SUT Journal of Mathematics
Vol. 41, No. 2 (2005), 97–116

The effect of non-normality on the distribution of sample fourth order cumulant under elliptical populations

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(Received April 11, 2005; Revised July 23, 2005)

Abstract. In this paper, we consider the effect of non-normality on the distribution of the sample fourth order cumulant in elliptical distributions. Asymptotic properties of the distribution of the sample fourth order cumulant is presented under non-normality, especially, the class of elliptical populations in detail. Asymptotic expansion formulas for the distributions are derived by a perturbation method. Finally, the numerical results are obtained.

AMS 2000 *Mathematics Subject Classification.* 62H10 (62E20, 62F12)

Key words and phrases. Asymptotic expansion, bias correction, cumulant, elliptical distribution, kurtosis parameter, moment parameter, non-normality

§1. Introduction

The distribution of the sample fourth order cumulant is studied when the parent population is elliptical. In the case of normal populations, the related discussions have been given by Mardia ([Ma70], [Ma74]) and Srivastava [S84]. Under the non-normal populations, some works have already been done (see Berkane and Bentler [BB90], Henze [H94], Seo and Toyama [ST96], Maruyama and Seo [MS03] and Maruyama [M05]). However, little work has been done regarding the effect of non-normality on the distribution of the sample fourth order cumulant itself. One purpose of the present paper is to investigate the effect of non-normality on the sample fourth order cumulant under elliptical populations. The other is to suggest the more accurate estimation. For our purposes, asymptotic expansion for the moments of the sample fourth order cumulant is derived up to the higher order.

Let a p-variate random vector $\boldsymbol{X}$ be distributed as a p-variate elliptical distribution with parameters $\boldsymbol{\mu}$ and Λ, i.e., $E_p(\boldsymbol{\mu}, \Lambda)$, where Λ is some positive definite symmetric matrix. If the probability density function exists, it has the form

$$f(\boldsymbol{x}) = c_p|\Lambda|^{-\frac{1}{2}}g\big({}^t(\boldsymbol{x}-\boldsymbol{\mu})\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\big)$$

for some non-negative function g, where c_p is the normalizing constant and ${}^t\boldsymbol{x}$ denotes a transposition of a vector $\boldsymbol{x}$. The characteristic function is $\phi(\boldsymbol{\theta}) = \exp[i\,{}^t\boldsymbol{\theta}\boldsymbol{\mu}]\psi({}^t\boldsymbol{\theta}\Lambda\boldsymbol{\theta})$ for some function ψ, where $i = \sqrt{-1}$. Note that $E(\boldsymbol{X}) = \boldsymbol{\mu}$ and $Cov(\boldsymbol{X}) = -2\psi'(0)\Lambda =: \Sigma$, respectively. For example, the multivariate normal, the multivariate t and the contaminated normal distributions belong to the class of elliptical distributions which is referred to e.g., Fang, Kotz and Ng [FKN90] and Anderson [A03]. We also define the kurtosis parameter by $\kappa := \psi''(0)/(\psi'(0))^2 - 1$, and in general let $\mathcal{K}_{(m)} := \psi^{(m)}(0)/(\psi'(0))^m - 1$ which is called the $2m$-th order moment parameter. In elliptical distributions, the fourth order cumulant is essentially equal to the kurtosis parameter from the relation between the moments and cumulants (see Stuart and Ord [SO94]). First, in Section 2, we study asymptotic properties of the distribution of the sample fourth order cumulant under elliptical populations. In Seo and Toyama [ST96] and Maruyama [M05], asymptotic expansions of the consistent estimators of the kurtosis parameter and the general moment parameter were derived up to the order n^{-1} as the size n of sample tends to infinity using the joint probability density function (j.p.d.f.) of the sample mean and the sample covariance matrix (see Wakaki [W94] and Iwashita [I97]). In this paper, we make use of the j.p.d.f. obtained by Wakaki [W97] in order to get an extension of their results, that is to say up to the order n^{-2}. Asymptotic expansion formulas for the distributions are derived by a perturbation method. Moreover we suggest the modified estimator of κ with the bias correction, which is the extension of the result discussed by Seo and Toyama [ST96]. Finally, simulation results are presented and the effect of non-normality on the distribution of the sample fourth order cumulant is discussed in Section 3.

§2. The main results

Suppose that $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ are independent and identically distributed random vectors according to $E_p(\boldsymbol{\mu}, \Lambda)$. An extension of the multivariate coefficient of kurtosis in the sense of Mardia [Ma70] has been discussed by Anderson [A03] under the elliptical populations. Note that $\beta_{2,p} := E[\{{}^t(\boldsymbol{X}-\boldsymbol{\mu})\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\}^2] = p(p+2)(\kappa+1)$. Then, we have a consistent estimator of the kurtosis parameter which is essentially equal to the fourth order cumulant given by

$$\hat{\kappa} = \frac{1}{p(p+2)}b_{2,p} - 1, \tag{2.1}$$

where

$$b_{2,p} := \frac{1}{n}\sum_{i=1}^{n}\left\{{}^{t}(\boldsymbol{X}_i - \overline{\boldsymbol{X}})U^{-1}(\boldsymbol{X}_i - \overline{\boldsymbol{X}})\right\}^2.$$

Also $\overline{\boldsymbol{X}}$ is the sample mean and U is the unbiased sample covariance matrix. Since $b_{2,p}$ is not affected by nonsingular affine transformations of the data $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$, we can assume without loss of generality that $\boldsymbol{\mu} = \boldsymbol{0}$ and $\Sigma = I_p$.

To start with, we consider the cumulants of $Y := \sqrt{n}(\hat{\kappa} - \kappa)$ up to the third order, which take the following forms:

$$\mathcal{K}_1(Y) = \frac{a_1}{\sqrt{n}} + O(n^{-\frac{3}{2}}), \quad \mathcal{K}_2(Y) = a_2^2 + O(n^{-1}), \quad \mathcal{K}_3(Y) = \frac{6a_3}{\sqrt{n}} + O(n^{-\frac{3}{2}}),$$

where a_1 and a_2 are obtained by Seo and Toyama [ST96]. Further, from a general theory of asymptotic expansions (see [SHF85]), we can expand the distribution of the sample fourth order cumulant as

$$\text{(2.2)}\quad P\left(\frac{\sqrt{n}(\hat{\kappa} - \kappa)}{a_2} \le y\right) = \Phi(y) - \frac{1}{\sqrt{n}}\left(\frac{a_1}{a_2}\Phi'(y) + \frac{a_3}{a_2^3}\Phi^{(3)}(y)\right) + O(n^{-1}),$$

where $\Phi(y)$ is the cumulative distribution function of the standard normal distribution $N(0,1)$ and $\Phi^{(k)}(y)$ is the k-th derivative of $\Phi(y)$. Therefore we have to calculate the third order cumulant $\mathcal{K}_3(Y)$. Note that

$$\begin{aligned}
\mathcal{K}_3(Y) &:= E\left[\big(Y - E(Y)\big)^3\right] = n\sqrt{n}\,E\left[\big(\hat{\kappa} - E(\hat{\kappa})\big)^3\right] \\
&= \frac{n\sqrt{n}}{p^3(p+2)^3}\,E\left[\big(b_{2,p} - E(b_{2,p})\big)^3\right] \\
&= \frac{n\sqrt{n}}{p^3(p+2)^3}\left\{E(b_{2,p}^3) - 3E(b_{2,p}^2)E(b_{2,p}) + 2\big(E(b_{2,p})\big)^3\right\}.
\end{aligned}$$

2.1. Asymptotic expansions of $b_{2,p}$, $b_{2,p}^2$ and $b_{2,p}^3$

Let $T_i := {}^{t}(\boldsymbol{X}_i - \overline{\boldsymbol{X}})U^{-1}(\boldsymbol{X}_i - \overline{\boldsymbol{X}})$, then note that $E(b_{2,p}) = E(T_i^2)$. In order to avoid the dependence of $\boldsymbol{X}_i$, $\overline{\boldsymbol{X}}$ and U, we define

$$\begin{aligned}
\overline{\boldsymbol{X}}_{(i)} &:= \frac{1}{n-1}\sum_{k\neq i}^{n}\boldsymbol{X}_k, \\
U_{(i)} &:= \frac{1}{n-2}\sum_{k\neq i}^{n}(\boldsymbol{X}_k - \overline{\boldsymbol{X}}_{(i)})\,{}^{t}(\boldsymbol{X}_k - \overline{\boldsymbol{X}}_{(i)}).
\end{aligned}$$

Then we can expand T_i^2 as

$$
\begin{aligned}
T_i^2 &= ({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2 + \frac{1}{\sqrt{n}} 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{Q}_1 \\
&\quad + \frac{1}{n}\left\{\mathbb{Q}_1^2 + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{Q}_2 - 4({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2 - 2({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^3\right\} \\
&\quad + \frac{1}{n\sqrt{n}}\left\{2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{Q}_3 - 8\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{Q}_1 - 6({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2\mathbb{Q}_1 + 2\mathbb{Q}_1\mathbb{Q}_2\right\} \\
&\quad + \frac{1}{n^2}\left\{\mathbb{Q}_2^2 - 8\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{Q}_2 - 6({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2\mathbb{Q}_2 - 6\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{Q}_1^2 + 2\mathbb{Q}_1\mathbb{Q}_3 - 4\mathbb{Q}_1^2\right. \\
&\qquad \left. + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{Q}_4 + 6({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2 + 8({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^3 + 3({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^4\right\} \\
&\quad + O(n^{-\frac{5}{2}}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{Q}_1 &= -2\,{}^t\boldsymbol{X}_i\boldsymbol{y}_1 - {}^t\boldsymbol{X}_iM_1\boldsymbol{X}_i, \\
\mathbb{Q}_2 &= {}^t\boldsymbol{y}_1\boldsymbol{y}_1 + 2\,{}^t\boldsymbol{X}_iM_1\boldsymbol{y}_1 + {}^t\boldsymbol{X}_iM_1^2\boldsymbol{X}_i + {}^t\boldsymbol{X}_i\boldsymbol{X}_i, \\
\mathbb{Q}_3 &= -{}^t\boldsymbol{X}_iM_1^3\boldsymbol{X}_i - 2\,{}^t\boldsymbol{X}_iM_1^2\boldsymbol{y}_1 - {}^t\boldsymbol{y}_1M_1\boldsymbol{y}_1 - 3\,{}^t\boldsymbol{X}_i\boldsymbol{y}_1 - \frac{3}{2}{}^t\boldsymbol{X}_iM_1\boldsymbol{X}_i, \\
\mathbb{Q}_4 &= {}^t\boldsymbol{X}_iM_1^4\boldsymbol{X}_i + 2\,{}^t\boldsymbol{X}_iM_1^3\boldsymbol{y}_1 + {}^t\boldsymbol{y}_1M_1^2\boldsymbol{y}_1 + 2\,{}^t\boldsymbol{y}_1\boldsymbol{y}_1 + 4\,{}^t\boldsymbol{X}_iM_1\boldsymbol{y}_1 + 2\,{}^t\boldsymbol{X}_iM_1^2\boldsymbol{X}_i \\
&\quad + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i,
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{y}_1 &:= \sqrt{n-1}\,\overline{\boldsymbol{X}}_{(i)}, \\
M_1 &:= \sqrt{n-1}(U_{(i)} - I_p).
\end{aligned}
$$

By using the asymptotic expanded j.p.d.f. of $\boldsymbol{y}_1$ and M_1 (see Wakaki [W97]), we can calculate the expectation for the expansion of T_i^2. Then we obtain

$$
E(b_{2,p}) = p(p+2)\left(\mathbb{A}_0 + \frac{1}{n}\mathbb{A}_1 + \frac{1}{n^2}\mathbb{A}_2\right) + O(n^{-3}),
$$

where

$$
\begin{aligned}
\text{(2.3)} \qquad \mathbb{A}_0 &= \kappa + 1, \\
\mathbb{A}_1 &= -2(p+4)(\mathcal{K}_{(3)}+1) + (2p+7)(\kappa+1)^2 - 5(\kappa+1) + 2, \\
\mathbb{A}_2 &= 3(p+4)(p+6)(\mathcal{K}_{(4)}+1) - 8(p+4)(p+5)(\kappa+1)(\mathcal{K}_{(3)}+1) \\
&\quad + 14(p+4)(\mathcal{K}_{(3)}+1) + (p+4)(5p+22)(\kappa+1)^3 \\
&\quad - 7(2p+7)(\kappa+1)^2 - (2p+3)(\kappa+1) + 2p+3.
\end{aligned}
$$

Next we consider the asymptotic expansion of $b_{2,p}^2$. Since $b_{2,p} = (1/n)\sum_{i=1}^n T_i^2$, we can write

$$E(b_{2,p}^2) = \frac{1}{n} E(T_i^4) + \left(1 - \frac{1}{n}\right) E(T_i^2 T_j^2).$$

In order to avoid the dependence of $\boldsymbol{X}_i$, $\boldsymbol{X}_j$, $\overline{\boldsymbol{X}}$ and U, we define

$$\overline{\boldsymbol{X}}_{(ij)} := \frac{1}{n-2} \sum_{k \neq i,j}^{n} \boldsymbol{X}_k,$$
$$U_{(ij)} := \frac{1}{n-3} \sum_{k \neq i,j}^{n} (\boldsymbol{X}_k - \overline{\boldsymbol{X}}_{(ij)})\, {}^t(\boldsymbol{X}_k - \overline{\boldsymbol{X}}_{(ij)}).$$

Further, let $\boldsymbol{y}_2 := \sqrt{n-2}\,\overline{\boldsymbol{X}}_{(ij)}$ and $M_2 := \sqrt{n-2}(U_{(ij)} - I_p)$. Then we can expand $T_i^2 T_j^2$ as

$$\begin{aligned} T_i^2 T_j^2 &= \mathbb{B}_1^2 + \frac{1}{\sqrt{n}}\mathbb{C}_1 + \frac{1}{n}(\mathbb{C}_2 - \mathbb{B}_1^2\mathbb{B}_2) + \frac{1}{n\sqrt{n}}\left\{\mathbb{C}_3 - 2\mathbb{B}_1^2(\mathbb{F}_1 + \mathbb{F}_5) - \mathbb{C}_1\mathbb{B}_2\right\} \\ &\quad + \frac{1}{n^2}\Big\{\mathbb{B}_1^2\big\{8\mathbb{B}_2 - 2(\mathbb{B}_1 + \mathbb{F}_2 + \mathbb{F}_6) - 36 + 3({}^t\boldsymbol{X}_i\boldsymbol{X}_i + {}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\big\} \\ &\qquad + \mathbb{C}_4 - \mathbb{C}_2\mathbb{B}_2 - 2(\mathbb{F}_1 + \mathbb{F}_5)\Big\} + O(n^{-\frac{5}{2}}), \end{aligned}$$

where

$$\begin{aligned} \mathbb{B}_1 &= {}^t\boldsymbol{X}_i\boldsymbol{X}_i\, {}^t\boldsymbol{X}_j\boldsymbol{X}_j, \\ \mathbb{B}_2 &= 2({}^t\boldsymbol{X}_i\boldsymbol{X}_i + {}^t\boldsymbol{X}_j\boldsymbol{X}_j + 4), \\ \mathbb{C}_1 &= 2\mathbb{B}_1({}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_1 + {}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_2), \\ \mathbb{C}_2 &= (\mathbb{F}_1^2 + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_2)({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2 + (\mathbb{F}_5^2 + 2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_6)({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2 + 4\mathbb{B}_1\mathbb{F}_1\mathbb{F}_5, \\ \mathbb{C}_3 &= 2(\mathbb{F}_1\mathbb{F}_2 + {}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_3)({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2 + 2(\mathbb{F}_5\mathbb{F}_6 + {}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_7)({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2 \\ &\quad + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_1(\mathbb{F}_5^2 + 2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_6) + 2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_5(\mathbb{F}_1^2 + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_2), \\ \mathbb{C}_4 &= (\mathbb{F}_2^2 + 2\mathbb{F}_1\mathbb{F}_3 + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_4)({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2 \\ &\quad + (\mathbb{F}_6^2 + 2\mathbb{F}_5\mathbb{F}_7 + 2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_8)({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2 + 4\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_1(\mathbb{F}_5\mathbb{F}_6 + {}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_7) \\ &\quad + 4\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_5(\mathbb{F}_1\mathbb{F}_2 + {}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_3) + (\mathbb{F}_1^2 + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{F}_2)(\mathbb{F}_5^2 + 2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{F}_6), \\ \mathbb{F}_1 &= -2\,{}^t\boldsymbol{X}_i\boldsymbol{y}_2 - {}^t\boldsymbol{X}_i M_2 \boldsymbol{X}_i, \\ \mathbb{F}_2 &= \mathbb{G}_{2(i,i)} - {}^t\boldsymbol{X}_i\boldsymbol{X}_i - ({}^t\boldsymbol{X}_i\boldsymbol{X}_j)^2 - 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_j, \\ \mathbb{F}_3 &= \mathbb{G}_{3(i,i)} + \mathbb{F}_1 - 2\mathbb{G}_{1(i,j)}({}^t\boldsymbol{X}_i\boldsymbol{X}_j + 1), \\ \mathbb{F}_4 &= \mathbb{G}_{4(i,i)} + \mathbb{G}_{2(i,i)} - \mathbb{G}_{1(i,j)}^2 - 2\mathbb{G}_{2(i,j)}({}^t\boldsymbol{X}_i\boldsymbol{X}_j + 1) - 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i + {}^t\boldsymbol{X}_j\boldsymbol{X}_j \\ &\quad + ({}^t\boldsymbol{X}_i\boldsymbol{X}_j + 2)({}^t\boldsymbol{X}_j\boldsymbol{X}_j - 2), \end{aligned}$$

$$
\begin{aligned}
\mathbb{G}_{1(i,j)} &= -{}^t\boldsymbol{X}_i M_2 \boldsymbol{X}_j - {}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)\boldsymbol{y}_2,\\
\mathbb{G}_{2(i,j)} &= {}^t\boldsymbol{X}_i \boldsymbol{X}_j + {}^t(\boldsymbol{X}_i + \boldsymbol{X}_j) M_2 \boldsymbol{y}_2 + {}^t\boldsymbol{X}_i M_2^2 \boldsymbol{X}_j + {}^t\boldsymbol{y}_2 \boldsymbol{y}_2,\\
\mathbb{G}_{3(i,j)} &= -2\,{}^t\boldsymbol{X}_i M_2 \boldsymbol{X}_j - {}^t\boldsymbol{X}_i M_2^3 \boldsymbol{X}_j - {}^t(\boldsymbol{X}_i + \boldsymbol{X}_j) M_2^2 \boldsymbol{y}_2 - 2\,{}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)\boldsymbol{y}_2\\
&\quad - {}^t\boldsymbol{y}_2 M_2 \boldsymbol{y}_2,\\
\mathbb{G}_{4(i,j)} &= 3\,{}^t\boldsymbol{X}_i M_2^2 \boldsymbol{X}_j + {}^t\boldsymbol{X}_i M_2^4 \boldsymbol{X}_j + {}^t\boldsymbol{y}_2 M_2^2 \boldsymbol{y}_2 + 3\,{}^t\boldsymbol{y}_2\boldsymbol{y}_2 + 3\,{}^t(\boldsymbol{X}_i + \boldsymbol{X}_j) M_2 \boldsymbol{y}_2\\
&\quad + {}^t(\boldsymbol{X}_i + \boldsymbol{X}_j) M_2^3 \boldsymbol{y}_2 + 3\,{}^t\boldsymbol{X}_i \boldsymbol{X}_j,
\end{aligned}
$$

and $\mathbb{F}_{k+4}$ $(k = 1, 2, 3, 4)$ are given by exchanging the subscripts i and j of $\mathbb{F}_k$. Calculating the expectation for the expansion of T_i^4 with respect to $\boldsymbol{X}_i$, $\boldsymbol{y}_1$ and M_1, and then doing for the expansion of $T_i^2 T_j^2$ with respect to $\boldsymbol{X}_i$, $\boldsymbol{X}_j$, $\boldsymbol{y}_2$ and M_2, we may obtain the variance.

Finally we derive an asymptotic expansion for $b_{2,p}^3$. Note that

$$
E(b_{2,p}^3) = \frac{1}{n^2} E(T_i^6) + 3\left(\frac{1}{n} - \frac{1}{n^2}\right) E(T_i^4 T_j^2) + \left(1 - \frac{3}{n} + \frac{2}{n^2}\right) E(T_i^2 T_j^2 T_l^2).
$$

In order to avoid the dependence of $\boldsymbol{X}_i$, $\boldsymbol{X}_j$, $\boldsymbol{X}_l$, $\overline{\boldsymbol{X}}$ and U, we define

$$
\begin{aligned}
\overline{\boldsymbol{X}}_{(ijl)} &:= \frac{1}{n-3}\sum_{k\neq i,j,l}^{n} \boldsymbol{X}_k,\\
U_{(ijl)} &:= \frac{1}{n-4}\sum_{k\neq i,j,l}^{n} (\boldsymbol{X}_k - \overline{\boldsymbol{X}}_{(ijl)})\,{}^t(\boldsymbol{X}_k - \overline{\boldsymbol{X}}_{(ijl)}).
\end{aligned}
$$

Further, let $\boldsymbol{y}_3 := \sqrt{n-3}\,\overline{\boldsymbol{X}}_{(ijl)}$ and $M_3 := \sqrt{n-3}(U_{(ijl)} - I_p)$. Then, $T_i^2 T_j^2 T_l^2$ can be expanded as

$$
\begin{aligned}
T_i^2 T_j^2 T_l^2 &= \mathbb{J}_1^2 + \frac{1}{\sqrt{n}}\mathbb{H}_1 + \frac{1}{n}(\mathbb{H}_2 - \mathbb{J}_1^2\mathbb{J}_2)\\
&\quad + \frac{1}{n\sqrt{n}}\big\{\mathbb{H}_3 - \mathbb{H}_1\mathbb{J}_2 - 2\mathbb{J}_1^2(\mathbb{R}_1 + \mathbb{R}_5 + \mathbb{R}_9)\big\}\\
&\quad + \frac{1}{n^2}\Big\{\mathbb{J}_1^2\big\{12\mathbb{J}_2 - 78 + 3({}^t\boldsymbol{X}_i\boldsymbol{X}_i + {}^t\boldsymbol{X}_j\boldsymbol{X}_j + {}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2 + \mathbb{J}_3\big\}\\
&\qquad + \mathbb{H}_4 - \mathbb{H}_2\mathbb{J}_2 - 2\mathbb{H}_1(\mathbb{R}_1 + \mathbb{R}_5 + \mathbb{R}_9)\Big\} + O(n^{-\frac{5}{2}}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{J}_1 &= {}^t\boldsymbol{X}_i\boldsymbol{X}_i\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l,\\
\mathbb{J}_2 &= 2({}^t\boldsymbol{X}_i\boldsymbol{X}_i + {}^t\boldsymbol{X}_j\boldsymbol{X}_j + {}^t\boldsymbol{X}_l\boldsymbol{X}_l + 6),\\
\mathbb{J}_3 &= -2({}^t\boldsymbol{X}_i\boldsymbol{X}_i\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j + {}^t\boldsymbol{X}_j\boldsymbol{X}_j\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l + {}^t\boldsymbol{X}_i\boldsymbol{X}_i\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{H}_1 &= 2\mathbb{J}_1({}^t\boldsymbol{X}_j\boldsymbol{X}_j\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_1 + {}^t\boldsymbol{X}_i\boldsymbol{X}_i\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_5 + {}^t\boldsymbol{X}_i\boldsymbol{X}_i\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_9),\\
\mathbb{H}_2 &= \mathbb{D}_{1(i)}({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2+\mathbb{D}_{2(j)}({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2+\mathbb{D}_{3(l)}({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\\
&\quad + 4\mathbb{J}_1({}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_5\mathbb{R}_9 + {}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_1\mathbb{R}_9 + {}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_1\mathbb{R}_5),\\
\mathbb{H}_3 &= \mathbb{E}_{1(i)}({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2+\mathbb{E}_{2(j)}({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2+\mathbb{E}_{3(l)}({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\\
&\quad + ({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_5\mathbb{D}_{3(l)} + ({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_9\mathbb{D}_{2(j)}\\
&\quad + ({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_1\mathbb{D}_{3(l)} + ({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_9\mathbb{D}_{1(i)}\\
&\quad + ({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_1\mathbb{D}_{2(j)} + ({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_5\mathbb{D}_{1(i)}\\
&\quad + 8\mathbb{J}_1\mathbb{R}_2\mathbb{R}_5\mathbb{R}_9,\\
\mathbb{H}_4 &= (\mathbb{R}_2^2 + 2\mathbb{R}_1\mathbb{R}_3 + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_4)({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2\\
&\quad + 4\mathbb{D}_{1(i)}\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_5\mathbb{R}_9\\
&\quad + (\mathbb{R}_6^2 + 2\mathbb{R}_5\mathbb{R}_7 + 2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_8)({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2\\
&\quad + 4\mathbb{D}_{2(j)}\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_1\mathbb{R}_9\\
&\quad + (\mathbb{R}_{10}^2 + 2\mathbb{R}_9\mathbb{R}_{11} + 2\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_{12})({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\\
&\quad + 4\mathbb{D}_{3(l)}\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_1\mathbb{R}_5\\
&\quad + 2\big\{({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_5\mathbb{E}_{3(l)} + ({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_9\mathbb{E}_{2(j)}\\
&\qquad + ({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_1\mathbb{E}_{3(l)} + ({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\,{}^t\boldsymbol{X}_l\boldsymbol{X}_l\mathbb{R}_9\mathbb{E}_{1(i)}\\
&\qquad + ({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2\,{}^t\boldsymbol{X}_j\boldsymbol{X}_j\mathbb{R}_5\mathbb{E}_{1(i)} + ({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_1\mathbb{E}_{2(j)}\big\}\\
&\quad + ({}^t\boldsymbol{X}_i\boldsymbol{X}_i)^2\mathbb{D}_{2(j)}\mathbb{D}_{3(l)} + ({}^t\boldsymbol{X}_j\boldsymbol{X}_j)^2\mathbb{D}_{1(i)}\mathbb{D}_{3(l)} + ({}^t\boldsymbol{X}_l\boldsymbol{X}_l)^2\mathbb{D}_{1(i)}\mathbb{D}_{2(j)},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{D}_{k(i)} &= \mathbb{R}_{4k-3}^2 + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_{4k-2},\\
\mathbb{E}_{k(i)} &= 2(\mathbb{R}_{4k-3}\mathbb{R}_{4k-2} + {}^t\boldsymbol{X}_i\boldsymbol{X}_i\mathbb{R}_{4k-1}),
\end{aligned}
$$

for $k = 1, 2, 3, 4$, what is more

$$
\begin{aligned}
\mathbb{R}_1 &= -2\,{}^t\boldsymbol{X}_i\boldsymbol{y}_3 - {}^t\boldsymbol{X}_iM_3\boldsymbol{X}_i,\\
\mathbb{R}_2 &= \mathbb{L}_{2(i,i)} + 2\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i - {}^t\boldsymbol{X}_i\boldsymbol{X}_j({}^t\boldsymbol{X}_i\boldsymbol{X}_j + 2) - {}^t\boldsymbol{X}_i\boldsymbol{X}_l({}^t\boldsymbol{X}_i\boldsymbol{X}_l + 2),\\
\mathbb{R}_3 &= \mathbb{L}_{3(i,i)} + 2\mathbb{L}_{1(i,i)} - 2\mathbb{L}_{1(i,j)}({}^t\boldsymbol{X}_i\boldsymbol{X}_j + 1) - 2\mathbb{L}_{1(i,l)}({}^t\boldsymbol{X}_i\boldsymbol{X}_l + 1),\\
\mathbb{R}_4 &= \mathbb{L}_{4(i,i)}+\mathbb{L}_{2(i,i)}-2\mathbb{L}_{2(i,j)}({}^t\boldsymbol{X}_i\boldsymbol{X}_j + 1)-2\mathbb{L}_{2(i,l)}({}^t\boldsymbol{X}_i\boldsymbol{X}_l + 1)-\mathbb{L}_{1(i,j)}^2-\mathbb{L}_{1(i,l)}^2\\
&\quad + {}^t\boldsymbol{X}_i\boldsymbol{X}_j({}^t\boldsymbol{X}_i\boldsymbol{X}_j + 2)({}^t\boldsymbol{X}_j\boldsymbol{X}_j - 2) + {}^t\boldsymbol{X}_i\boldsymbol{X}_l({}^t\boldsymbol{X}_i\boldsymbol{X}_l + 2)({}^t\boldsymbol{X}_l\boldsymbol{X}_l - 2)\\
&\quad - 2\Big\{{}^t\boldsymbol{X}_i\boldsymbol{X}_j({}^t\boldsymbol{X}_i\boldsymbol{X}_j - {}^t\boldsymbol{X}_j\boldsymbol{X}_l + 1) + {}^t\boldsymbol{X}_i\boldsymbol{X}_l({}^t\boldsymbol{X}_i\boldsymbol{X}_l - {}^t\boldsymbol{X}_j\boldsymbol{X}_l + 1)\\
&\qquad - {}^t\boldsymbol{X}_i\boldsymbol{X}_l\,{}^t\boldsymbol{X}_i\boldsymbol{X}_j({}^t\boldsymbol{X}_j\boldsymbol{X}_l+ 1) - {}^t\boldsymbol{X}_j\boldsymbol{X}_l+ 1\Big\} - 7\,{}^t\boldsymbol{X}_i\boldsymbol{X}_i+{}^t\boldsymbol{X}_j\boldsymbol{X}_j+{}^t\boldsymbol{X}_l\boldsymbol{X}_l,
\end{aligned}
$$

where

$$\begin{aligned}
\mathbb{L}_{1(i,j)} &= -{}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)\boldsymbol{y}_3 - {}^t\boldsymbol{X}_i M_3 \boldsymbol{X}_j,\\
\mathbb{L}_{2(i,j)} &= {}^t\boldsymbol{X}_i\boldsymbol{X}_j + {}^t\boldsymbol{X}_i M_3^2 \boldsymbol{X}_j + {}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)M_3\boldsymbol{y}_3 + {}^t\boldsymbol{y}_3\boldsymbol{y}_3,\\
\mathbb{L}_{3(i,j)} &= -\frac{5}{2}\,{}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)\boldsymbol{y}_3 - \frac{5}{2}{}^t\boldsymbol{X}_i M_3 \boldsymbol{X}_j - {}^t\boldsymbol{X}_i M_3^3 \boldsymbol{X}_j - {}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)M_3^2\boldsymbol{y}_3\\
&\quad - {}^t\boldsymbol{y}_3 M_3 \boldsymbol{y}_3,\\
\mathbb{L}_{4(i,j)} &= 4\,{}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)M_3\boldsymbol{y}_3 + 4\,{}^t\boldsymbol{X}_i M_3^2 \boldsymbol{X}_j + 4\,{}^t\boldsymbol{y}_3\boldsymbol{y}_3 + 4\,{}^t\boldsymbol{X}_i\boldsymbol{X}_j\\
&\quad + {}^t(\boldsymbol{X}_i + \boldsymbol{X}_j)M_3^3\boldsymbol{y}_3 + {}^t\boldsymbol{y}_3 M_3^2 \boldsymbol{y}_3 + {}^t\boldsymbol{X}_i M_3^4 \boldsymbol{X}_j,
\end{aligned}$$

and $\mathbb{R}_{k+4}$ $(k = 1, 2, 3, 4)$ are obtained by permuting $(i, j, l) \to (j, l, i)$ in $\mathbb{R}_k$. In the same way, $\mathbb{R}_{k+8}$ are done with the permutation $(i, j, l) \to (l, i, j)$. Calculating the expectation for the expansion of T_i^6 with respect to $\boldsymbol{X}_i$, $\boldsymbol{y}_1$ and M_1, and for the expansion of $T_i^4 T_j^2$ with respect to $\boldsymbol{X}_i$, $\boldsymbol{X}_j$, $\boldsymbol{y}_2$ and M_2, and then doing for the expansion of $T_i^2 T_j^2 T_l^2$ with respect to $\boldsymbol{X}_i$, $\boldsymbol{X}_j$, $\boldsymbol{X}_l$, $\boldsymbol{y}_3$ and M_3, we may come by the skewness.

2.2. Distribution of the sample fourth order cumulant

Therefore after a great deal of calculation, the first three cumulants of $Y := \sqrt{n}(\hat{\kappa} - \kappa)$ can be expressed as

$$\begin{aligned}
\mathcal{K}_1(Y) &= \frac{1}{\sqrt{n}}a_{11} + \frac{1}{n\sqrt{n}}a_{12} + O(n^{-\frac{5}{2}}),\\
\mathcal{K}_2(Y) &= a_2^2 + O(n^{-1}),\\
\mathcal{K}_3(Y) &= \frac{6}{\sqrt{n}}a_3 + O(n^{-\frac{3}{2}}),
\end{aligned}$$

where the coefficients a_{11} $(= a_1$ in page 99) and a_{12} are equal to $\mathbb{A}_1$ and $\mathbb{A}_2$ obtained by (2.3), respectively,

$$a_2^2 = b_1(\mathcal{K}_{(4)} + 1) + b_2(\mathcal{K}_{(3)} + 1)(\kappa + 1) + b_3(\kappa + 1)^3 - (\kappa + 1)^2,$$

here

$$b_1 = \frac{(p+4)(p+6)}{p(p+2)}, \quad b_2 = -\frac{4(p+4)}{p}, \quad b_3 = \frac{4(p+2)}{p},$$

and

$$\begin{aligned}
a_3 = {}& d_1(\mathcal{K}_{(6)} + 1) + d_2(\mathcal{K}_{(5)} + 1)(\kappa + 1) + d_3(\mathcal{K}_{(4)} + 1)(\kappa + 1)\\
&+ d_4(\mathcal{K}_{(4)} + 1)(\kappa + 1)^2 + d_5(\mathcal{K}_{(4)} + 1)(\mathcal{K}_{(3)} + 1) + d_6(\mathcal{K}_{(3)} + 1)(\kappa + 1)^2\\
&+ d_7(\mathcal{K}_{(3)} + 1)(\kappa + 1)^3 + d_8(\mathcal{K}_{(3)} + 1)^2(\kappa + 1) + d_9(\kappa + 1)^5 + d_{10}(\kappa + 1)^4\\
&+ \frac{2}{3}(\kappa + 1)^3,
\end{aligned}$$

where

$$d_1 = \frac{(p+4)(p+6)(p+8)(p+10)}{6p^2(p+2)^2}, \quad d_2 = -\frac{(p+4)(p+6)(p+8)}{p^2(p+2)},$$
$$d_3 = -\frac{3(p+4)(p+6)}{2p(p+2)}, \quad d_4 = \frac{6(p+4)(p+6)}{p^2}, \quad d_5 = -\frac{2(p+4)^2(p+6)}{p^2(p+2)},$$
$$d_6 = \frac{6(p+4)}{p}, \quad d_7 = -\frac{64(p+2)(p+4)}{3p^2}, \quad d_8 = \frac{7(p+4)^2}{p^2},$$
$$d_9 = \frac{12(p+2)^2}{p^2}, \quad d_{10} = -\frac{6(p+2)}{p}.$$

We can see that $a_{11}\,(= a_1)$ and a_2^2 coincide with the expressions obtained by Seo and Toyama [ST96]. Further, if the underlying distribution is normal then $\mathcal{K}_{(m)} = 0$, the coefficients are equal to

$$a_{11} = -4, \quad a_{12} = 7, \quad a_2^2 = \frac{8}{p(p+2)}, \quad a_3 = \frac{32(p+8)}{3p^2(p+2)^2}.$$

Note that a_2^2 is essentially the same as the result given by Mardia [Ma70].

It may be noticed that more efficient estimate could be found by some transformations, for example, the variance stabilizing transformation, the normalizing transformation, etc. These are generally derived from solving differential equations; however, we have difficulty in finding such transformations because asymptotic expansions given here include a lot of moment parameters which are practically unknown. In addition, the above mentioned equations themselves are complicated so that it is hard to solve them.

2.3. Bias correction for $\hat{\kappa}$

In this section, we shall correct the bias of the estimator $\hat{\kappa}$ given by (2.1). First, it follows from (2.3) that the bias in $\hat{\kappa}$ is of order n^{-1}. Then an estimator with the bias correction has been by Seo and Toyama [ST96], which is as follows

$$\tilde{\kappa} := \hat{\kappa} - \frac{1}{n}\hat{\mathbb{A}}_1, \tag{2.4}$$

where $\hat{\mathbb{A}}_1$ is given by replacing $\mathcal{K}_{(m)}$ with $\hat{\mathcal{K}}_{(m)}$ in $\mathbb{A}_1$, that is

$$\hat{\mathbb{A}}_1 = -2(p+4)(\hat{\mathcal{K}}_{(3)} + 1) + (2p+7)(\hat{\kappa}+1)^2 - 5(\hat{\kappa}+1) + 2.$$

Remark that the bias in $\tilde{\kappa}$ is of order n^{-2} with the same variance up to the order n^{-1} as that of $\hat{\kappa}$.

Further, we may obtain another estimator as

$$\breve{\kappa} := \tilde{\kappa} - \frac{1}{n^2}(\hat{\mathbb{A}}_2 - \hat{\mathbb{P}}), \tag{2.5}$$

where $\hat{\mathbb{A}}_2$ is given by replacing $\mathcal{K}_{(m)}$ with $\hat{\mathcal{K}}_{(m)}$ in $\mathbb{A}_2$, and also $\hat{\mathbb{P}}$ is done by calculation the expectation of $\hat{\mathbb{A}}_1$ up to the order n^{-1} with asymptotic results in Maruyama [M05]. Note that the bias in $\breve{\kappa}$ is improved, that is order n^{-3} with the same variance of $\hat{\kappa}$ up to the order n^{-1} and the same third order cumulant up to the order n^{-2}.

§3. Numerical results

In order to examine the accuracy of the obtained approximations, we give simulation results of the mean, variance and skewness of the distribution of sample fourth order cumulant. Computations are made for selected values: $p = 2$, 5 and $n = 100$, 200, 500, 4000. We consider the following six types of distributions for elliptical populations,

M1 : Contaminated normal ($\omega = 0.1$, $\tau = 3$),
M2 : Contaminated normal ($\omega = 0.4$, $\tau = 3$),
M3 : Contaminated normal ($\omega = 0.7$, $\tau = 3$),
M4 : Multivariate normal,
M5 : Multivariate t ($\nu = 13$),
M6 : Compound normal $U(1, 2)$,

where the random vector $\boldsymbol{X}$ from M6 is the product of a normal vector $\boldsymbol{Z}$ which has the standard normal distribution $N_p(\mathbf{0}, I_p)$ and the inverse of a random variable according to the uniform distribution on the interval $[1, 2]$. We note that the theoretical values of $\mathcal{K}_{(m)}$ are computed easily using formulas, i.e., for the contaminated normal distribution,

$$\mathcal{K}_{(m)} = \frac{1 + \omega(\tau^{2m} - 1)}{\{1 + \omega(\tau^2 - 1)\}^m} - 1, \quad 0 \le \omega \le 1,$$

for the multivariate t distribution with ν degrees of freedom is given by

$$\mathcal{K}_{(m)} = \frac{(\nu - 2)^m}{2^m(\frac{\nu}{2} - m)_m} - 1, \quad \nu > 2m,$$

where $(\nu)_m := \nu(\nu+1)\cdots(\nu+m-1)$ and for the compound normal distribution M6,

$$\mathcal{K}_{(m)} = \frac{2^m(2^{-2m+1} - 1)}{-2m + 1} - 1.$$

Table 1: Theoretical values of $\mathcal{K}_{(m)}$.

	κ	$\mathcal{K}_{(3)}$	$\mathcal{K}_{(4)}$	$\mathcal{K}_{(5)}$	$\mathcal{K}_{(6)}$
M1	1.78	11.65	61.58	311.54	1561.52
M2	0.87	2.94	7.43	17.07	37.72
M3	0.31	0.77	1.42	2.30	3.50
M4	0	0	0	0	0
M5	0.22	0.92	3.22	14.49	169.42
M6	0.16	0.55	1.26	2.54	4.81

Since we need them up to the sixth order for computing, the moment parameters of each distribution are given by Table 1.

Table 2 shows the asymptotic approximations for the mean, variance and skewness of $\hat{\kappa}$ obtained by

$$M_{\hat{\kappa}} := \kappa + \frac{1}{n}a_{11} + \frac{1}{n^2}a_{12}, \quad V_{\hat{\kappa}} := \frac{1}{n}a_2^2, \quad \gamma_{\hat{\kappa}} := \frac{6}{n^2}V_{\hat{\kappa}}^{-\frac{3}{2}}a_3, \tag{3.1}$$

respectively, in the case $p = 2$. On the other hand, when we may assume that $\Sigma = I_p$ without any loss of generality, the sample mean, variance and skewness for $\hat{\kappa}$ based on 10,000 simulations are also obtained. Table 3 gives the results in the case $p = 5$. Also Figure 1-3 present the relative error from the limiting distribution based on (2.2) for $p = 3$. It may be seen from Tables 2 and 3 that the simulation results nearly coincide with the approximate values for the mean under these elliptical distributions. It can also turn out to be that the approximations to the variance and the skewness are better estimates as n is large. But in the models which have large values of the moment parameters meaning the non-normality, for instance, M1 and M5, the convergence is slow and the approximate expressions (3.1) are not always precise. In the other models, both values agree for a sufficiently large n. The value obtained for the estimator was acceptable for a large n and improved as ω increased for the contaminated normal M1, M2 and M3, and when n increased for M5 and the compound normal M6. It may be noted that the largeness of p does not have much effect on the mean of $\hat{\kappa}$. In addition, when p is large, $V_{\hat{\kappa}}$ and $\gamma_{\hat{\kappa}}$ decrease monotonously. As regards the limiting distribution, it can be seen from Figures that the distribution of the sample fourth order cumulant tends to the normal distribution as $n \to \infty$. It may be also noted that the errors in the case M1 and M5 are pretty large for small n because of the large values of the moment parameters. In conclusion, it can be noted that the asymptotic expansion for the distribution of a function of the sample fourth order cumulant with the coefficients a_{11}, a_{12}, a_2^2 and a_3 are not so bad approximations under the class of elliptical population.

Next we shall focus our concentration on the bias and the mean-squared error (MSE) for the estimators of the kurtosis parameter. Here make a comparison between the consistent estimator given by (2.1) and others by (2.4) and (2.5) about the bias and the MSE. Table 4 gives the results mentioned above based on 20,000 simulations in the case $p = 2$. Also Table 5 presents the results in the case $p = 5$. It may be seen from simulation results in Tables 4 and 5 that the expectation of the any estimators converges to the kurtosis parameter when the sample size is large. Especially in normal case M4, it is noted that the modified estimator $\breve{\kappa}$ rapidly approaches κ than $\tilde{\kappa}$ in addition to $\hat{\kappa}$. Also from Tables 4 and 5, we notice that $\hat{\kappa}$ is underestimated for elliptical populations. The bias of $\breve{\kappa}$ is actually smaller than that of $\hat{\kappa}$ and $\tilde{\kappa}$ in magnitude. As for the MSE, it may be noted that the MSE of $\tilde{\kappa}$ is small as compared with that of $\hat{\kappa}$ and $\breve{\kappa}$ is even smaller when the size of p is large. In particular when the sample size is moderately small, the fact mentioned above is true for populations with large moment parameters (e.g., M1 and M5) rather than the normal case M4. Therefore, it can be seen that the bias for $\tilde{\kappa}$ and $\breve{\kappa}$ is reduced as well as the MSE when the sample size is large. As far as we can judge these results, $\breve{\kappa}$ is better than $\hat{\kappa}$ and $\tilde{\kappa}$.

Acknowledgments

The author would like to express his sincere gratitude to Professor Takashi Seo for his kind guidance and constant encouragement. The author also thanks the referee for the comments that were useful very much for the revision of the paper.

Table 2: Simulations and approximations to mean, variance and skewness for $p = 2$.

	n	mean	$M_{\hat{\kappa}}$	variance	$V_{\hat{\kappa}}$	skewness	$\gamma_{\hat{\kappa}}$
M1	100	1.147	1.169	0.421	1.175	0.247	2.636
	200	1.429	1.430	0.339	0.587	0.187	0.659
	500	1.623	1.629	0.187	0.235	0.620*	0.105
	4000	1.760	1.760	0.287*	0.293*	0.156**	0.164**
M2	100	0.715	0.712	0.721*	0.109	0.196*	0.516*
	200	0.794	0.790	0.447*	0.546*	0.826**	0.129*
	500	0.838	0.838	0.200*	0.219*	0.175**	0.206**
	4000	0.867	0.866	0.270**	0.273**	0.301†	0.323†
M3	100	0.240	0.241	0.216*	0.287*	0.286**	0.543**
	200	0.274	0.275	0.125*	0.143*	0.107**	0.135**
	500	0.294	0.296	0.542**	0.574**	0.179***	0.217***
	4000	0.307	0.308	0.702***	0.718***	0.340††	0.339††
M4	100	−0.414*	−0.393*	0.819**	0.100*	0.631***	0.100**
	200	−0.201*	−0.198*	0.449**	0.500**	0.196***	0.250***
	500	−0.799**	−0.797**	0.189**	0.200**	0.375†	0.400†
	4000	−0.788***	−0.999***	0.246***	0.250***	0.513†††	0.625†††
M5	100	0.122	0.133	0.349*	0.102	0.155*	1.257
	200	0.167	0.171	0.282*	0.513*	0.123*	0.314
	500	0.193	0.199	0.144*	0.205*	0.552**	0.503*
	4000	0.211	0.217	0.249**	0.256**	0.431***	0.786***
M6	100	0.945*	0.887*	0.215*	0.317*	0.424**	0.102*
	200	0.130	0.123	0.127*	0.158*	0.136**	0.255**
	500	0.153	0.144	0.603**	0.634**	0.321***	0.408***
	4000	0.165	0.158	0.814***	0.793***	0.567††	0.638††

* : value$\times 10^{-1}$, ** : value$\times 10^{-2}$, *** : value$\times 10^{-3}$, †: value$\times 10^{-4}$, ††: value$\times 10^{-5}$, †††: value$\times 10^{-6}$.

Table 3: Continued. for $p = 5$.

	n	mean	$M_{\hat{\kappa}}$	variance	$V_{\hat{\kappa}}$	skewness	$\gamma_{\hat{\kappa}}$
M1	100	0.960	0.996	0.107	0.363	0.942**	0.214
	200	1.306	1.313	0.929*	0.181	0.912**	0.537*
	500	1.576	1.575	0.570*	0.726*	0.411**	0.859**
	4000	1.751	1.753	0.862**	0.908**	0.146***	0.134***
M2	100	0.685	0.684	0.225*	0.388*	0.201**	0.600**
	200	0.778	0.776	0.147*	0.194*	0.751***	0.150**
	500	0.834	0.832	0.714**	0.776**	0.213***	0.240***
	4000	0.866	0.865	0.964***	0.970***	0.438††	0.375††
M3	100	0.236	0.237	0.696**	0.968**	0.346***	0.588***
	200	0.271	0.273	0.408**	0.484**	0.968†	0.147***
	500	0.294	0.295	0.181**	0.193**	0.201†	0.235†
	4000	0.306	0.308	0.238***	0.242***	0.239†††	0.367†††
M4	100	−0.388*	−0.393*	0.173**	0.228**	0.354†	0.679†
	200	−0.198*	−0.198*	0.101**	0.114**	0.143†	0.169†
	500	−0.784**	−0.797**	0.426***	0.457***	0.228††	0.271††
	4000	−0.107**	−0.999***	0.566†	0.571†	0.570‡	0.424‡
M5	100	0.109	0.126	0.928**	0.376*	0.139**	0.207
	200	0.156	0.163	0.756**	0.188*	0.100**	0.517*
	500	0.189	0.194	0.467**	0.752**	0.887***	0.828**
	4000	0.211	0.216	0.859***	0.940***	0.147***	0.129***
M6	100	0.870*	0.789*	0.527**	0.864**	0.259***	0.904***
	200	0.125	0.117	0.340**	0.423**	0.111***	0.226***
	500	0.150	0.142	0.166**	0.179**	0.300†	0.361†
	4000	0.164	0.157	0.229***	0.216***	0.512†††	0.565†††

* : value$\times 10^{-1}$, ** : value$\times 10^{-2}$, *** : value$\times 10^{-3}$, †: value$\times 10^{-4}$, ††: value$\times 10^{-5}$, †††: value$\times 10^{-6}$, ‡: value$\times 10^{-7}$.

Table 4: Simulation results for bias and MSE of the estimators in the case $p = 2$.

	n	bias $\hat{\kappa}$	bias $\tilde{\kappa}$	bias $\breve{\kappa}$	MSE $\hat{\kappa}$	MSE $\tilde{\kappa}$	MSE $\breve{\kappa}$
M1	100	−0.631	−0.277	−0.224	0.817	0.868	0.877
	200	−0.361	−0.107	−0.718*	0.465	0.548	0.551
	500	−0.148	−0.168*	−0.651**	0.211	0.235	0.238
	4000	−0.209*	−0.169**	−0.149**	0.291*	0.285*	0.286*
M2	100	−0.155	−0.321*	−0.276*	0.969*	0.114	0.127
	200	−0.785*	−0.811**	−0.572**	0.512*	0.582*	0.594*
	500	−0.314*	−0.247**	−0.221**	0.211*	0.227*	0.232*
	4000	−0.431**	−0.291***	−0.281***	0.274**	0.277**	0.279**
M3	100	−0.681*	−0.833**	−0.822**	0.265*	0.310*	0.324*
	200	−0.349*	−0.271**	−0.243**	0.137*	0.152*	0.156*
	500	−0.136*	−0.751***	−0.718***	0.553**	0.584**	0.590**
	4000	−0.188**	−0.136***	−0.129***	0.709***	0.713***	0.715***
M4	100	−0.399*	−0.466**	−0.451**	0.977**	0.111*	0.121*
	200	−0.205*	−0.189**	−0.161**	0.495**	0.533**	0.538**
	500	−0.771**	−0.303***	−0.301***	0.199**	0.198**	0.200**
	4000	−0.117**	−0.175***	−0.168***	0.249***	0.248***	0.250***
M5	100	−0.980*	−0.294*	−0.241*	0.456*	0.637*	0.683*
	200	−0.537*	−0.124*	−0.975**	0.297*	0.406*	0.414*
	500	−0.197*	−0.401**	−0.304**	0.149*	0.176*	0.182*
	4000	−0.705**	−0.297**	−0.194**	0.248**	0.245**	0.247**
M6	100	−0.704*	−0.124*	−0.102*	0.261*	0.324*	0.344*
	200	−0.363*	−0.364**	−0.255**	0.141*	0.165*	0.173*
	500	−0.149*	−0.749***	−0.647***	0.618**	0.632**	0.636**
	4000	−0.238**	−0.533***	−0.526***	0.815***	0.822***	0.826***

$*$: value$\times 10^{-1}$, $**$: value$\times 10^{-2}$, $***$: value$\times 10^{-3}$.

Table 5: Continued. in the case $p = 5$.

		bias			MSE		
	n	$\hat{\kappa}$	$\tilde{\kappa}$	$\breve{\kappa}$	$\hat{\kappa}$	$\tilde{\kappa}$	$\breve{\kappa}$
M1	100	−0.820	−0.385	−0.317	0.781	0.378	0.371
	200	−0.467	−0.132	−0.910*	0.311	0.182	0.178
	500	−0.206	−0.291*	−0.167*	0.969*	0.785*	0.787*
	4000	−0.273*	−0.924***	−0.648***	0.948**	0.920**	0.919**
M2	100	−0.185	−0.422*	−0.236*	0.564*	0.395*	0.396*
	200	−0.932*	−0.113*	−0.102*	0.234*	0.205*	0.202*
	500	−0.378*	−0.225**	−0.218**	0.844**	0.814**	0.815**
	4000	−0.535**	−0.678***	−0.650***	0.984***	0.975***	0.973***
M3	100	−0.730*	−0.210**	−0.109**	0.121*	0.100*	0.103*
	200	−0.366*	−0.155**	−0.992***	0.537**	0.502**	0.504**
	500	−0.152*	−0.952***	−0.750***	0.203**	0.198**	0.196**
	4000	−0.193**	−0.945†	−0.934†	0.238***	0.237***	0.235***
M4	100	−0.394*	−0.207*	−0.110*	0.329**	0.260**	0.261**
	200	−0.197*	−0.246**	−0.987***	0.138**	0.123**	0.121**
	500	−0.794**	−0.344***	−0.123***	0.500***	0.481***	0.480***
	4000	−0.101**	−0.198†	−0.175†	0.578†	0.575†	0.574†
M5	100	−0.113	−0.366*	−0.357*	0.220*	0.206*	0.207*
	200	−0.654*	−0.178*	−0.162*	0.120*	0.139*	0.136*
	500	−0.325*	−0.995**	−0.918**	0.601**	0.627**	0.624**
	4000	−0.117*	−0.861**	−0.851**	0.958***	0.944***	0.943***
M6	100	−0.800*	−0.162*	−0.156*	0.115*	0.913**	0.911**
	200	−0.416*	−0.488**	−0.430**	0.516**	0.480**	0.476**
	500	−0.163*	−0.164***	−0.135***	0.191**	0.198**	0.194**
	4000	−0.218**	−0.589†	−0.576†	0.232***	0.232***	0.231***

* : value$\times 10^{-1}$, ** : value$\times 10^{-2}$, *** : value$\times 10^{-3}$, †: value$\times 10^{-4}$.

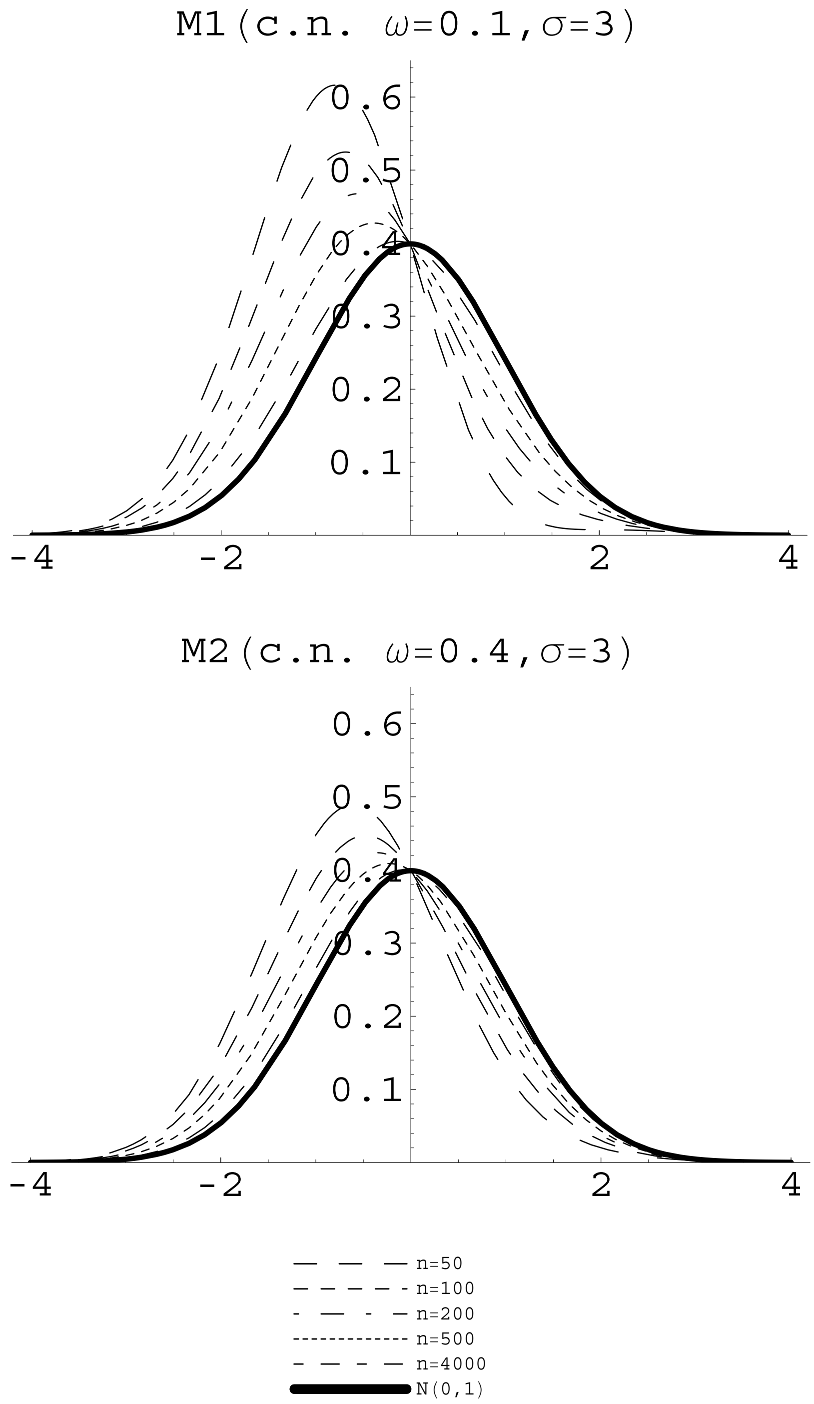

Figure 1: Distribution of $\sqrt{n}(\hat{\kappa} - \kappa)/a_2$

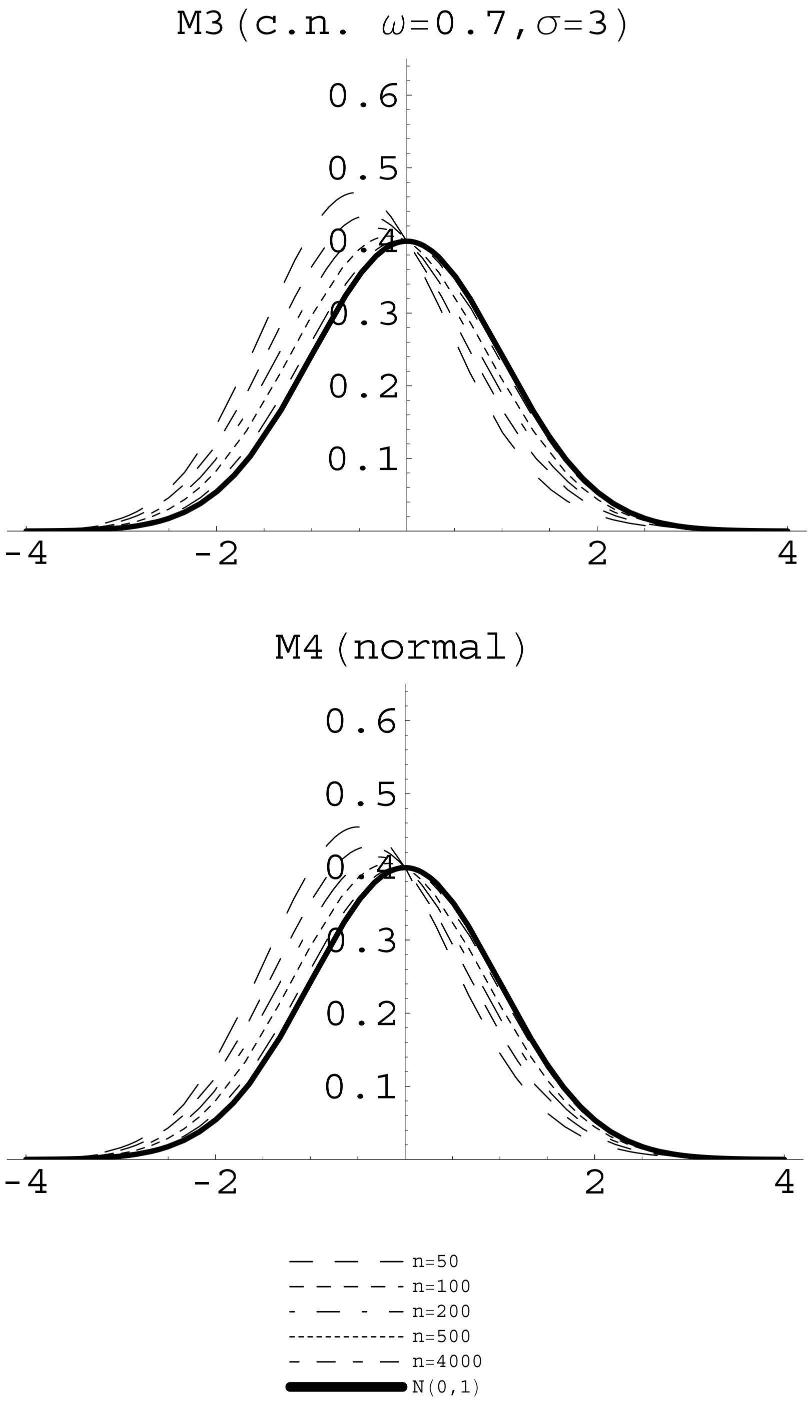

Figure 2: Continued.

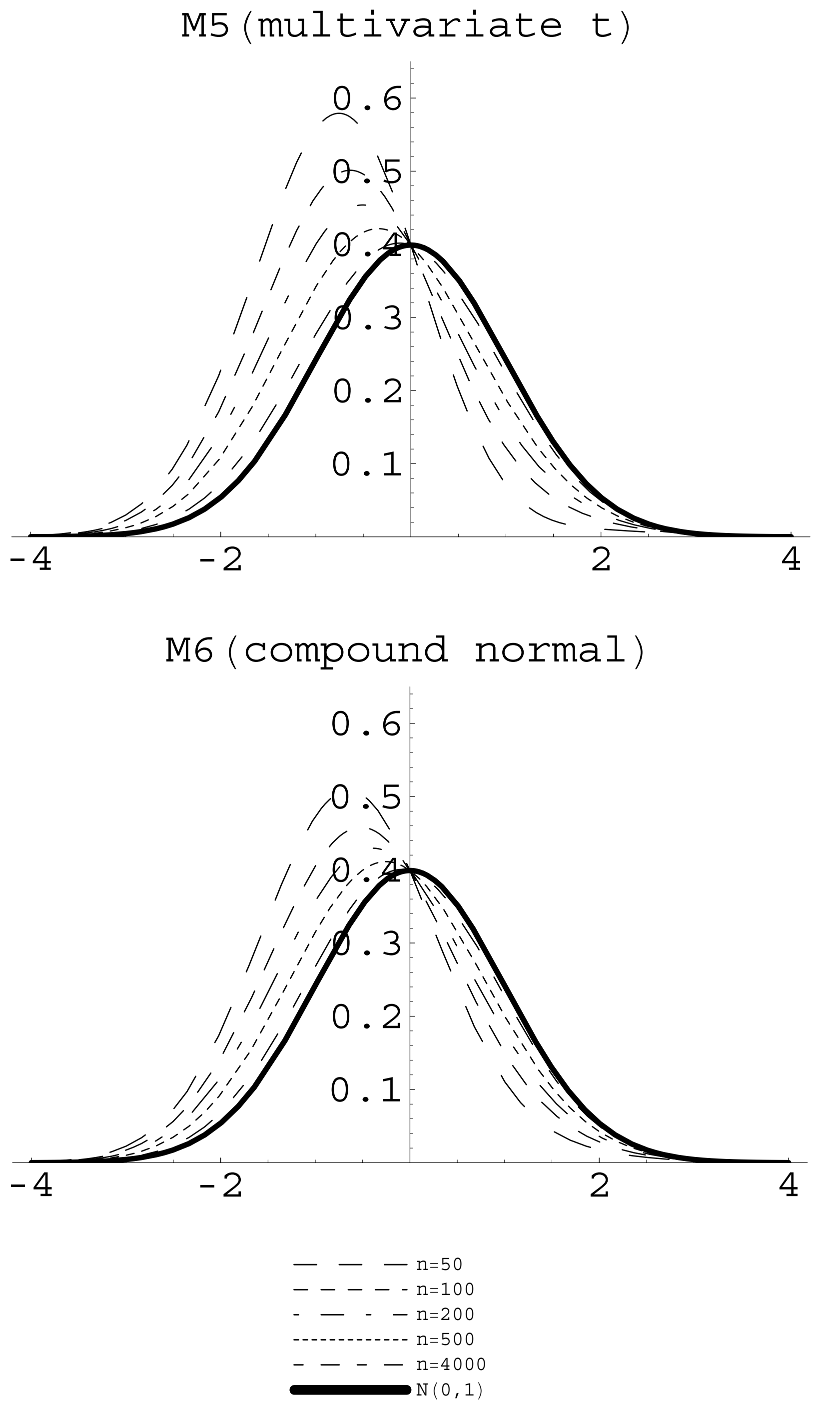

Figure 3: Continued.

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