mathcal{N}_A(f_1)} \mathcal{N}_A(P_0) \xrightarrow{\mathcal{N}_A(f_0)} \mathcal{N}_A(M) \longrightarrow 0,$$

where $0 \to \tau_A(M) \to \mathcal{N}_A(P_1) \xrightarrow{\mathcal{N}_A(f_1)} \mathcal{N}_A(P_0)$ is a minimal injective A-copresentation. Here, since $\mathcal{N}_A$ is an equivalence, we easily obtain isomorphisms $\tau_A(M) \simeq \Omega_A^2\mathcal{N}_A(M) \simeq \mathcal{N}_A\Omega_A^2(M)$ of left A-modules.

For each $M \in \mathrm{mod}\,(A)$, we put $\tau_A^0(M) := M$ and $\tau_A^i(M) := \tau_A\left(\tau_A^{i-1}(M)\right)$ for $i \geqslant 1$. A left A-module N is τ_A-*periodic* if $\tau_A^m(N) \simeq N$ for some positive integer m. Then the τ_A-*period* of N is the smallest positive integer n with $\tau_A^n(N) \simeq N$. Similarly, for each $M \in \mathrm{mod}\,(A)$, we define $\mathcal{N}_A^0(M) := M$ and $\mathcal{N}_A^i(M) := \mathcal{N}_A\left(\mathcal{N}_A^{i-1}(M)\right)$ for $i \geqslant 1$. A left A-module N said to be $\mathcal{N}_A$-*periodic* if $\mathcal{N}_A^m(N) \simeq N$ for some positive integer m. Then we call the smallest positive integer n with $\mathcal{N}_A^n(N) \simeq N$ the $\mathcal{N}_A$-*period* of N.

Let C be any finite dimensional algebra over a field K, $\alpha\colon C \to C$ an algebra automorphism, and M a left C^e-module, equivalently C-bimodule. Then we will define the left C^e-module ${}_1M_\alpha$ as follows: ${}_1M_\alpha$ has the underlying K-space M, and the action of C on M from the left is the usual one. The action $*$ of C on M from the right is defined as $m * b = m\alpha(b)$ for $m \in {}_1M_\alpha$ and $b \in C$. Moreover, for each C^e-homomorphism $f\colon M \to N$, we define a C^e-homomorphism ${}_1f_\alpha\colon {}_1M_\alpha \to {}_1N_\alpha$ by ${}_1f_\alpha(m) = f(m)$ for each $m \in {}_1M_\alpha$. Then, by setting $F_\alpha(X) := {}_1X_\alpha$ for each object X in $\mathrm{mod}\,(C^e)$ and $F_\alpha(f) := {}_1f_\alpha$ for each morphism f in $\mathrm{mod}\,(C^e)$, we have the functor $F_\alpha\colon \mathrm{mod}\,(C^e) \to \mathrm{mod}\,(C^e)$. It is easy to check that $F_{\alpha^{-1}}F_\alpha = F_\alpha F_{\alpha^{-1}} = 1_{\mathrm{mod}\,(C^e)}$ holds. So F_α is an isomorphism of the categories. In particular, if $\psi\colon P \to M$ is a projective cover in $\mathrm{mod}\,(C^e)$, then $F_\alpha(\psi) = {}_1\psi_\alpha\colon {}_1P_\alpha \to {}_1M_\alpha$ is also a projective cover in $\mathrm{mod}\,(C^e)$.

§3. A self-injective Nakayama algebra and its enveloping algebra

Let K be a field, s a positive integer and Γ the cyclic quiver with s vertices $e_1, \dots, e_s$ and s arrows $a_1, \dots, a_s$. Denote the sum of all arrows in the path algebra $K\Gamma$ by X: $X = a_1 + \cdots + a_s$. Then $X^j e_i = e_{i+j}X^j = a_{i+j-1}\cdots a_i$, the path of length j for $j \geqslant 1$, where we regard the subscripts i of e_i modulo s.

We denote the algebra $K\Gamma/(X^k)$ by B, where k is a positive integer with $k \geqslant 2$. Note that the set $\left\{X^j e_i \,\middle|\, 1 \leqslant i \leqslant s,\ 0 \leqslant j \leqslant k-1\right\}$ is a K-basis of B, so

$\dim_K B = ks$. In this section, we consider the dual module $D(e_iB \otimes Be_j)$ $(1 \leqslant i, j \leqslant s)$ of the indecomposable projective right B^e-module $e_iB \otimes Be_j$.

First we consider the dual modules $D(Be_m)$ and $D(e_mB)$ for each m $(1 \leqslant m \leqslant s)$. Clearly the set $\{X^je_m \,|\, 0 \leqslant j \leqslant k-1\}$ gives a K-basis of Be_m and the set $\{e_mX^j \,|\, 0 \leqslant j \leqslant k-1\}$ gives a K-basis of e_mB. We take the dual basis $\{(X^je_m)^* \,|\, 0 \leqslant j \leqslant k-1\}$ of $D(Be_m)$, that is, each $(X^je_m)^* \in D(Be_m)$ $(0 \leqslant j \leqslant k-1)$ satisfies that $((X^je_m)^*)(X^qe_m) = 1$ if $q = j$, 0 if $q \neq j$. Similarly, we take the dual basis $\{(e_mX^j)^* \,|\, 0 \leqslant j \leqslant k-1\}$ of $D(e_mB)$.

Lemma 3.1. *Let j and m, n be integers with $0 \leqslant j \leqslant k-1$ and $1 \leqslant m, n \leqslant s$. Then, for $(X^je_m)^* \in D(Be_m)$, we have*

$$(X^je_m)^*X = \begin{cases} 0 & \text{if } j = 0, \\ (X^{j-1}e_m)^* & \text{if } 1 \leqslant j \leqslant k-1, \end{cases}$$

$$(X^je_m)^*e_n = \begin{cases} 0 & \text{if } n \not\equiv m+j \pmod{s}, \\ (X^je_m)^* & \text{if } n \equiv m+j \pmod{s}. \end{cases}$$

Moreover, for $(e_mX^j)^* \in D(e_mB)$, we obtain

$$X(e_mX^j)^* = \begin{cases} 0 & \text{if } j = 0, \\ (e_mX^{j-1})^* & \text{if } 1 \leqslant j \leqslant k-1, \end{cases}$$

$$e_n(e_mX^j)^* = \begin{cases} 0 & \text{if } n \not\equiv m+j \pmod{s}, \\ (e_mX^j)^* & \text{if } n \equiv m+j \pmod{s}. \end{cases}$$

Proof. We will show that the first equation holds. For $0 \leqslant q \leqslant k-2$, we obtain $((e_m)^*X)(X^qe_m) = (e_m)^*\left(X^{q+1}e_m\right) = 0$. Also, we have $((e_m)^*X)\left(X^{k-1}e_m\right) = (e_m)^*\left(X^ke_m\right) = (e_m)^*(0) = 0$. So we get $((e_m)^*X)(X^qe_m) = 0$ for all q $(0 \leqslant q \leqslant k-1)$, which implies $(e_m)^*X = 0$. If $1 \leqslant j \leqslant k-1$, then we have $\left((X^je_m)^*X\right)(X^{j-1}e_m) = (X^je_m)^*(X^je_m) = 1$. Moreover, for $0 \leqslant q \leqslant k-1$ with $q \neq j-1$, we have $\left((X^je_m)^*X\right)(X^qe_p) = (X^je_m)^*(X^{q+1}e_p) = 0$. Therefore we obtain $(X^je_m)^*X = (X^{j-1}e_m)^*$.

Next, we will verify that the second equation holds. First we deal with the case $n \not\equiv m+j \,(\mathrm{mod}\, s)$. Then, for $0 \leqslant p \leqslant k-1$ with $m \equiv n-p \,(\mathrm{mod}\, s)$, we have $e_m = e_{n-p}$ and $p \neq j$. So we obtain $\left((X^je_m)^*e_n\right)(X^pe_m) = (X^je_m)^*(e_nX^pe_m) = (X^je_m)^*(X^pe_{n-p}e_m) = (X^je_m)^*(X^pe_m) = 0$. Moreover, for $0 \leqslant p \leqslant k-1$ with $m \not\equiv n-p \,(\mathrm{mod}\, s)$, we have $e_m \neq e_{n-p}$. So we obtain $\left((X^je_m)^*e_n\right)(X^pe_m) = (X^je_m)^*(e_nX^pe_m) = (X^je_m)^*(X^pe_{n-p}e_m) = (X^je_m)^*(0) = 0$. Hence we get $\left((X^je_m)^*e_n\right)(X^pe_m) = 0$ for all p $(0 \leqslant p \leqslant k-1)$, that is, $(X^je_m)^*e_n = 0$. Next we deal with the case $n \equiv m + j \,(\mathrm{mod}\, s)$. Then we have $e_n = e_{m+j}$. So, it follows that $\left((X^je_m)^*e_n\right)(X^je_m) = (X^je_m)^*(e_{m+j}X^je_m) = (X^je_m)^*(X^je_m) = 1$. Furthermore, for $0 \leqslant p \leqslant k-1$

with $p \neq j$ and $p \equiv j \,(\operatorname{mod} s)$, we clearly have $e_n = e_{p+m}$. Thus we obtain $\left((X^j e_m)^* e_n\right)(X^p e_m) = (X^j e_m)^*(e_{p+m} X^p e_m) = (X^j e_m)^*(X^p e_m) = 0$. Also, for $0 \leqslant p \leqslant k-1$ with $p \not\equiv j \,(\operatorname{mod} s)$, we get $e_m \neq e_{n-p}$. So we obtain $\left((X^j e_m)^* e_n\right)(X^p e_m) = (X^j e_m)^*(e_n X^p e_m) = (X^j e_m)^*(X^p e_{n-p} e_m) = (X^j e_m)^*(0) = 0$. Therefore we have $(X^j e_m)^* e_n = (X^j e_m)^*$.

The rest of the lemma is shown in a similar way above. □

Since B is a self-injective algebra, we get $D(Be_m) \simeq e_t B$ as right B-modules for some $1 \leqslant t \leqslant s$ and $D(e_m B) \simeq Be_r$ as left B-modules for some $1 \leqslant r \leqslant s$. In fact, we have the following lemma.

Lemma 3.2. *Let m be an integer with $1 \leqslant m \leqslant s$. Then the following homomorphism of K-spaces is the isomorphism of right B-modules*:

$$\Phi : D(Be_m) \longrightarrow e_{m+k-1}B; \quad (X^j e_m)^* \longmapsto e_{m+k-1}X^{k-j-1} \quad (0 \leqslant j \leqslant k-1).$$

Also, the following homomorphism of K-spaces is the isomorphism of left B-modules:

$$\Psi : D(e_m B) \longrightarrow Be_{m-k+1}; \quad (e_m X^j)^* \longmapsto X^{k-j-1}e_{m-k+1} \quad (0 \leqslant j \leqslant k-1).$$

Proof. Clearly Φ is an isomorphism of K-spaces. We prove that Φ is a homomorphism of right B-modules. Since B is generated by $e_i \, (1 \leqslant i \leqslant s)$ and X, it suffices to verify that $\Phi\left((X^j e_m)^* X\right) = \Phi\left((X^j e_m)^*\right) X$ and $\Phi\left((e_m X)^* e_n\right) = \Phi\left((e_m X^j)^*\right) e_n$ hold for $0 \leqslant j \leqslant k-1$ and $1 \leqslant n \leqslant s$. We will show that the first equation holds. If $j = 0$, then by Lemma 3.1 the left hand side equals $\Phi(0) = 0$ and the right hand side equals $e_{m-k-1}X^{k-1}X = e_{m-k-1}X^k = 0$. If $1 \leqslant j \leqslant k-1$, then by Lemma 3.1 the left hand side equals $\Phi\left((X^{j-1}e_m)^*\right) = e_{m-k-1}X^{k-j}$ and the right hand side equals $e_{m+k-1}X^{k-j-1}X = e_{m+k-1}X^{k-j}$. Next we will show the second equation holds. If $n \not\equiv m + j \,(\operatorname{mod} s)$, then by Lemma 3.1 the left hand side equals $\Phi(0) = 0$. On the other hand, since $e_n \neq e_{m+j}$, by Lemma 3.1 the right hand side equals $\left(e_{m+k-1}X^{k-j-1}\right) e_n = X^{k-j-1}e_{m+j}e_n = 0$. If $n \equiv m + j \,(\operatorname{mod} s)$, by Lemma 3.1 the left hand side equals $\Phi\left((X^j e_m)^*\right) = e_{m+k-1}X^{k-j-1} = X^{k-j-1}e_{m+j}$. On the other hand, since $e_n = e_{m+j}$, by Lemma 3.1 the right hand side equals $\left(e_{m+k-1}X^{k-j-1}\right) e_n = X^{k-j-1}e_{m+j}e_n = X^{k-j-1}e_{m+j}$.

Similarly, it is shown by Lemma 3.1 that Ψ is an isomorphism of left B-modules. □

It is known that the set $\{e_m \otimes e_n^\circ \,|\, 1 \leqslant m, n \leqslant s\}$ is a complete set of the primitive orthogonal idempotents of B^e (see [H]). Therefore $Be_m \otimes e_n B$ ($\simeq B^e(e_m \otimes e_n^\circ)$) is an indecomposable projective left B^e-module and $e_m B \otimes Be_n$ ($\simeq (e_m \otimes e_n^\circ)\, B^e$) is an indecomposable projective right B^e-module for each $1 \leqslant m, n \leqslant s$. Since B is a basic self-injective algebra, B^e is also a basic self-injective algebra (cf. [P1]). Hence $D(e_mB \otimes Be_n) \simeq Be_t \otimes e_rB$ for some $1 \leqslant t, r \leqslant s$. In fact, we have the following lemma.

Proposition 3.3. *Let m, n be integers with $1 \leqslant m, n \leqslant s$. Then, we have the following isomorphism of left B^e-modules*:

$$\begin{aligned} &D(e_mB \otimes Be_n) \longrightarrow Be_{m-k+1} \otimes e_{n+k-1}B;\\ &(e_mX^i \otimes X^je_n)^* \longmapsto X^{k-i-1}e_{m-k+1} \otimes e_{n+k-1}X^{k-j-1} \quad (0 \leqslant i, j \leqslant k-1). \end{aligned}$$

Proof. By [M, Chapter V, Proposition 4.3], we get the isomorphism $F\colon D(e_mB) \otimes D(Be_n) \to D(e_mB \otimes Be_n)$ of K-vector spaces given by $F(f \otimes g)(x \otimes y) = f(x)g(y)$ for $f \in D(e_mB)$, $g \in D(Be_n)$, $x \in e_mB$ and $y \in Be_n$. We will show that F is an isomorphism of left B^e-modules. For $a \otimes b^\circ \in B^e$ $(a, b \in B)$, $f \in D(e_mB)$, $g \in D(Be_n)$, $x \in e_mB$ and $y \in Be_n$, we get $F\left((a \otimes b^\circ)(f \otimes g)\right)(x \otimes y) = F((af) \otimes (gb))(x \otimes y) = ((af)(x))\,((gb)(y)) = f(xa)g(by) = F(f \otimes g)(xa \otimes by) = F(f \otimes g)\left((x \otimes y)(a \otimes b^\circ)\right) = \left((a \otimes b^\circ)F(f \otimes g)\right)(x \otimes y)$. This implies that $F\left((a \otimes b^\circ)(f \otimes g)\right) = (a \otimes b^\circ)F(f \otimes g)$ holds for all $a \otimes b^\circ \in B^e$ and $f \otimes g \in D(e_mB) \otimes D(Be_n)$.

Now, it is easy to check that F is an isomorphism of K-spaces given by $F((e_mX^i)^* \otimes (X^je_n)^*) = (e_mX^i \otimes X^je_n)^*$ for each $0 \leqslant i, j \leqslant k-1$. So $F^{-1}\colon D(e_mB \otimes Be_n) \to D(e_mB) \otimes D(Be_n)$ is an isomorphism of K-spaces given by $F^{-1}\left((e_mX^i \otimes X^je_n)^*\right) = (e_mX^i)^* \otimes (X^je_n)^*$. Furthermore, by Lemma 3.2, we easily obtain the isomorphism $G\colon D(e_mB) \otimes D(Be_n) \to Be_{m-k+1} \otimes e_{n+k-1}B$ of left B^e-modules given by $G((e_mX^i)^* \otimes (X^je_n)^*) = X^{k-i-1}e_{m-k+1} \otimes e_{n+k-1}X^{k-j-1}$. Consequently, we get the isomorphism

$$\begin{aligned} GF^{-1}\colon D(e_mB \otimes Be_n) &\longrightarrow Be_{m-k+1} \otimes e_{n+k-1}B;\\ (e_mX^i \otimes X^je_n)^* &\longmapsto X^{k-i-1}e_{m-k+1} \otimes e_{n+k-1}X^{k-j-1}\\ &\qquad\qquad (0 \leqslant i, j \leqslant k-1) \end{aligned}$$

of left B^e-modules. □

§4. The modules $\tau^i_{B^e}(B)$ and $\mathcal{N}^i_{B^e}(B)$

In this section, we describe the structures of the left B^e-modules $\tau^i_{B^e}(B)$ and $\mathcal{N}^i_{B^e}(B)$ for $i \geqslant 0$, and we compute the τ_{B^e}-period and the $\mathcal{N}_{B^e}$-period of the K-algebra $B = K\Gamma/(X^k)$ $(k \geqslant 2)$.

We define the projective left B^e-modules

$$P_0 = \bigoplus_{i=1}^{s} Be_i \otimes e_iB, \qquad P_1 = \bigoplus_{i=1}^{s} Be_{i+1} \otimes e_iB.$$

Then we obtain the following exact sequence of B^e-modules ([EH, F]):

$$0 \longrightarrow {}_1B_{\beta^{-k}} \xrightarrow{\ \kappa\ } P_1 \xrightarrow{\ \phi\ } P_0 \xrightarrow{\ \pi\ } B \longrightarrow 0, \tag{4.1}$$

where left B^e-homomorphisms ϕ and κ are given by

$$\phi\,(e_{i+1} \otimes e_i) = e_{i+1}\,(X \otimes 1 - 1 \otimes X)\,e_i,$$
$$\kappa(e_i) = e_i \left(\sum_{j=0}^{k-1} X^j \otimes X^{k-j-1} \right) e_{i-k} \quad \text{for } 1 \leqslant i \leqslant s,$$

and π is the multiplication, and $P_1 \xrightarrow{\phi} P_0 \xrightarrow{\pi} B \to 0$ is a minimal projective B^e-presentation of B. We define an algebra automorphism $\beta\colon B \to B$ by $e_i \mapsto e_{i-1}$, $a_i \mapsto a_{i-1}$ $(1 \leqslant i \leqslant s)$. Here, we note that the order of β equals s.

Let n be any integer with $n \geqslant 0$. First, we give a minimal projective B^e-presentation of ${}_1B_{\beta^n}$. We define projective left B^e-modules

$$Q_0 = \bigoplus_{i=1}^{s} Be_i \otimes e_{i+n}B, \qquad Q_1 = \bigoplus_{i=1}^{s} Be_{i+1} \otimes e_{i+n}B.$$

Lemma 4.1. *We have the following exact sequence of left B^e-modules*:

$$0 \longrightarrow {}_1B_{\beta^{n-k}} \xrightarrow{\ \rho\ } Q_1 \xrightarrow{\ \psi\ } Q_0 \xrightarrow{\ \theta\ } {}_1B_{\beta^n} \longrightarrow 0, \tag{4.2}$$

where the left B^e-homomorphisms θ, ψ and ρ are given by

$$\theta(e_i \otimes e_{i+n}) = e_i, \quad \psi(e_{i+1} \otimes e_{i+n}) = e_{i+1}\,(X \otimes 1 - 1 \otimes X)\,e_{i+n}$$

and

$$\rho(e_i) = e_i \left(\sum_{l=0}^{k-1} X^l \otimes X^{k-l-1} \right) e_{i+n-k} \quad \textit{for } 1 \leqslant i \leqslant s.$$

Moreover, $Q_1 \xrightarrow{\psi} Q_0 \xrightarrow{\theta} {}_1B_{\beta^n} \to 0$ is the minimal projective B^e-presentation of ${}_1B_{\beta^n}$.

Proof. Applying the functor F_{β^n} to the exact sequence (4.1) we have the following exact sequence:

$$0 \longrightarrow {}_1B_{\beta^{n-k}} \xrightarrow{{}_1\kappa_{\beta^n}} {}_1(P_1)_{\beta^n} \xrightarrow{{}_1\phi_{\beta^n}} {}_1(P_0)_{\beta^n} \xrightarrow{{}_1\pi_{\beta^n}} {}_1B_{\beta^n} \longrightarrow 0,$$

where ${}_1(P_1)_{\beta^n} \xrightarrow{{}_1\phi_{\beta^n}} {}_1(P_0)_{\beta^n} \xrightarrow{{}_1\pi_{\beta^n}} {}_1B_{\beta^n} \to 0$ is the minimal projective B^e-presentation of ${}_1B_{\beta^n}$.

Let $g_0\colon {}_1(P_0)_{\beta^n} \to Q_0$ and $g_1\colon {}_1(P_1)_{\beta^n} \to Q_1$ be B^e-homomorphisms given by the followings respectively:

$$g_0\,(e_j \otimes e_j) = e_j \otimes e_{j+n}, \quad g_1\,(e_{j+1} \otimes e_j) = e_{j+1} \otimes e_{j+n} \quad \text{for } 1 \leqslant j \leqslant s.$$

Then it is easy to see that g_0 and g_1 are isomorphisms of left B^e-modules. Also, by setting $\theta := {}_1\pi_{\beta^n} \circ g_0^{-1}$, $\psi := g_0 \circ {}_1\phi_{\beta^n} \circ g_1^{-1}$ and $\rho := g_1 \circ {}_1\kappa_{\beta^n}$, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow {}_1B_{\beta^{n-k}} & \xrightarrow{{}_1\kappa_{\beta^n}} & {}_1(P_1)_{\beta^n} & \xrightarrow{{}_1\phi_{\beta^n}} & {}_1(P_0)_{\beta^n} & \xrightarrow{{}_1\pi_{\beta^n}} & {}_1B_{\beta^n} \longrightarrow 0 \\
\Vert & & \wr\downarrow g_1 & & \wr\downarrow g_0 & & \Vert \\
0 \longrightarrow {}_1B_{\beta^{n-k}} & \xrightarrow{\rho} & Q_1 & \xrightarrow{\psi} & Q_0 & \xrightarrow{\theta} & {}_1B_{\beta^n} \longrightarrow 0
\end{array}$$

of left B^e-modules. Furthermore, for each $j\,(1 \leqslant j \leqslant s)$ we get

$$\theta\,(e_j \otimes e_{j+n}) = {}_1\pi_{\beta^n}\,(e_j \otimes e_j) = e_j,$$

$$\begin{aligned}
\psi\,(e_{j+1} \otimes e_{j+n}) &= (g_0 \circ {}_1\phi_{\beta^n})\,(e_{j+1} \otimes e_j) \\
&= g_0\,(e_{j+1}\,(X \otimes 1 - 1 \otimes X)\,e_j) \\
&= e_{j+1}\,(X \otimes 1 - 1 \otimes X)\,e_{j+n},
\end{aligned}$$

and

$$\begin{aligned}
\rho(e_j) &= g_1\left(e_j\left(\sum_{l=0}^{k-1} X^l \otimes X^{k-l-1}\right)e_{j-k}\right) \\
&= e_j\left(\sum_{l=0}^{k-1} X^l \otimes X^{k-l-1}\right)e_{j+n-k}.
\end{aligned}$$

Hence (4.2) is exact and $Q_1 \xrightarrow{\psi} Q_0 \xrightarrow{\theta} {}_1B_{\beta^n} \to 0$ is the minimal projective B^e-presentation of ${}_1B_{\beta^n}$. So the lemma is proved. □

Now, consider the right B^e-module $(Be_m \otimes e_nB)^\vee := \mathrm{Hom}_{B^e}(Be_m \otimes e_nB, B^e)$ for $1 \leqslant m, n \leqslant s$. We identify B^e with $B \otimes B$ as left B^e-modules via the isomorphism $B^e \to B \otimes B$; $x \otimes y^\circ \mapsto x \otimes y$ of left B^e-modules. Then we easily obtain the following.

Lemma 4.2. *Let m and n be integers such that $1 \leqslant m, n \leqslant s$. Then the map $\Theta : (Be_m \otimes e_nB)^\vee \to e_mB \otimes Be_n$ given by $\Theta(u) = u(e_m \otimes e_n)$ ($u \in (Be_m \otimes e_nB)^\vee$) is an isomorphism of right B^e-modules.*

Proof. By [ARS, Chapter I, Proposition 4.9], Θ is an isomorphism of K-vector spaces. Then it is easy to see that Θ is an isomorphism of right B^e-modules. □

Next we will give a minimal projective $(B^e)^\circ$-presentation of $\mathrm{Tr}_{B^e}({}_1B_{\beta^n})$. We define the projective right B^e-modules

$$R_0 = \bigoplus_{i=1}^{s} e_iB \otimes Be_{i+n}, \qquad R_1 = \bigoplus_{i=1}^{s} e_{i+1}B \otimes Be_{i+n}.$$

Lemma 4.3. *We have the following exact sequences of right B^e-modules:*

$$0 \longrightarrow ({}_1B_{\beta^n})^\vee \xrightarrow{\ \eta\ } R_0 \xrightarrow{\ \chi\ } R_1 \longrightarrow \mathrm{Tr}_{B^e}({}_1B_{\beta^n}) \longrightarrow 0, \tag{4.3}$$

where the B^e-homomorphisms η and χ are given by

$$\eta(f) = f(1) \quad \textit{for } f \in ({}_1B_{\beta^n})^\vee,$$
$$\chi(e_j \otimes e_{j+n}) = e_{j+1}X \otimes e_{j+n} - e_j \otimes Xe_{j+n-1} \quad \textit{for } 1 \leqslant j \leqslant s.$$

Moreover, $R_0 \xrightarrow{\chi} R_1 \to \mathrm{Tr}_{B^e}({}_1B_{\beta^n}) \to 0$ is the minimal projective $(B^e)^\circ$-presentation of $\mathrm{Tr}_{B^e}({}_1B_{\beta^n})$.

Proof. Applying the duality $(\ )^\vee = \mathrm{Hom}_{B^e}(\ , B^e)$ to (4.2), we have the exact sequence

$$0 \longrightarrow ({}_1B_{\beta^n})^\vee \xrightarrow{\ \theta^\vee\ } Q_0^\vee \xrightarrow{\ \psi^\vee\ } Q_1^\vee \longrightarrow \mathrm{Tr}_{B^e}({}_1B_{\beta^n}) \longrightarrow 0$$

of right B^e-modules, where $Q_0^\vee \xrightarrow{\psi^\vee} Q_1^\vee \to \mathrm{Tr}_{B^e}({}_1B_{\beta^n}) \to 0$ is the minimal projective $(B^e)^\circ$-presentation of $\mathrm{Tr}_{B^e}({}_1B_{\beta^n})$. By Lemma 4.2, we have the isomorphisms

$$h_0 \colon Q_0^\vee \xrightarrow{\ \sim\ } \bigoplus_{i=1}^{s} (Be_i \otimes e_{i+n}B)^\vee \xrightarrow{\ \sim\ } R_0,$$
$$h_1 \colon Q_1^\vee \xrightarrow{\ \sim\ } \bigoplus_{i=1}^{s} (Be_{i+1} \otimes e_{i+n}B)^\vee \xrightarrow{\ \sim\ } R_1.$$

of right B^e-modules. Here, note that $\left(h_0^{-1}(e_i \otimes e_{i+n})\right)(e_j \otimes e_{j+n}) = e_i \otimes e_{i+n}$ if $j = i$, 0 if $j \neq i$, and $h_1(u) = \sum_{m=1}^{s} u(e_{m+1} \otimes e_{m+n})$ for $u \in Q_1^\vee$. Furthermore, these isomorphisms yield the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & ({}_1B_{\beta^n})^\vee & \xrightarrow{\ \theta^\vee\ } & Q_0^\vee & \xrightarrow{\ \psi^\vee\ } & Q_1^\vee & \longrightarrow & \mathrm{Tr}_{B^e}({}_1B_{\beta^n}) & \longrightarrow 0 \\
& \| & & \wr\downarrow h_0 & & \wr\downarrow h_1 & & \| & \\
0 \longrightarrow & ({}_1B_{\beta^n})^\vee & \xrightarrow{\ \eta\ } & R_0 & \xrightarrow{\ \chi\ } & R_1 & \longrightarrow & \mathrm{Tr}_{B^e}({}_1B_{\beta^n}) & \longrightarrow 0
\end{array}$$

of right B^e-modules, where we set $\chi := h_1 \circ \psi^\vee \circ h_0^{-1}$ and $\eta := h_0 \circ \theta^\vee$. Also, for each $f \in ({}_1B_{\beta^n})^\vee$, we obtain

$$\eta(f) = h_0(f \circ \theta) = \sum_{m=1}^{s} (f \circ \theta)(e_m \otimes e_{m+n}) = \sum_{m=1}^{s} f(e_m) = f(1)$$

and, for each $1 \leqslant j \leqslant s$, we get

$$\begin{aligned}\chi(e_j \otimes e_{j+n}) &= h_1\left(h_0^{-1}(e_j \otimes e_{j+n}) \circ \psi\right)\\ &= \sum_{m=1}^{s} \left(h_0^{-1}(e_j \otimes e_{j+n}) \circ \psi\right)(e_{m+1} \otimes e_{m+n})\\ &= \sum_{m=1}^{s} h_0^{-1}(e_j \otimes e_{j+n})\left(e_{m+1}(X \otimes 1 - 1 \otimes X)\, e_{m+n}\right)\\ &= \sum_{m=1}^{s} h_0^{-1}(e_j \otimes e_{j+n})\left(Xe_m \otimes e_{m+n} - e_{m+1} \otimes e_{m+n+1}X\right)\\ &= e_{j+1}X \otimes e_{j+n} - e_j \otimes Xe_{j+n-1}.\end{aligned}$$

So it is verified that (4.3) is exact and $R_0 \xrightarrow{\chi} R_1 \to \mathrm{Tr}_{B^e}({}_1B_{\beta^n}) \to 0$ is the minimal projective $(B^e)^\circ$-presentation of $\mathrm{Tr}_{B^e}({}_1B_{\beta^n})$. Hence, the lemma is proved. □

Next, we will give the minimal injective B^e-copresentation of $\tau_{B^e}({}_1B_{\beta^n}) := D\mathrm{Tr}_{B^e}({}_1B_{\beta^n})$. We define projective left B^e-modules

$$L_0 = \bigoplus_{i=1}^{s} Be_i \otimes e_{i+n+2(k-1)}B, \qquad L_1 = \bigoplus_{i=1}^{s} Be_{i+1} \otimes e_{i+n+2(k-1)}B.$$

Lemma 4.4. *We have the following exact sequence of left B^e-modules:*

$$0 \longrightarrow \tau_{B^e}({}_1B_{\beta^n}) \longrightarrow L_1 \xrightarrow{\sigma} L_0 \longrightarrow \mathcal{N}_{B^e}({}_1B_{\beta^n}) \longrightarrow 0, \tag{4.4}$$

where the left B^e-homomorphism σ is given by

$$\sigma(e_{i+1} \otimes e_{i+n+2(k-1)}) = e_{i+1}(X \otimes 1 - 1 \otimes X)\, e_{i+n+2(k-1)} \quad \text{for } 1 \leqslant i \leqslant s.$$

Furthermore, $0 \to \tau_{B^e}({}_1B_{\beta^n}) \to L_1 \xrightarrow{\sigma} L_0$ is the minimal injective B^e-copresentation of $\tau_{B^e}({}_1B_{\beta^n})$.

Proof. Applying the duality $D = \mathrm{Hom}_K(\ , K)$ to the exact sequence (4.3), we have the exact sequence

$$0 \longrightarrow \tau_{B^e}({}_1B_{\beta^n}) \longrightarrow D(R_1) \xrightarrow{D(\chi)} D(R_0) \xrightarrow{D(\eta)} \mathcal{N}_{B^e}({}_1B_{\beta^n}) \longrightarrow 0$$

of left B^e-modules, where $0 \to \tau_{B^e}({}_1B_{\beta^n}) \to D(R_1) \overset{D(\chi)}{\to} D(R_0)$ is the minimal injective B^e-copresentation of $\tau_{B^e}({}_1B_{\beta^n})$. Moreover, by Proposition 3.3, we obtain the isomorphisms

$$g_0\colon D(R_0) \xrightarrow{\sim} \bigoplus_{i=1}^{s} D\left(e_iB \otimes Be_{i+n}\right) \xrightarrow{\sim} L_0,$$
$$g_1\colon D(R_1) \xrightarrow{\sim} \bigoplus_{i=1}^{s} D\left(e_{i+1}B \otimes Be_{i+n}\right) \xrightarrow{\sim} L_1$$

of left B^e-modules. Here, we note that

$$g_1^{-1}\left(e_{i+1} \otimes e_{i+n+2(k-1)}\right) = \left(e_{i+k}X^{k-1} \otimes X^{k-1}e_{i+n+k-1}\right)^*$$

holds for $1 \leqslant i \leqslant s$. Using these isomorphisms, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \tau_{B^e}({}_1B_{\beta^n}) & \longrightarrow & D(R_1) & \xrightarrow{D(\chi)} & D(R_0) & \xrightarrow{D(\eta)} & \mathcal{N}_{B^e}({}_1B_{\beta^n}) & \longrightarrow 0 \\
 & \| & & \wr\downarrow g_1 & & \wr\downarrow g_0 & & \| & \\
0 \longrightarrow & \tau_{B^e}({}_1B_{\beta^n}) & \longrightarrow & L_1 & \xrightarrow{\sigma} & L_0 & \xrightarrow{\rho} & \mathcal{N}_{B^e}({}_1B_{\beta^n}) & \longrightarrow 0
\end{array}$$

of left B^e-modules, where we set $\sigma := g_0 \circ D(\chi) \circ g_1^{-1}$ and $\rho := D(\eta) \circ g_0^{-1}$.

Since for $1 \leqslant i, l \leqslant s$ and $0 \leqslant p, q \leqslant k-1$ we get

$$\begin{aligned}
&\left(\left(D(\chi) \circ g_1^{-1}\right)\left(e_{i+1} \otimes e_{i+n+2(k-1)}\right)\right)\left(e_lX^p \otimes X^qe_{l+n}\right) \\
&= \left(D(\chi) \circ \left(e_{i+k}X^{k-1} \otimes X^{k-1}e_{i+n+k-1}\right)^*\right)\left(e_lX^p \otimes X^qe_{l+n}\right) \\
&= \left(\left(e_{i+k}X^{k-1} \otimes X^{k-1}e_{i+n+k-1}\right)^* \circ \chi\right)\left(e_lX^p \otimes X^qe_{l+n}\right) \\
&= \left(e_{i+k}X^{k-1} \otimes X^{k-1}e_{i+n+k-1}\right)^*\left(e_{l+1}X^{p+1} \otimes X^qe_{l+n} - e_lX^p \otimes X^{q+1}e_{l+n-1}\right) \\
&= \begin{cases} 1 & \text{if } p = k-2,\ q = k-1 \text{ and } l \equiv i+k-1 \pmod{s}, \\ -1 & \text{if } p = k-1,\ q = k-2 \text{ and } l \equiv i+k \pmod{s}, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

it follows that

$$\begin{aligned}
&\left(D(\chi) \circ g_1^{-1}\right)\left(e_{i+1} \otimes e_{i+n+2(k-1)}\right) \\
&= \left(e_{i+k-1}X^{k-2} \otimes X^{k-1}e_{i+n+k-1}\right)^* - \left(e_{i+k}X^{k-1} \otimes X^{k-2}e_{i+n+k}\right)^*.
\end{aligned}$$

Therefore, by Proposition 3.3, for $1 \leqslant i \leqslant s$ we have

$$\begin{aligned}
&\sigma\left(e_{i+1} \otimes e_{i+n+2(k-1)}\right)\\
&= g_0\left(\left(e_{i+k-1}X^{k-2} \otimes X^{k-1}e_{i+n+k-1}\right)^* - \left(e_{i+k}X^{k-1} \otimes X^{k-2}e_{i+n+k}\right)^*\right)\\
&= Xe_i \otimes e_{i+n+2(k-1)} - e_{i+1} \otimes e_{i+n+2k-1}X\\
&= e_{i+1}\left(X \otimes 1 - 1 \otimes X\right)e_{i+n+2(k-1)}.
\end{aligned}$$

Hence (4.4) is an exact sequence of left B^e-modules and $0 \to \tau_{B^e}({}_1B_{\beta^n}) \to L_1 \xrightarrow{\sigma} L_0$ is the minimal injective B^e-copresentation of $\tau_{B^e}({}_1B_{\beta^n})$. Therefore, the lemma is proved. □

The following lemma is easily shown by Lemmas 4.1, 4.4.

Lemma 4.5. *Let n be any integer with $n \geqslant 0$. Then, we obtain the following exact sequence of left B^e-modules:*

$$0 \longrightarrow {}_1B_{\beta^{n+k-2}} \xrightarrow{\ \iota\ } L_1 \xrightarrow{\ \sigma\ } L_0 \longrightarrow {}_1B_{\beta^{n+2(k-1)}} \longrightarrow 0,$$

where ι is given by

$$\iota(e_i) = e_i\left(\sum_{j=0}^{k-1} X^j \otimes X^{k-j-1}\right)e_{i+n+k-2} \quad \textit{for } 1 \leqslant i \leqslant s.$$

Furthermore, $0 \to {}_1B_{\beta^{n+k-2}} \xrightarrow{\iota} L_1 \xrightarrow{\sigma} L_0$ is the minimal injective B^e-copresentation of ${}_1B_{\beta^{n+k-2}}$. Hence we obtain the isomorphisms of left B^e-modules

$$\tau_{B^e}({}_1B_{\beta^n}) \simeq {}_1B_{\beta^{n+k-2}} \quad \textit{and} \quad \mathcal{N}_{B^e}({}_1B_{\beta^n}) \simeq {}_1B_{\beta^{n+2(k-1)}}.$$

Now, we easily have the following structures of $\tau^i_{B^e}(B)$ and $\mathcal{N}^i_{B^e}(B)$ for $i \geqslant 0$ by induction on n.

Theorem. *We have the isomorphisms of left B^e-modules*

$$\tau^i_{B^e}(B) \simeq {}_1B_{\beta^{i(k-2)}} \quad \textit{and} \quad \mathcal{N}^i_{B^e}(B) \simeq {}_1B_{\beta^{2i(k-1)}}$$

for all $i \geqslant 0$.

Corollary 4.6. *The left B^e-module B is τ_{B^e}-periodic and $\mathcal{N}_{B^e}$-periodic, and the τ_{B^e}-period is*

$$\begin{cases} 1 & \textit{if } k = 2,\\ \dfrac{\operatorname{lcm}(k-2, s)}{k-2} & \textit{if } k \geqslant 3 \end{cases}$$

and the $\mathcal{N}_{B^e}$-period is

$$\frac{\operatorname{lcm}\big(2(k-1), s\big)}{2(k-1)}.$$

Proof. If $k = 2$, then obviously the τ_{B^e}-period of B is 1. Also, if $k \geqslant 3$, then since the order of β is s, the order of β^{k-2} equals $s/\gcd(k-2, s) = \mathrm{lcm}(k-2, s)/(k-2)$. Similarly the order of $\beta^{2(k-1)}$ equals $\mathrm{lcm}(2(k-1), s)/(2(k-1))$. This completes the proof. □

Remark. The τ_{B^e}-period of B is given in [P2, Theorem 2].

Corollary 4.7. *Let s and k be integers with $s \geqslant 1$ and $k \geqslant 2$. Then the τ_{B^e}-period of B is* 1 *if and only if $k \equiv 2 \pmod{s}$, and the $\mathcal{N}_{B^e}$-period of B is* 1 *if and only if $2(k-1) \equiv 0 \pmod{s}$.*

Appendix

In this Appendix, we will give an alternative proof of Theorem in Section 4. Throughout this Appendix, we keep the notation in Sections 3 and 4.

First we will investigate the Nakayama automorphism of the enveloping algebra $B^e := B \otimes B^\circ$ of $B = K\Gamma/(X^k)$ $(k \geqslant 2)$. We identify B^e with $B \otimes B$ as left B^e-modules via the isomorphism $B^e \to B \otimes B$; $x \otimes y^\circ \mapsto x \otimes y$ of left B^e-modules. Define the algebra automorphism $\nu : B^e \to B^e$ by $\beta^{1-k} \otimes \beta^{k-1} : B^e \to B^e$.

For any integer m and n with $1 \leqslant m, n \leqslant s$, by Proposition 3.3, we have the isomorphism

$$\begin{aligned} Be_m \otimes e_n B &\longrightarrow D(e_{m+k-1}B \otimes Be_{n-k+1}); \\ X^i e_m \otimes e_n X^j &\longmapsto (e_{m+k-1}X^{k-i-1} \otimes X^{k-j-1}e_{n-k+1})^* \\ &\qquad (0 \leqslant i, j \leqslant k-1) \end{aligned}$$

of left B^e-modules. By means of these isomorphisms, we obtain the isomorphisms $\Psi : B^e \to D(B^e)$ of left B^e-modules. Then we have the following:

Lemma A.1. The map $\Psi : B^e \to {}_1D(B^e)_\nu$ is the isomorphism of B^e-bimodules. So ν is the Nakayama automorphism of B^e.

Proof. It suffices to show that $\Psi \colon B^e \to {}_1D(B^e)_\nu$ is the isomorphism of right B^e-modules. Since $\{e_p \otimes e_q^\circ, Xe_p \otimes e_q^\circ, e_p \otimes (e_q X)^\circ | 1 \leqslant p, q \leqslant s\}$ generates B^e as an algebra and Ψ is the isomorphism of left B^e-modules, it suffices to check that the following equations hold: $\Psi(e_p \otimes e_q) = \Psi(e_p \otimes e_q)\nu(e_p \otimes e_q^\circ)$, $\Psi(Xe_p \otimes e_q) = \Psi(e_{p+1} \otimes e_q)\nu(Xe_p \otimes e_q^\circ)$, $\Psi(e_p \otimes e_q X) = \Psi(e_p \otimes e_{q-1})\nu(e_p \otimes (e_q X)^\circ)$ for $p, q\ (1 \leqslant p, q \leqslant s)$.

We prove that the first equation holds. Take any $e_m X^r \otimes X^t e_n \in B^e\ (1 \leqslant m, n \leqslant s; 0 \leqslant r, t \leqslant k-1)$. Note that $\Psi(e_p \otimes e_q) = (e_{p+k-1}X^{k-1} \otimes X^{k-1}e_{q-k+1})^*$

holds. By direct calculation, we have the equation

$$\big(\Psi(e_p \otimes e_q)\nu(e_p \otimes e_q^\circ)\big)\,(e_m X^r \otimes X^t e_n) = \begin{cases} 1 & \text{if } m \equiv p+k-1 \,(\mathrm{mod}\, s),\ n \equiv q-k+1 \,(\mathrm{mod}\, s) \text{ and } r=t=k-1, \\ 0 & \text{otherwise.} \end{cases}$$

So we get $\Psi(e_p \otimes e_q)\nu(e_p \otimes e_q^\circ) = (e_{p+k-1}X^{k-1} \otimes X^{k-1}e_{q-k+1})^*$. This equals $\Psi(e_p \otimes e_q)$. So the desired equation is proved.

Next we prove the second equation holds. Note that $\Psi(e_{p+1} \otimes e_q) = (e_{p+k}X^{k-1} \otimes X^{k-1}e_{q-k+1})^*$ holds. Take any $e_m X^r \otimes X^t e_n \in B^e\,(1 \leqslant m, n \leqslant s; 0 \leqslant r, t \leqslant k-1)$. Then, by direct calculation, we have

$$\big(\Psi(e_{p+1} \otimes e_q)\nu(Xe_p \otimes e_q^\circ)\big)\,(e_m X^r \otimes X^t e_n) = \begin{cases} 1 & \text{if } m \equiv p+k-1 \,(\mathrm{mod}\, s),\ n \equiv q-k+1 \,(\mathrm{mod}\, s), \\ & r = k-2 \text{ and } t = k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have $\Psi(e_{p+1} \otimes e_q)\nu(Xe_p \otimes e_q^\circ) = (e_{p+k}X^{k-2} \otimes X^{k-1}e_{q-k+1})^*$. Clearly this equals $\Psi(Xe_p \otimes e_q)$. So the desired equation is proved.

Similarly, it is shown that the third equation holds. So we get the isomorphism $\Psi : B^e \to {}_1D(B^e)_\nu$ of left B^e-modules. Hence, by [Y, Theorem 2.4.1], ν is the Nakayama automorphism of B^e. □

There exists the isomorphism $\gamma = \{\gamma_X | X \in \mathrm{mod}\,(B^e)\}$ of the functors between $D(B^e) \otimes_{B^e} -$ and $\mathcal{N}_{B^e}$, where $\gamma_X : D(B^e) \otimes_{B^e} X \to \mathcal{N}_{B^e}(X)$ is given by $\gamma_X(f \otimes x)(\phi) = (f \circ \phi)(x)$ for $f \in D(B^e)$, $x \in X$ and $\phi \in X^\vee$. Moreover by Lemma A.1 the functor $D(B^e) \otimes_{B^e} -$ is isomorphic to the functor ${}_\nu(\ )$, where the functor ${}_\nu(\ ) \colon \mathrm{mod}\,(B^e) \to \mathrm{mod}\,(B^e)$ is given as follows: For any $M \in \mathrm{mod}\,(B^e)$, ${}_\nu M$ has the underlying K-vector space M, and the left operation $*$ of B^e is given by $x * m = \nu(x)m$ for $x \in B^e$ and $m \in {}_\nu M$. And, for any M, $N \in \mathrm{mod}\,(B^e)$ and any $f \in \mathrm{Hom}_{B^e}(M, N)$, the left B^e-homomorphism ${}_\nu f \colon {}_\nu M \to {}_\nu N$ is given by ${}_\nu f(m) = f(m)$ for $m \in {}_\nu M$. Hence $\mathcal{N}_{B^e}$ is isomorphic to ${}_\nu(\ )$ (see [G, Section 2.1], [Y, Section 2.4]). Then we have the following:

Lemma A.2. Let n be any integer. Then we have an isomorphism ${}_\nu({}_1B_{\beta^n}) \simeq {}_1B_{\beta^{n+2(k-1)}}$ of left B^e-modules. Hence $\mathcal{N}_{B^e}({}_1B_{\beta^n}) \simeq {}_1B_{\beta^{n+2(k-1)}}$ as left B^e-modules.

Proof. Let $\xi : {}_\nu({}_1B_{\beta^n}) \to {}_1B_{\beta^{n+2(k-1)}}$ be the map given by $\xi(x) = \beta^{k-1}(x)$ for $x \in {}_\nu({}_1B_{\beta^n})$. Then it is easy to check that ξ is an isomorphism of left B^e-modules. □

It is shown in [EH] that $\Omega^{2i}_{B^e}(B) \simeq {}_1B_{\beta^{-ik}}$ as left B^e-modules for each $i \geqslant 0$. From this fact and Lemma A.2, we have an alternative proof of Theorem:

Alternative proof of Theorem. By Lemma A.2, we easily obtain the isomorphism $\mathcal{N}^i_{B^e}(B) \simeq {}_1B_{\beta^{2i(k-1)}}$ of left B^e-modules for each $i \geqslant 0$. Furthermore, we get the isomorphism $\tau^i_{B^e}(B) \simeq (\mathcal{N}_{B^e}\Omega^2_{B^e})^i(B) \simeq \mathcal{N}^i_{B^e}\Omega^{2i}_{B^e}(B) \simeq \mathcal{N}^i_{B^e}({}_1B_{\beta^{-ik}}) \simeq {}_1B_{\beta^{i(k-2)}}$ of left B^e-modules. □

Acknowledgment

I would like to thank my supervisor, Professor Katsunori Sanada, for many valuable discussions and valuable comments. Also I would like to thank the referee for the valuable suggestions and comments.

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