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On the non-trivial cycles in Collatz's problem

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Abstract. In this note, we improve upon results of Steiner [9] and Mimuro [8] concerning the structure of non-trivial cycles in Collatz's problem.

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§1. Introduction

Let f be the function defined on the set of all positive integers given by $f(m) = m/2$ if m is even and $f(m) = (3m+1)/2$ otherwise. A well-known conjecture of Collatz asserts that for every positive integer m there exists a positive integer n such that $f^n(m) = 1$, where

$$f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}$$

is the nth fold composition of f with itself. Collatz [5] invented this question in 1928 in the context of dealing with iteration problems represented using associated graphs and hypergraphs. The exact origin of Collatz's conjecture goes back to the past and it is also sometimes called the *Syracuse conjecture* or the $3x+1$ *problem.* This conjecture circulated orally among the mathematical community for many years. In spite of the considerable work which has been done around this problem (a large database of research papers dealing with this problem is annotated in [7]), this conjecture is still open. There are several prizes offered for its solution (\$50 by Coxeter in 1970, \$500 by Erdős, and £1000 by Thwaites [10]).

In this paper, we extend a result of Mimuro [8] concerning this conjecture. We follow the notation and terminology from [8].

We call m to be a *cycle-number* if $f^n(m) = m$ holds with some positive integer n. Note that Collatz's conjecture suggests that 1 and 2 are the only cycle numbers. Furthermore, removal of cycle numbers would lead to an efficient construction of the Collatz tree as defined in [1] and [2].

Let m be a cycle-number. We assume that m is odd since every even number is mapped into an odd number by iterating f. Since $m = 1$ generates the cycle $\{1,\ 2\}$ (possibly the unique such), we assume that $m > 1$. Such a cycle will be called *non-trivial*. For $u \geq 1$, we define ℓ_u and m_u as follows:

(i) We put $m_1 = m$.

(ii) For $u \geq 1$, ℓ_u is the largest positive integer such that $f^{\ell_u}(m_u)$ is odd.

(iii) We put $m_{u+1} = f^{\ell_u}(m_u)$.

Note that

$$m_{u+1} = \frac{3m_u + 1}{2^{\ell_u}}. \tag{1.1}$$

If $m = m_1 = m_{k+1}$, we then write

$$\mathcal{C}(m) = \langle \ell_1, \ell_2, \ldots, \ell_k \rangle$$

to record the successive compositions in the cycle generated by m. It is easy to see that

$$\mathcal{C}(m_1) = \langle \ell_1, \ell_2, \ldots, \ell_k \rangle,$$

while

$$\mathcal{C}(m_u) = \langle \ell_u, \ell_{u+1}, \ldots, \ell_k, \ell_1, \ldots, \ell_{u-1} \rangle \qquad \text{for all } u = 2, \ldots, k.$$

We put

$$n = \sum_{u=1}^{k} \ell_u.$$

We may assume that $m = m_1 = \min\{m_u : u = 1, \ldots, k\}$. Lemma 2.1 in [8] shows that $\ell_1 = 1$. Furthermore, we can assume that $\ell_k \neq 1$. Indeed, it is clear that $\mathcal{C}(m)$ cannot be $\langle 1, \ldots, 1 \rangle$ (as in this case $m_{u+1} = (3m_u+1)/2 > m_u$ holds for all $u = 1, \ldots, k-1$, which is impossible because $m_1 = m_k$), and if $\ell_k = 1$, then

$$\mathcal{C}(m_1) = \langle 1, \ldots, \ell_{k-1}, 1 \rangle > m_k = \langle 1, 1, \ldots, \ell_{k-1} \rangle,$$

contradicting the fact that m_1 is minimal among the m_u's. We let $\kappa_0 = 0$ and $1 \leq \kappa_1 < \ldots < \kappa_j = k$ be such that

$$\ell_{\kappa_{i-1}+1} = \ell_{\kappa_i+2} = \cdots = \ell_{\kappa_i} \neq \ell_{\kappa_i+1} \qquad \text{for all } i = 1, \ldots, j-1.$$

For $i = 1, \dots, j$, we write $N_i = \kappa_i - \kappa_{i-1}$, and we put $L_i = \ell_{\kappa_{i-1}+1}$. Hence, we have

$$\mathcal{C}(m) = \langle \underbrace{L_1, \dots, L_1}_{N_1 \text{ times}}, \dots, \underbrace{L_j, \dots, L_j}_{N_j \text{ times}} \rangle, \tag{1.2}$$

where

$$N_1 + \cdots + N_j = k, \tag{1.3}$$

and where $L_1 = 1$ and $L_i \neq L_{i+1}$ for all $i = 1, \dots, j$. Here, we set $L_{j+1} = L_1$. Note that

$$n = \sum_{i=1}^{j} L_i N_i. \tag{1.4}$$

With these notations, Steiner [9] proved that there is no such m with $j = 2$ and $N_2 = 1$; i.e., that there is no odd cycle-number whose associated compositions are of the form $\langle 1, \dots, 1, \ell \rangle$. This result was generalized by Mimuro in [8] who proved that there are at most finitely many odd cycle-numbers m whose associated compositions have $j = 2$.

In this note, we generalize Mimuro's result from [8]. Our precise results are stated in Section 2 and proved in Section 3.

Throughout this paper, we use the Vinogradov symbols $\ll$ and $\gg$ with their usual meaning. Recall that if A and B are two functions defined on the set of positive real numbers we say that $A \ll B$ if there exists a positive constant c such that the inequality $|A(x)| < c|B(x)|$ holds for all positive real numbers x. The notation $A \gg B$ is equivalent to $B \ll A$. The constants implied by them are absolute. We also use $c_0,\ c_1,\ \dots$ for positive computable positive constants which are absolute and which are labeled increasingly throughout the paper. For a finite set $\mathcal{A}$ we use $\#\mathcal{A}$ to denote its cardinality.

§2. Results

Let $m > 1$ be an odd cycle-number given by (1.2) where the positive integers L_i and N_i satisfy (1.3) and (1.4). We write

$$\mathcal{A}(m) = \{i \in \{1, \dots, j\} \ : \ L_i = 1\},$$

and put $t = \#\mathcal{A}(m)$. Our main result is the following:

Theorem 2.1. *The following inequality holds:*

$$t \gg \log n. \tag{2.1}$$

The following corollary to Theorem 2.1 generalizes the result from [8]; i.e., the result from [8] is obtained from the corollary below with $c = 1$.

Corollary 2.2. *Let c be any positive constant. Then there exist at most finitely many odd cycle-numbers m having $t \le c$.*

§3. Proofs

We start with some general considerations.

It is clear that we may assume that $m = m_1$ is as large as we wish.

We recall that if $u \in \{1, \ldots, k\}$, then there exists a unique value of $i \in \{0, \ldots, j-1\}$ such that $u \in \{k_i + 1, \ldots, k_{i+1}\}$. In this case, $\ell_u = L_i$. Furthermore (see [3], or formula (1.2) in [8]),

$$m = \frac{\sum_{u=1}^{k} 3^{k-s} \cdot 2^{\sum_{v=0}^{u-1} \ell_v}}{2^n - 3^k}. \tag{3.1}$$

Since $m > 0$, it follows that $2^n > 3^k$. In particular, if n is bounded, then (see formula (1.4)), there are only finitely many possibilities for k and the k-tuple $(\ell_1, \ldots, \ell_k)$; hence, for m as well. Thus, from now on we may assume that both n and $m = m_1$ are as large as we wish.

The next result gives a non-trivial bound for m in terms of n.

Lemma 3.1. *There exist positive constants $c_0,\ c_1,\ c_2$ such that if $n > c_0$, then*

(i) $m < n^{c_1}$;

(ii)

$$\sum_{i \in \mathcal{A}(m)} N_i \ge c_2 n. \tag{3.2}$$

Proof. By formula (1.1), if $u \in \{1, \ldots, k-1\}$, then

$$m_{u+1} = \frac{3m_u + 1}{2^{\ell_u}} = \left(\frac{3}{2^{\ell_u}}\right) m_u \left(1 + \frac{1}{3m_u}\right). \tag{3.3}$$

Multiplying relations (3.3) for $u = 1, \ldots, k$, we get

$$m_1 = m_{k+1} = \frac{3^k}{2^n} m_1 \prod_{u=1}^{k} \left(1 + \frac{1}{3m_u}\right). \tag{3.4}$$

Using the fact that the inequality $1 + x < \exp x$ holds for all $x > 0$, we get

$$\prod_{u=1}^{k} \left(1 + \frac{1}{3m_u}\right) < \exp\left(\sum_{u=1}^{k} \frac{1}{3m_u}\right). \tag{3.5}$$

Clearly,

$$\begin{aligned} \sum_{u=1}^{k} \frac{1}{3m_u} &\leq \frac{1}{3} \sum_{\substack{m_1 \leq \lambda \leq m_1+k-1 \\ \lambda \text{ integer}}} \frac{1}{\lambda} \\ &< \frac{1}{3m_1} + \int_{m_1}^{m_1+k-1} \frac{d\nu}{3\nu} \\ &\leq \frac{1}{3m_1} + \frac{1}{3}\log\left(1 + \frac{k-1}{m_1}\right) \\ &< \log\left(1 + \frac{2}{3m_1}\right) + \log\left(\left(1 + \frac{k-1}{m_1}\right)^{1/3}\right) \\ &< \log\left(1 + \frac{2}{3m_1}\right) + \log\left(1 + \frac{k-1}{3m_1}\right) \\ &= \log\left(1 + \frac{k+1}{3m_1} + \frac{2(k-1)}{9m_1^2}\right) \\ &< \log\left(1 + \frac{2k}{3m_1}\right) < \log\left(1 + \frac{n}{m_1}\right). \end{aligned} \tag{3.6}$$

In the above chain of inequalities we used, aside from the fact that $k < n$ and that

$$\frac{1}{k} \leq \int_{k-1}^{k} \frac{d\nu}{\nu} \qquad \text{for all } k \geq 2,$$

also the fact that $(1 + x)^{1/3} < 1 + x/3$ holds for all $x > 0$, as well as the fact that $x < \log(1 + 2x)$ holds for all $x \in (0, 1)$. Thus, from estimates (3.5) and (3.6), we get

$$\prod_{u=1}^{k} \left(1 + \frac{1}{3m_u}\right) < 1 + \frac{n}{m_1}. \tag{3.7}$$

From equation (3.4) and estimate (3.7), we get that

$$m_1 = \frac{3^k}{2^n} m_1 + \eta,$$

where η is a positive number such that $\eta < 3^k n/2^n$. Thus, we get that

$$m_1 = \frac{2^n \eta}{2^n - 3^k} < \frac{3^k n}{2^n - 3^k}.$$

Using a linear form in logarithms á la Baker (see [4], for example), it follows that there exists a computable positive constant c_3 such that the inequality

$$2^n - 3^k > 2^n n^{-c_3}$$

holds whenever $2^n > 3^k$ and $n \geq 2$. We thus get

$$m_1 < \frac{3^k n^{c_3+1}}{2^n} < n^{c_3+1},$$

which takes care of (i) above with $c_1 = c_3 + 1$ and $c_0 \geq 2$.

To deal with (ii), we let $\delta > 0$ be sufficiently small, and assume that

$$S(m) = \sum_{i \in \mathcal{A}(m)} N_i < \delta n.$$

Then

$$\begin{aligned} k &= \sum_{i=1}^{j} N_i = S(m) + \sum_{i \notin \mathcal{A}(m)} N_i \leq S(m) + \frac{1}{2} \sum_{i \notin \mathcal{A}(m)} L_i N_i \\ &= S(m) + \frac{1}{2}\left(n - S(m)\right) = \frac{1}{2}(S(m) + n) < \frac{(1+\delta)n}{2}. \end{aligned}$$

In the above chain of inequalities we used the fact that $L_i \geq 2$ if $i \notin \mathcal{A}(m)$. We choose δ such that $(1+\delta)/2 = \log 2/\log 3 - \delta$. This leads to the choice $\delta = (2\log 2/\log 3 - 1)/3 = \log(4/3)/(3\log 3)$. We then get that

$$3^k < 3^{(1+\delta)n/2} = 3^{(\log 2/\log 3 - \delta)n} = \frac{2^n}{3^{\delta n}}.$$

Hence,

$$m_1 < \frac{n}{3^{\delta n}},$$

inequality which leads to the absurd conclusion that $m_1 < 1$ for $n > c_0 \geq 2$. Thus, if $n > c_0$, then inequality (3.2) holds with $c_2 = \delta$. □

In what follows, we label the elements of

$$\mathcal{A}(m) = \{i_1, \ldots, i_t\}$$

increasingly so that $1 = i_1 < \ldots < i_t$. We put

$$\lambda_s = \kappa_{i_s - 1} + 1 \qquad \text{for all } s = 1, \ldots, t.$$

Lemma 3.2. *There exist positive constants c_4, c_5 such that if $n > c_0$, then both inequalities*

(i) $m_{\lambda_s} < n^{c_4^s}$ *and*

(ii) $N_{i_s} < c_5^s \log n$

hold for all $s \in \{1, \ldots, t\}$.

Proof. We use induction on $s = 1, \ldots, t$, to show that assertion (i) above for s implies both assertion (ii) above for s as well as assertion (i) above for $s + 1$. The fact that (i) above holds with $s = 1$ whenever $n \geq c_0$ and $c_4 \geq c_1$ follows from (i) of Lemma 3.1 by noting that $\lambda_1 = \kappa_0 + 1 = 1$. We assume that $s \leq t$ is such that the inequality

$$m_{\lambda_s} \leq n^{C(s)} \tag{3.8}$$

holds whenever $n \geq c_0$, where $C(s)$ is some function of s with $C(1) \geq c_1$ which we will determine later. Note that

$$\mathcal{C}(m_{\lambda_s}) = \langle \underbrace{1, \ldots, 1}_{N_{i_s} \text{ times}}, \ldots, L_{i_s - 1} \rangle,$$

while

$$\frac{3m_{\lambda_s} + 1}{2} = m_{\lambda_s + 1},$$

so

$$\mathcal{C}(m_{\lambda_s + 1}) = \langle \underbrace{1, \ldots, 1}_{N_{i_s} - 1 \text{ times}}, \ldots, L_{i_s - 1}, 1 \rangle.$$

We now observe that if

$$\mathcal{C}(m') = \langle \underbrace{1, \ldots, 1}_{T \text{ times}}, \ldots \rangle,$$

then $m' \equiv -1 \pmod{2^T}$. Indeed, this can be easily proved by induction on T. If $T = 1$, then m' is odd; hence, $m' \equiv -1 \pmod{2^T}$, in this case. Assume now, by induction, that the above assertion is true for T, and let

$$\mathcal{C}(m') = \langle \underbrace{1, \ldots, 1}_{T+1 \text{ times}}, \ldots \rangle.$$

Then

$$\frac{3m' + 1}{2} = m'',$$

therefore

$$\mathcal{C}(m'') = \langle \underbrace{1, \ldots, 1}_{T \text{ times}}, \ldots \rangle,$$

so, by the induction hypothesis, $m'' \equiv -1 \pmod{2^T}$. This gives

$$\frac{3m'+1}{2} \equiv -1 \pmod{2^T},$$

which leads to the desired conclusion that $m' \equiv -1 \pmod{2^{T+1}}$.

Hence, we get that $m_{\lambda_s} \equiv -1 \pmod{2^{N_{i_s}}}$ and $m_{\lambda_s+1} \equiv -1 \pmod{2^{N_{i_s}-1}}$. We thus arrive at

$$2^{N_{i_s}-1} | m_{\lambda_s+1} - m_{\lambda_s}.$$

This leads to

$$2^{N_{i_s}-1} \leq m_{\lambda_s+1} - m_{\lambda_s} = \frac{m_{\lambda_s}+1}{2} \leq m_{\lambda_s} \leq n^{C(s)},$$

therefore, since $n \geq c_0 \geq 2$, we have

$$N_{i_s} \leq \frac{(C(s)+1)}{\log 2} \log n. \tag{3.9}$$

We now use recurrence (3.3) for $u = \lambda_s = \kappa_{i_s-1}+1, \ldots, \lambda_s + N_s - 1 = \kappa_{i_s}$, and multiply the resulting relations keeping in mind that $\ell_u = 1$ for such values of u, to get that

$$\begin{aligned} m_{\lambda_s+N_{i_s}} &= \left(\frac{3}{2}\right)^{N_{i_s}} m_{\lambda_s} \prod_{\lambda_s \leq u \leq \lambda_s+N_{i_s}-1} \left(1 + \frac{1}{3m_u}\right) \\ &\leq \left(\frac{3}{2}\right)^{N_{i_s}} (m_{\lambda_s}+n) \\ &\leq nm_{\lambda_s} \left(\frac{3}{2}\right)^{N_{i_s}}. \end{aligned}$$

In the last inequality above we used the analog of estimate (3.7) with the sequence of numbers $m_1 < \ldots < m_k$ replaced by the sequence of numbers $m_{\lambda_s} < \ldots < m_{\lambda_s+N_s-1}$, together with the fact that $n+m_{\lambda_s} \leq nm_{\lambda_s}$, inequality which holds because $\min\{n, m_1\} \geq 2$. We now note that for all $u \in \{\lambda_s+N_{i_s}+1, \ldots, \lambda_{s+1}\}$, we have that $\mathcal{C}(m_u) = \langle \ell_u, \ldots, \rangle$, where $\ell_u > 1$, therefore

$$m_u = \frac{3m_{u-1}+1}{2^{\ell_u}} \leq \frac{3m_{u-1}+1}{4} < m_{u-1}.$$

In particular,

$$m_{\lambda_{s+1}} < m_{\lambda_s+N_{i_s}} < nm_{\lambda_s} \left(\frac{3}{2}\right)^{N_{i_s}}. \tag{3.10}$$

From inequalities (3.10), (3.8) and (3.9), we get that

$$m_{\lambda_{s+1}} \le n^{1+C(s)+(C(s)+1)\log(3/2)/\log 2}.$$

Hence, inequality (3.8) holds with s replaced by $s+1$ provided that

$$C(s+1) \ge (C(s)+1)\left(1+\frac{\log(3/2)}{\log 2}\right).$$

The above inequality is satisfied if

$$C(s+1)+1 \ge (C(s)+1)\left(2+\frac{\log(3/2)}{\log 2}\right).$$

With equality above, we note that the sequence $a(s) := C(s)+1$ becomes simply the geometrical progression of ratio $2+\log(3/2)/\log 2$. Since we also need that $a(1) = C(1)+1 \ge c_1+1$, it follows easily that if we take $C(s)$ such that $C(s)+1 = \gamma^s$, where

$$\gamma \ge \max\left\{c_1+1, 2+\frac{\log(3/2)}{\log 2}\right\}, \tag{3.11}$$

then, by (3.8) and the above argument, both inequalities (i) and (ii) are satisfied with $c_4 = \gamma$ and $c_5 = \gamma\log 2$, respectively, which completes the proof of Lemma 3.2. □

We are now ready to prove our results.

Proof of Theorem 2.1. We may, of course, assume that $n > c_0$. Then, by (ii) of Lemma 3.2, the inequality

$$N_{i_s} \le c_5^s \log n$$

holds for all $s \in \{1,\dots,t\}$. Thus, by (ii) of Lemma 3.1, we have

$$c_2 n \le \sum_{i\in\mathcal{A}(m)} N_i \le \sum_{s=1}^{t} N_{i_s} \le \sum_{s=1}^{t} c_5^s \log n < c_5^{t+1}\log n,$$

because $c_5 > 2$ (see (3.11)). Hence,

$$t+1 > \frac{1}{\log c_5}\left(\log n - \log\log n + \log c_2\right),$$

which implies the conclusion of Theorem 2.1. Note that the implied constant in the inequality (2.1) can be taken to be $c_6 = 1/(2\log c_5)$ provided that n is sufficiently large. □

Proof of Corollary 2.2. If c is fixed and $t \le c$, then Theorem 2.1 implies that $n < C$, where C depends on c. By relation (1.4), we conclude that k and $(\ell_1,\dots,\ell_k)$ can take only finitely many values, and now relation (3.1) shows that m can take only finitely many values as well. □

§4. Conclusion

As we pointed out in the Introduction, Steiner [9] proved that there do not exist odd positive integers m whose associated compositions are $\mathcal{C}(m) = \langle 1, \dots, 1, \ell \rangle$, while Mimuro [8] proved that there exist only finitely many whose compositions have the form $\mathcal{C}(m) = \langle 1, \dots, 1, \ell, \dots, \ell \rangle$. Our results generalize those of Steiner and Mimuro, yet Collatz's conjecture implies that there should be no such cycles at all. An important next step towards Collatz's conjecture will be to prove that there are finitely many possible cycles altogether. It could be that our method might be of some help here, but we could not prove such a statement.

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