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Application of local linking to asymptotically linear wave equations with resonance II

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Abstract. Existence of a time-periodic solution to a non-linear wave equation with resonance is established by a variational method. We consider the 2π-periodic weak solution to a wave equation $\Box u(x,t) = h(x,t,u(x,t))$ of space dimension 1, where $h(x,t,\xi)$ is asymptotically linear in ξ both as $\xi \to 0$ or $\xi \to \infty$, with the co-efficient as $\xi \to \infty$ belonging to $\sigma(\Box)$. It was proved that there are some cases, where the difference of $h(t,x,\xi)$ from its linear approximation is not bounded, that guarantee the existence of a non-trivial weak solution. In this paper, we show that the restriction in our previous result imposed on these co-efficients can be further relaxed.

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§1. Introduction

The purpose of this paper is to extend the result of [12], which is concerned with the existence of a non-trivial time-periodic solution to the following non-linear wave equation (WE) with asymptotically linear non-linear term h ($\Box := \partial^2/\partial t^2 - \partial^2/\partial x^2$):

$$\text{(WE)} \qquad \begin{cases} \Box u(x,t) = h(x,t,u(x,t)), & (0 < x < \pi,\ t \in \mathbb{R}), \\ u(0,t) = u(\pi,t) = 0, & (t \in \mathbb{R}), \\ u(x,t+2\pi) = u(x,t), & (0 < x < \pi,\ t \in \mathbb{R}). \end{cases}$$

Many authors treated this problem by variational methods under various conditions on $h(x,t,\xi)$ ([1]-[5], [7]-[9], [11], [12], [14], [16]-[18]). In this paper we treat the case where the non-linear term h is asymptotically linear at both

0 and ∞ in the following sense: There exist constants b_0 and b for which

$$g_0(x,t,\xi) := h(x,t,\xi) - b_0\xi = o(|\xi|) \text{ as } \xi \to 0 \text{ uniformly in } (x,t),$$
$$g(x,t,\xi) := h(x,t,\xi) - b\xi = o(|\xi|) \quad \text{as } |\xi| \to \infty \text{ uniformly in } (x,t).$$

Li and Szulkin [8], Kryszewski and Szulkin [7] and Bartsch and Ding [2] already considered the case where $h(x,t,\xi)$ is asymptotically linear in ξ both as $\xi \to 0$ and $|\xi| \to \infty$. However, they all assume that $h(x,t,\xi) - b\xi$ is bounded when $b \in \sigma(\Box)$ ("resonant" case). Miyajima and Tanaka [12] treated the case where $h(x,t,\xi) - b\xi$ is not bounded and $b, b_0 \in \sigma(\Box)$. Their results shows, for example, that (WE) has a non-trivial periodic solution for $h(x,t,\xi) = b\xi + |\xi|^\alpha \operatorname{sgn} \xi$ with $0 < \alpha < 1$. (To be rigorous, $|\xi|^\alpha \operatorname{sgn} \xi$ should be deformed to a C^2 class function in a neighborhood of 0.)

The main purpose of this paper is to show that the condition imposed on b_0 and b can be further relaxed under the same assumption (C2) in [12] (see Section 3) on the nonlinear term (see Remark 21). Another purpose is to prove the existence of a non-trivial solution to (WE) when $h(x,t,\xi)$ is odd in ξ, by using Krasnoselskii genus. Note that Bartsch and Ding [2] considered the case of even functional corresponding to (WE) using the equivariant limit category.

In the following Section 2, we firstly obtain an abstract existence theory of a non-trivial critical point for C^1-class functional, its proof is based on that of Bartsch and Ding [2]. We also have Lusternik-Schnirelmann-type results for even functional. In Section 3, we show the existence of a non-trivial weak solution to (WE).

§2. Some results on critical point theory

Throughout this section, E denotes a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and Φ denotes a C^1 class functional on E. The gradient $\nabla\Phi(u)$ ($u \in E$) is considered to be an element of E through the Riesz representation theorem. A subset $\tilde{E}$ is defined by $\tilde{E} := \{ u \in E : \nabla\Phi(u) \neq 0 \}$. Then recall that a map $V : \tilde{E} \to E$ is called a pseudo-gradient vector field (abbreviated to "p.g.v.f.") for Φ if V satisfies the following conditions for every $u \in \tilde{E}$:

$$\begin{cases} \|V(u)\| \leq \frac{3}{2} \|\nabla\Phi(u)\|, \\ \langle \nabla\Phi(u), V(u) \rangle \geq \frac{1}{2} \|\nabla\Phi(u)\|^2. \end{cases}$$

It is well known that there exists a locally Lipschitz continuous pseudo-gradient vector field V for every C^1 class functional Φ ([13, Lemma 6.1]). For such V, the ordinary differential equation

$$\text{(2.1)} \qquad \frac{du(t)}{dt} = V(u(t)), \quad u(0) = u_0 \qquad (u_0 \in \tilde{E})$$

has a unique solution which is maximally defined in the positive direction of t.

2.1. An invariant set with respect to gradient flow and minimax method

Definition 1 *Let U be a subset of E. We say that U is an invariant set with respect to p.g.v.f. V for Φ if $\eta(t,u) \in U$ holds for every $u \in U \cap \tilde{E}$ and $t \geq 0$, where $\eta(t,u)$ is the maximal solution of differential equation* (2.1) *starting from u.*

Now we consider a condition for a subset U to be invariant with respect to some p.g.v.f. V for Φ. The following proposition was shown in [12].

Proposition 2 ([12, Proposition 1]) *Let $f\colon E \to \mathbb{R}$ be a C^1 class functional on E and let $U := \{x \in E \mid f(x) < 0\}$ be non-empty. Suppose that Φ satisfies*

$$\langle \nabla\Phi\,,\, \nabla f(u)\rangle > 0 \quad \text{on } \partial U. \tag{2.2}$$

Then there exists a locally Lipschitz continuous pseudo-gradient vector field V for Φ such that U is invariant with respect to $-V$.

Next we prepare the following lemma, which is a variation of minimax argument using some family of invariant set. (*see* [15, Theorem 4.2.], [19, Theorem 2.8.])

Lemma 3 *Let M be a compact subset of E and M_0 a closed subset of M. Moreover suppose that U is an invariant subset of E with respect to some p.g.v.f. V for Φ. Then we define;*

$$\Sigma := \{\gamma \in C\left([0,1]\times M, E\right) \mid \gamma \text{ satisfies the following condition (A) }\}$$

$$c := \sup_{\gamma\in\Sigma}\ \min_{(t,u)\in[0,1]\times M} \Phi\left(\gamma(t,u)\right), \quad c_0 := \min_{u\in M}\Phi(u),$$

$$\text{(A)}\begin{cases} \text{(a) } \gamma(0,u) = u \text{ for } u \in M \text{ and } \gamma(1,M) \subset U \\ \text{(b) there exists some constant } t_\gamma \in (0,1) \text{ such that} \\ \qquad \Phi(\gamma(t,u)) \geq \Phi(u) \text{ for } (t,u) \in [0,t_\gamma]\times M_0 \\ \qquad \text{and } \gamma(t,u) \in U \text{ for } (t,u) \in [t_\gamma,1]\times M_0. \end{cases}$$

If $\Sigma \neq \emptyset$ and $c_0 > c$ hold, then for every $\varepsilon > 0$ there exists a $v \in E$ such that

$$c - 4\varepsilon \leq \Phi(v) \leq c + 4\varepsilon, \quad \|\nabla\Phi(v)\| \leq 4\varepsilon.$$

Proof. Without loss of generality, we can assume that $\varepsilon > 0$ satisfies $c_0 > c + 4\varepsilon$. Suppose that

$$\|\nabla\Phi(v)\| > 4\varepsilon \quad \text{for every } v \in \Phi^{-1}([c-4\varepsilon, c+4\varepsilon]).$$

By the definition of c, there exists some $\gamma_0 \in \Sigma$ such that

$$\min_{(t,u)\in[0,1]\times M} \Phi(\gamma_0(t,u)) \geq c - \varepsilon. \tag{2.3}$$

Let $f : E \to [0,1]$ be a locally Lipschitz continuous function such that

$$f(u) := \begin{cases} 1 & \text{if } u \in \Phi^{-1}([c-2\varepsilon, c+2\varepsilon]), \\ 0 & \text{if } u \notin \Phi^{-1}([c-3\varepsilon, c+3\varepsilon]). \end{cases}$$

We consider the differential equation with the initial value $u \in E$

$$\begin{cases} \dfrac{d}{ds}\sigma(s,u) = f(\sigma(s,u))\dfrac{V(\sigma(s,u))}{\|V(\sigma(s,u))\|}, \\ \sigma(0,u) = u, \end{cases}$$

and let $\sigma(s,u)$ be the maximal solution of the above differential equation. It is verified that $\sigma(s,u)$ is well defined for every $(s,u) \in [0,\infty) \times E$. We define $g(t,u) := \sigma(1, \gamma_0(t,u))$ for $(t,u) \in [0,1] \times M$.

Then we can show that g satisfies (A). Indeed, for every $u \in M$, we have $\Phi(\gamma_0(0,u)) = \Phi(u) \geq c_0 > c + 4\varepsilon$. Therefore we obtain $g(0,u) = \sigma(1,\gamma_0(0,u)) = \sigma(1,u) = u$ for every $u \in M$. For every $u \in M$, we have $g(1,u) = \sigma(1,\gamma_0(1,u)) \in U$ because U is an invariant subset with respect to V. Moreover since $\gamma_0(t,u) \in U$ for every $(t,u) \in [t_{\gamma_0}, 1] \times M_0$, we obtain $g(t,u) = \sigma(1,\gamma_0(t,u)) \in U$ for every $(t,u) \in [t_{\gamma_0}, 1] \times M_0$. Finally because $\Phi(\sigma(s,u))$ is non-decreasing in s, we obtain $\Phi(g(t,u)) \geq \Phi(\gamma_0(t,u)) \geq \Phi(u)$ for every $(t,u) \in [0, t_{\gamma_0}] \times M_0$. Hence g satisfies (A).

On the other hand if $c - \varepsilon \leq \Phi(\sigma(s,\gamma_0(t,u))) \leq c + 2\varepsilon$ for every $s \in [0,1]$, then we obtain

$$\begin{aligned} \Phi(g(t,u)) &= \Phi(\sigma(1,\gamma_0(t,u))) \\ &\geq \Phi(\gamma_0(t,u)) + \int_0^1 \frac{1}{3}\|\nabla\Phi(\sigma(s,\gamma_0(t,u)))\|\, ds \\ &\geq c - \varepsilon + 4\varepsilon/3 = c + \varepsilon/3 \end{aligned}$$

from $\Phi(\gamma_0(t,u)) \geq c - \varepsilon$, which holds by (2.3) and the definition of V. Moreover, if there exists some $s \in [0,1]$ such that $\Phi(\sigma(s,\gamma_0(t,u))) \geq c + 2\varepsilon$, then we have $\Phi(g(t,u)) \geq \Phi(\sigma(s,\gamma_0(t,u))) \geq c + 2\varepsilon$. Therefore we obtain $\Phi(g(t,u)) \geq c + \frac{\varepsilon}{3}$ for every $(t,u) \in [0,1] \times M$. This is a contradiction to the definition of c. ∎

For later use, we prepare the following slight generalization of Proposition 2 for a U with specific piecewise smooth boundary.

Proposition 4 *Suppose that there exists an orthogonal decomposition $E = V_\infty \oplus W_\infty$, and Φ satisfies the following condition* (R)*. Then there exists a locally Lipschitz continuous pseudo-gradient vector field V for Φ on $\tilde{E}$ satisfying condition* (R) *with $\nabla\Phi$ replaced by V, for which the region*

$$U := \{\, (v_\infty, w_\infty) \mid \|v_\infty\| > \max\{R_1, \delta\|w_\infty\|^\lambda\}\,\} \tag{2.4}$$

is invariant with respect to V.

(R) *The following* (i) *and* (ii) *hold for some $\lambda \geq 0$, $\delta > 0$, and $R_1 > 0$, where*

$$u = w_\infty + v_\infty \quad (\, w_\infty \in W_\infty \,,\, v_\infty \in V_\infty\,).$$

(i) $\left\langle \nabla\Phi(u)\,,\, v_\infty - \lambda\delta^2 \dfrac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0 \quad$ (if $\|v_\infty\| = \delta\|w_\infty\|^\lambda, \|v_\infty\| \geq R_1$).

(ii) $\langle \nabla\Phi(u)\,,\, v_\infty \rangle > 0 \quad$ (if $\|v_\infty\| \geq \delta\|w_\infty\|^\lambda\,,\, \|v_\infty\| = R_1$).

We omit the proof because we can prove Proposition 4 by the same argument as that in the proof of Proposition 4 of [12].

Remark 5 In [12], it was proved that there exists a p.g.v.f. V for Φ satisfying condition (R) with $\nabla\Phi$ replaced by V, and the region

$$\widetilde{U} := \{\, (v_\infty, w_\infty) \mid \|v_\infty\| < \max\{R_1, \delta\|w_\infty\|^\lambda\}\,\} \tag{2.5}$$

is invariant with respect to $-V$.

Note that $\widetilde{U}$ is the complement of closure of U given by (2.4), and condition (R) says that the inner product of $\nabla\Phi$ and outward normal vector to U is negative on ∂U.

2.2. Local linking and the main result

Let us recall the definition of local linking.

Definition 6 *Let Φ be a C^1 functional on E. Then we define as follows.*

(i) *If there exist an orthogonal decomposition* $E = V_0 \oplus W_0$ *and an* $r > 0$ *satisfying the following condition,* Φ *is said to have a local linking at 0 with respect to* (V_0, W_0) :

$$\begin{cases} \Phi(u) \geq 0 \ (\forall u \in B_r V_0), \\ \Phi(u) \leq 0 \ (\forall u \in B_r W_0), \end{cases} \tag{2.6}$$

where, $B_r V_0 := \{u \in V_0 \,:\, \|u\| \leq r\}$, $B_r W_0 := \{u \in W_0 \,:\, \|u\| \leq r\}$.

(ii) Φ *is said to have a strong local linking at 0 with respect to* (V_0, W_0) *if there exist an* $r > 0$ *satisfying* (2.6)*, and the following properties hold for some* $\varepsilon > 0$:

$$\begin{cases} \Phi(u) \geq \varepsilon \quad \text{on} \quad \partial B_r V_0, \\ \Phi(u) \leq -\varepsilon \ \text{on} \quad \partial B_r W_0. \end{cases} \tag{2.7}$$

We state the $(WPS)_c^*$ condition introduced by [2], which is a generalization of the usual Palais–Smale condition.

Definition 7 *Suppose that a sequence* $\{E_n\}_n$ *of finite dimensional subspaces of* E *satisfy*

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset E, \qquad E = \overline{\bigcup_{n=1}^{\infty} E_n}, \tag{2.8}$$

and let P_n *denote the orthogonal projection from* E *onto* E_n*. Then,*

(i) *a sequence* $\{u_j\}_j$ *is called a* $(PS)_c^*$ *sequence (with respect to* Φ *and* $\{E_n\}_n$*) provided* $u_j \in E_{n_j}$, $n_j \to \infty$, $\Phi(u_j) \to c$ *and* $P_{n_j}(\nabla\Phi(u_j)) \to 0$ *(as* $j \to \infty$*);*

(ii) Φ *is said to satisfy the* $(WPS)_c^*$ *condition if every* $(PS)_c^*$ *sequence has a subsequence weakly convergent to a critical point* u *of* Φ *with* $\Phi(u) = c$.

The following easy paraphrase is useful in proving the existence of a non-trivial critical point.

Lemma 8 *Let* $\{E_n\}_n$ *be as in Definition* 7 *and let* Φ *satisfy the* $(WPS)_c^*$ *condition for every* $c \in \mathbb{R}$*. Moreover, suppose that* 0 *is the only critical value of* Φ*. Then for any* $\varepsilon > 0$ *and* $M > 0$*, there exist some* $b > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\|\nabla\Phi_n(u)\| \geq b, \quad \forall u \in \Phi_n^{-1}\big([-M, -\varepsilon]\big) \cup \Phi_n^{-1}\big([\varepsilon, M]\big) \tag{2.9}$$

holds for every $n \geq n_0$*, where* $\Phi_n := \Phi|_{E_n}$*. (Note that* $\nabla\Phi_n(u) = P_n(\nabla\Phi(u))$ *for* $u \in E_n$*.)*

Now let us collect the conditions relevant to our main result about the critical points.

(Φ1) With respect to a sequence $\{E_n\}_n$ of finite dimensional subspaces satisfying (2.8), Φ satisfies $(\,WPS\,)^*_c$ condition for every $c \in \mathbb{R}$.

(Φ2) Φ is bounded on every bounded set.

(Φ3) There exists an orthogonal decomposition $E = V_0 \oplus W_0$ that satisfies one of the following conditions:

(i) Φ has a strong local linking at 0 with respect to (V_0, W_0).

(ii) Φ has a local linking at 0 with respect to (V_0, W_0), and for some $r > 0$ with the property (2.6), every $(PS)^*_0$ sequence in $B_{2r}E$ has a strongly convergent subsequence.

(Φ4) There exists an orthogonal decomposition $E = V_\infty \oplus W_\infty$ that satisfies the following (i)$\sim$(iii) for some $\lambda \geq 0$, $\delta > 0$, $R_1 > 0$:

(i) $\left\langle \nabla\Phi(u)\,,\, v_\infty - \lambda\delta^2 \dfrac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0, \quad$ (if $\|v_\infty\| = \delta\|w_\infty\|^\lambda\,,\ \|v_\infty\| \geq R_1$),

(ii) $\langle \nabla\Phi(u)\,,\, v_\infty\rangle > 0, \quad$ (if $\|v_\infty\| \geq \delta\|w_\infty\|^\lambda\,,\ \|v_\infty\| \geq R_1$),

(iii) for every $c < 0$ there exists an $R > 0$ such that $\Phi(u) < c$ provided $\|v_\infty\| \leq \delta\|w_\infty\|^\lambda$ and $\|w_\infty\| \geq R$,

where

$$u = w_\infty + v_\infty \quad (\,w_\infty \in W_\infty\,,\, v_\infty \in V_\infty\,).$$

Remark 9 We denote that if Φ has a local linking at 0 with respect to (V_0, W_0), then 0 is a critical point of Φ. Therefore if the assumption (Φ3) holds, then 0 is a critical point of Φ.

When we assume the conditions (Φ1) and (Φ3), we adopt the following notations:

$$\begin{aligned}
&\Phi_n := \Phi|_{E_n}, \\
&\Phi_n^c := \big\{u \in E_n \,:\, \Phi(u) \leq c\big\}, \quad &&(\Phi_n)_c := \big\{u \in E_n \,:\, \Phi(u) \geq c\big\}, \\
&V_0^n := E_n \cap V_0, &&W_0^n := E_n \cap W_0, \\
&V_\infty^n := E_n \cap V_\infty, &&W_\infty^n := E_n \cap W_\infty.
\end{aligned}$$

We say that the sequence $\{E_n\}_n$ in (Φ1) is *compatible* with the orthogonal decomposition $V_0 \oplus W_0$ [*resp.* $V_\infty \oplus W_\infty$] (Φ1) if

(2.10) $\quad E_n = (E_n \cap V_0) \oplus (E_n \cap W_0) \quad [\textit{resp.}\ E_n = (E_n \cap V_\infty) \oplus (E_n \cap W_\infty)]$

holds for every n. The next lemma is found as Lemma 6.5 in [13] and it can be proved by the standard deformation argument. (*cf.* [19, Lemma 2.3])

Lemma 10 (Deformation Lemma) *Suppose* (Φ1) *and* (ii) *of* (Φ3) *hold and there exists no non-zero critical point of* Φ *in* $B_{2r}E$*, where* $r > 0$ *satisfies* (2.6) *in* (Φ3)*. Then there exist some* $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *there exist continuous deformations* $\xi_n, \eta_n \in C([0,1] \times E_n, E_n)$ *satisfying the following conditions:*

(i) $\xi_n(0,\cdot) = \eta_n(0,\cdot) = id$,

(ii) $\xi_n(t,\cdot), \eta_n(t,\cdot)$ *are homeomorphisms from* E_n *to* E_n *for every* $t \in [0,1]$

(iii) $\|\xi_n(t,u) - u\| \leq \frac{r}{2}$, $\|\eta_n(t,u) - u\| \leq \frac{r}{2}$ *for every* $(t,u) \in [0,1] \times E_n$,

(iv) $\sup \Phi \circ \xi_n([0,1] \times B_r W_0^n) = \inf \Phi \circ \eta_n([0,1] \times B_r V_0^n) = 0$,

(v) $\Phi \circ \xi_n(t,\cdot)\,|_{\partial B_r W_0^n} < 0$, $\Phi \circ \eta_n(t,\cdot)\,|_{\partial B_r V_0^n} > 0$ *for every* $t \in (0,1]$,

(vi) $\xi_n(1,u) \subset \Phi_n^{-\varepsilon}$ *for every* $u \in \left(B_{\frac{3}{2}r}E \cap \Phi_n^{\varepsilon}\right) \backslash B_{\frac{r}{3}}E$,

(vii) $\eta_n(1,u) \subset (\Phi_n)_{\varepsilon}$ *for every* $u \in \left(B_{\frac{3}{2}r}E \cap (\Phi_n)_{-\varepsilon}\right) \backslash B_{\frac{r}{3}}E$.

Now we prepare the following definition and lemma to prove our main result.

Definition 11 *Suppose that* (Φ3) *holds, then we define as follows for* $n \in \mathbb{N}$*. Note that condition* (A2) *makes sense for sufficiently large* n*, since there appears the mapping* η_n *in Lemma* 10*.*

(i) *In the case where* (i) *of* (Φ3) *holds, a mapping* $\gamma \in C([0,1] \times B_r V_0^n, E_n)$ *is said to satisfy the condition* (A1) *for a subset* U *if the following conditions* (a) *and* (b) *hold.*

  (a) $\gamma(0,u) = u$ *for every* $u \in B_r V_0^n$ *and* $\gamma(1, B_r V_0^n) \subset U$.

  (b) *There exists some* $t_\gamma > 0$ *such that* $\Phi(\gamma(t,u)) \geq \Phi(u)$ *for every* $(t,u) \in [0,t_\gamma] \times \partial B_r V_0^n$ *and* $\gamma(t,u) \in U$ *for every* $(t,u) \in [t_\gamma, 1] \times \partial B_r V_0^n$

(ii) *Assume that* 0 *is the only critical point of* Φ *in* $B_{2r}E$*. In the case where* (ii) *of* (Φ3) *holds, a mapping* $\gamma \in C([0,1] \times B_r V_0^n, E_n)$ *is said to satisfy the condition* (A2) *for a subset* U *if the following conditions* (a) *and* (b) *hold, where* η_n *denotes a mapping satisfying* (i) $\sim$ (vii) *in Lemma* 10 *and* $r > 0$ *denotes a constant satisfying* (2.6)*.*

  (a) $\gamma(0,u) = u$ *for every* $u \in B_r V_0^n$ *and* $\gamma(1, B_r V_0^n) \subset U$.

(b) *There exist some* $t_\gamma^1 \in (0,1)$ *and* $t_\gamma^2 \in [t_\gamma^1, 1]$ *such that* $\gamma(t,u) = \eta_n(t/t_\gamma^1, u)$ *for every* $(t,u) \in [0, t_\gamma^1] \times \partial B_r V_0^n$, $\Phi(\gamma(t,u)) \geq \Phi(\eta_n(1,u))$ *for every* $(t,u) \in [t_\gamma^1, t_\gamma^2] \times \partial B_r V_0^n$ *and* $\gamma(t,u) \in U$ *for every* $(t,u) \in [t_\gamma^2, 1] \times \partial B_r V_0^n$

Now we state a linking lemma where condition (A1) or (A2) above is concerned. The proof is based on a standard argument using degree theory (*cf.* [2, Lemma3.2.], [10])

Lemma 12 *Suppose that* (Φ3) *holds and* $\{E_n\}$ *is compatible with respect to* $(V_0 \oplus W_0)$. *Let* U *be a subset of* E *satisfying* $\mathrm{dist}(0,U) \geq 2r$, *where* $r > 0$ *satisfies* (2.6). *Then the following* (i) *and* (ii) *hold:*

(i) *If the case* (i) *of* (Φ3) *holds, then* $\gamma \in C([0,1] \times B_r V_0^n, E_n)$ *satisfying (A1) for* U *has the property*

$$\gamma([0,1] \times B_r V_0^n) \cap \partial B_r W_0^n \neq \emptyset$$

for every $n \in \mathbb{N}$.

(ii) *Assume that* 0 *is the only critical point in* $B_{2r}E$. *If the case* (ii) *of* (Φ3) *holds, then* $\gamma \in C([0,1] \times B_r V_0^n, E_n)$ *satisfying* (A2) *for* U *has the property*

$$\gamma([0,1] \times B_r V_0^n) \cap \xi_n(1, \partial B_r W_0^n) \neq \emptyset$$

for every $n \geq n_0$, *where* ξ_n *denotes a mapping satisfying* (i) $\sim$ (vi) *in Lemma* 10 *and* $n_0 \in \mathbb{N}$ *denotes a natural number in Lemma* 10.

Proof. First we prove the case (i) by contradiction. So we suppose that $\gamma \in C([0,1] \times B_r V_0^n, E_n)$ satisfies (A1) for U and

$$\gamma([0,1] \times B_r V_0^n) \cap \partial B_r W_0^n = \emptyset. \tag{2.11}$$

Set $\Omega_n := \mathrm{int}(B_r V_0^n \times B_r W_0^n)$. Since γ satisfies (A1) for U, we have for every $(u_1, u_2) \in \partial B_r V_0^n \times B_r W_0^n$

$$\Phi(\gamma(t,u_1)) \geq \Phi(u_1) > 0 \geq \Phi(u_2) \quad \text{if } t \in [0, t_\gamma]$$

and $\gamma(t,u_1) \in U$ hence $\|\gamma(t,u_1)\| \geq 2r$ if $t \in [t_\gamma, 1]$. Therefore $\gamma(t,u_1) \neq u_2$ for every $t \in [0,1]$ and every $(u_1,u_2) \in \partial B_r V_0^n \times B_r W_0^n$. We define $F_t(u_1,u_2) := \gamma(t,u_1) - u_2$ for $t \in [0,1]$, $(u_1,u_2) \in \Omega_n$. Then the above observation says that $F_t(u_1,u_2) \notin 0$ for every $t \in [0,1]$, $(u_1,u_2) \in \partial B_r V_0^n \times B_r W_0^n$. Moreover the property (2.11) implies that $\gamma(t,u_1) \neq u_2$ for every $t \in [0,1]$, $(u_1,u_2) \in B_r V_0^n \times \partial B_r W_0^n$. So we obtain

$$F_t(u_1,u_2) = \gamma(t,u_1) - u_2 \neq 0 \quad \text{for every } t \in [0,1], (u_1,u_2) \in \partial\Omega_n.$$

Hence by the homotopy invariance of the degree,

$$0 \neq \deg(P^1 - P^2, \Omega_n, 0) = \deg(F_0, \Omega_n, 0) = \deg(F_1, \Omega_n, 0), \tag{2.12}$$

where $P^1 \colon E_n \to V_0^n$ and $P^2 \colon E_n \to W_0^n$ are the orthogonal projections. On the other hand because of $F_1(u_1, u_2) = \gamma(1, u_1) - u_2$ and $\gamma(1, u_1) \in U$ for every $u_1 \in B_r V_0^n$, we have $F_1(u_1, u_2) \neq 0$ for every $(u_1, u_2) \in \Omega_n$. Hence $\deg(F_1, \Omega_n, 0) = 0$ by the property of the degree. So this is a contradiction.

In the case of (ii), we put $G_t(u_1, u_2) := \xi_n(t, u_2) - u_1$ and $F_t(u_1, u_2) := \xi_n(1, u_2) - \gamma(t, u_1)$ for $t \in [0,1]$, $(u_1, u_2) \in \Omega_n$. If we suppose that γ satisfies (A2) for U and

$$\gamma\left([0,1] \times B_r V_0^n\right) \cap \xi_n\left(1, \partial B_r W_0^n\right) = \emptyset,$$

then we can similarly prove that

$$0 \neq \deg(G_0, \Omega_n, 0) = \deg(G_1, \Omega_n, 0) = \deg(F_0, \Omega_n, 0) = \deg(F_1, \Omega_n, 0) = 0,$$

which is a contradiction to (2.12). ∎

The following lemma which is necessary to prove our main result was stated in [12].

Lemma 13 ([12, Lemma 11]) *If Φ satisfies $(\Phi 4)$ with $\{E_n\}_n$ being compatible with respect to $(V_\infty \oplus W_\infty)$, then $\Phi|_{E_n}$ satisfies $(\Phi 4)$ with $(V_\infty \cap E_n, W_\infty \cap E_n)$ instead of $(V_\infty \oplus W_\infty)$ for every $n \in \mathbb{N}$.*

Now we state our main result.

Theorem 14 *Let Φ be a C^1 class functional on a Hilbert space E and let the conditions $(\Phi 1)$ to $(\Phi 4)$ be satisfied with $\{E_n\}_n$ in $(\Phi 1)$ compatible with the decomposition $V_0 \oplus W_0$ in $(\Phi 3)$ and $V_\infty \oplus W_\infty$ in $(\Phi 4)$ (cf.(2.10)). Moreover, suppose*

$$\limsup_{n\to\infty} \left| \dim E_n \cap V_\infty - \dim E_n \cap V_0 \right| > 0. \tag{2.13}$$

holds. Then Φ has at least one non-zero critical point.

Remark 15 In [12], under the assumption (2.13) replaced by the assumption

$$\limsup_{n\to\infty} [\dim E_n \cap W_\infty - \dim E_n \cap W_0] > 0, \tag{2.14}$$

it is shown that Φ has at least one non-zero critical point. Therefore we shall only prove Theorem 14 under the assumption

$$\limsup_{n\to\infty} [\dim E_n \cap V_\infty - \dim E_n \cap V_0] > 0$$

by the compatibility of $\{E_n\}_n$ with the orthogonal decomposition $V_\infty \oplus W_\infty$.

Proof. By Remark 15, we may assume that

$$\limsup_{n\to\infty}[\dim E_n \cap V_\infty - \dim E_n \cap V_0] > 0. \tag{2.15}$$

We prove this theorem by contradiction. So suppose that there exist no critical points other than the origin. We fix $R_2 \geq \max\{R_1, 2r\}$ where R_1 is a constant satisfying (i), (ii) of (Φ4). We define $U_1 := \{(v_\infty, w_\infty) \in V_\infty \oplus W_\infty\,;\, \|v_\infty\| < \max\{R_2, \delta\|w_\infty\|^\lambda\}\}$ and $U_2 := \{(v_\infty, w_\infty) \in V_\infty \oplus W_\infty\,;\, \|v_\infty\| > \max\{R_2, \delta\|w_\infty\|^\lambda\}\}$. Because of the assumption (Φ2) and (Φ4), $C_0 := \sup\{\Phi(u) |\, u \in \overline{U_1}\,\}$ is well defined.

First we consider the easier case where (i) of (Φ3) holds. Then there exist an $r > 0$ and an $\varepsilon > 0$ satisfying (2.7). Suppose $\dim V_0 > 0$. Then $\dim E_n \cap V_0 > 0$ for large n because of the compatibility of $\{E_n\}_n$ with the orthogonal decomposition $V_0 \oplus W_0$. By the assumption (2.15), there exists an increasing sequence $\{n_j\}_j$ of natural numbers satisfying $\dim E_{n_j} \cap V_\infty - \dim E_{n_j} \cap V_0 > 0$. We may also assume that $\dim E_{n_j} \cap V_0 > 0$. By Lemma 8, there exist some $b_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$\|\nabla\Phi_n(u)\| \geq b_1 \quad \text{if } u \in \Phi_n^{-1}([-C_0 - 1, -\varepsilon/2]) \cup \Phi_n^{-1}([\varepsilon/2, C_0 + 1])$$

for every $n \geq n_1$. By Proposition 4 and Lemma 13, there exists some p.g.v.f. V_n for Φ_n satisfying the conditions (i), (ii) of (Φ4) with $\nabla\Phi_n$ replaced by V_n and $U_2 \cap E_n$ is invariant with respect to V_n for every $n \in \mathbb{N}$.

We consider the following differential equation for large j

$$\begin{cases} \dfrac{d}{dt}\eta_j(t,u) = \dfrac{V_{n_j}(\eta_j(t,u))}{\|V_{n_j}(\eta_j(t,u))\|}, \\ \\ \eta_j(0,u) = u \in \partial B_r V_0^{n_j}. \end{cases}$$

We note that $\eta_j(t,u)$ is well defined for every $t \in [0,\infty)$ by $\|\frac{d}{dt}\eta_j(t,u)\| = 1$. Now we choose a T with $T > 3C_0/b_1$ and set $\sigma_j(t,u) := \eta_j(Tt,u)$. Then we obtain $C_0 < \Phi_{n_j}(\eta_j(T,u))$ and $\|\sigma_j(t,u)\| \leq r + T =: R_3$ for every $(t,u) \in [0,1] \times \partial B_r V_0^{n_j}$. Therefore $\sigma_j(1,u) \in U_2$ by the definition of C_0.

We define for $u \in \partial B_r V_0^{n_j}$

$$\sigma_j(t,u) := \begin{cases} (t-1)P_\infty\sigma_j(1,u) + (2-t)\sigma_j(1,u) & \text{if } t \in [1,2], \\ (3-t)P_\infty\sigma_j(1,u) + (t-2)R_3\dfrac{P_\infty\sigma_j(1,u)}{\|P_\infty\sigma_j(1,u)\|} & \text{if } t \in [2,3], \end{cases}$$

where P_∞ is the orthogonal projection onto V_∞. We note that $\sigma_j(t,u) \in U_2$ for every $(t,u) \in [1,3] \times \partial B_r V_0^{n_j}$ and $\|\sigma_j(t,u)\| \leq R_3$ for every $(t,u) \in [0,3] \times \partial B_r V_0^{n_j}$.

Then $\sigma_j(3,\cdot)$ is a continuous map from $\partial B_r V_0^{n_j}$ to $\partial B_{R_3} V_\infty^{n_j}$. Therefore $\sigma_j(3,\cdot)$ is homotopic in $\partial B_r V_\infty^{n_j}$ to a constant map because of $\dim V_\infty^{n_j} > \dim V_0^{n_j}$. (*cf.*[6]) Denoting this homotopy by $H_j(t,u)$ for $(t,u) \in [0,1] \times \partial B_r V_0^{n_j}$ with $H_j(0,u) = \sigma_j(3,u)$ and $H_j(1,u) = a_j$ for a fixed point $a_j \in \partial B_{R_3} V_\infty^{n_j}$. Now we define for $(t,u) \in \partial\left([0,1] \times B_r V_0^{n_j}\right)$

$$\gamma_j(t,u) := \begin{cases} u & \text{if } u \in B_r V_0^{n_j},\ t = 0, \\ \sigma_j(4t,u) & \text{if } u \in \partial B_r V_0^{n_j},\ t \in (0,3/4], \\ H_j(4t-3,u) & \text{if } u \in \partial B_r V_0^{n_j},\ t \in (3/4,1], \\ a_j & \text{if } u \in B_r V_0^{n_j},\ t = 1. \end{cases}$$

Then γ_j is a continuous map from $\partial\left([0,1] \times B_r V_0^{n_j}\right)$ to E_{n_j}. Note that by the Dugundij extension theorem, there exists $\rho_j \in C\left([0,1] \times B_r V_0^{n_j}, E_{n_j}\right)$ such that

$$\rho_j(t,u) = \gamma_j(t,u) \quad \text{if } (t,u) \in \partial\left([0,1] \times B_r V_0^{n_j}\right),$$
$$\|\rho_j(t,u)\| \le R_3 \quad \text{for every } (t,u) \in [0,1] \times B_r V_0^{n_j}.$$

Because of the assumption (Φ2), $C_1 := \inf\{\Phi(u) |\, \|u\| \le R_3\}$ is well defined. Set

$$\Sigma_{n_j}^1 := \left\{\gamma \in C\left([0,1] \times B_r V_0^{n_j}, E_{n_j}\right) \mid \gamma \text{ satisfies (A1) for } U_2\right\},$$
$$c_{n_j}^1 := \sup_{\gamma \in \Sigma_{n_j}^1} \min_{(t,u)} \Phi_{n_j}(\gamma(t,u)).$$

Since ρ_j constructed above belongs to $\Sigma_{n_j}^1$, $\Sigma_{n_j}^1 \ne \emptyset$ and $c_{n_j}^1 \ge C_1$. Using Lemma 12, we have $\gamma\left([0,1] \times B_r V_0^{n_j}\right) \cap \partial B_r W_0^{n_j} \ne \emptyset$ for every $\gamma \in \Sigma_{n_j}^1$. Hence we obtain $\min_{B_r V_0} \Phi = 0 > -\varepsilon = \sup_{\partial B_r W_0^{n_j}} \Phi \ge c_{n_j}^1 \ge C_1$. Note that we may apply Lemma 3 with E, Φ, M, M_0, U replaced by E_{n_j}, Φ_{n_j}, $B_r V_0^{n_j}$, $\partial B_r V_0^{n_j}$, $U_2 \cap E_{n_j}$ respectively, since $U_2 \cap E_{n_j}$ is an invariant set with respect to V_{n_j}. Hence for every j large enough, there exists some $v_j \in E_{n_j}$ such that

$$-\frac{\varepsilon}{2} \ge \Phi_{n_j}(v_j) \ge C_1 - 1 \text{ and } \|\nabla \Phi_{n_j}(v_j)\| \le \frac{1}{j}.$$

This contradicts Lemma 8.

In the case where $V_0 = \{0\}$, then $\Phi(0) = 0$ and $\Phi(u) \le -\varepsilon$ for every u with $\|u\| = r$. Set

$$\Sigma_j := \left\{\gamma \in C([0,1], E_{n_j}); \gamma(0) = 0, \gamma(1) \in U_2\right\} \ne \emptyset.$$

Then we have $\gamma([0,1]) \cap \partial B_r E_{n_j} \ne \emptyset$ for every $\gamma \in \Sigma_j$. Hence we obtain

$$-\varepsilon \ge c_j := \sup_{\gamma \in \Sigma_j} \min_{t \in [0,1]} \Phi(\gamma(t,u)) \ge \inf\{\Phi(u) |\, \|u\| \le R_2 + 1\} > -\infty.$$

By applying Lemma 3 with $M = \{0\}$ and $M_0 = \emptyset$, we can show that for every j large enough, there exists some $v_j \in E_{n_j}$ such that

$$-\frac{\varepsilon}{2} \geq \Phi_{n_j}(v_j) \geq \inf\{\Phi(u) \mid \|u\| \leq R_2 + 1\} - 1 \text{ and } \|\nabla\Phi_{n_j}(v_j)\| \leq \frac{1}{j}$$

Therefore the same contradiction occurs.

The case where (ii) of (Φ3) holds could be treated similarly by using a deformation. Assume that 0 is the only critical point, then we let ξ_{n_j} and η_{n_j} be mappings satisfying (i) $\sim$ (vii) in Lemma 10. Set

$$\Sigma^2_{n_j} := \left\{\gamma \in C([0,1] \times B_r V_0^{n_j}) \mid \gamma \text{ satisfies (A2) for } U_2\right\}.$$

Then we can similarly obtain $\Sigma^2_{n_j} \neq \emptyset$ and the proof similarly goes on by using Lemma 12 and Lemma 3. ∎

2.3. The existence of critical points for even functional

Now we consider the case where Φ is even. At first, we recall the definition of the Krasnoselskii genus.

Definition 16 ([15, 5.1 Definition]) *Let E be a Hilbert space and set*

$$\Sigma := \{A \subset E; A \text{ is closed}, A = -A\}.$$

We define for $A \in \Sigma$

$$i(A) := \begin{cases} \inf\{m \mid \text{there exists an odd mapping } h \in C(A, \mathbb{R}^m \setminus \{0\})\} \\ 0, \quad \text{if } A = \emptyset \\ \infty, \quad \text{otherwise} \end{cases}$$

It is well known that the genus has the following properties.

Proposition 17 ([15, 5.4 Proposition]) *Let A, $B \in \Sigma$ and $h \in C(E,E)$ be an odd map. Then the following hold:*

(i) $i(A) \leq i(B)$ *if* $A \subset B$.

(ii) $i(A \cup B) \leq i(A) + i(B)$.

(iii) $i(A) \leq i(\overline{h(A)})$.

(iv) *If A is compact, there exists a neighborhood $N \in \Sigma$ such that $A \subset N^\circ \subset N$ and $i(N) = i(A)$.*

(v) *If F is a linear subspace of E with $\dim F = n$, and if $A \subset F$ is bounded, open and symmetric neighborhood of the origin in F, then $i(\partial A) = n$.*

We also recall the usual Palais-Smale condition.

Definition 18 *Let X be a Hilbert space and let $\Phi : X \to \mathbb{R}$ be a C^1 class functional. We say that Φ satisfies $(PS)_c$ condition for $c \in \mathbb{R}$ if every $\{u_n\} \subset X$ satisfying $\Phi(u_n) \to c$ and $\nabla\Phi(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.*

Now we state an assumption to describe our result when Φ is even:

(Φ0) $\Phi_n \in C^1(E_n, \mathbb{R})$ satisfies $(PS)_c$ condition for every $n \in \mathbb{N}$ and $c \in \mathbb{R}$.

We introduce the following notations:

$$K := \{u \in E \mid \nabla\Phi(u) = 0\}, \qquad K_c := \{u \in K \mid \Phi(u) = c\}$$
$$K([a,b]) := \{u \in K \mid \Phi(u) \in [a,b]\}.$$

Theorem 19 *Let $\Phi \in C^1(E, \mathbb{R})$ be even and let (Φ0), (Φ1) and (Φ3) hold. Moreover, suppose that there exists a subspace V_∞ with $\inf_{V_\infty} \Phi > -\infty$ and*

$$k := \limsup_{n\to\infty} \left[\, \dim E_n \cap W_0 - \dim E_n \cap W_\infty \right] > 0, \tag{2.16}$$

where $W_\infty := V_\infty^\perp$. Also assume that $\{E_n\}_n$ in (Φ1) is compatible with the orthogonal decompositions $V_\infty \oplus W_\infty$ and $V_0 \oplus W_0$ (cf.(2.10)). Then Φ has at least k pairs of non-trivial critical points in $\Phi^{-1}((-\infty, 0])$.

Proof. We may assume that Φ has only finitely many critical points in $\Phi^{-1}((-\infty, 0])$. Because of the assumption (2.16), there exists an increasing sequence $\{n_j\}_j \subset \mathbb{N}$ such that $\dim(E_{n_j} \cap W_0) = k + \dim(E_{n_j} \cap W_\infty)$.

We define for j sufficiently large and $1 \le l \le k$

$$\Sigma_j^l := \left\{ A \subset E_{n_j}\,;\, \text{compact},\ A = -A,\ \infty > i(A) \ge \dim(E_{n_j} \cap W_\infty) + l \right\},$$

$$c_j^l := \inf_{A \in \Sigma_j^l} \max_{u \in A} \Phi(u).$$

We set $-M := \inf_{V_\infty} \Phi$. Then we can prove $M > 0$, that is $0 > \inf_{V_\infty} \Phi$. Indeed, suppose that this is not the case, then $M = 0$. Note $\inf_{V_\infty} \Phi = 0$. If the case of (i) in (Φ3) holds, we assume that $\varepsilon > 0$ is a constant satisfying (2.7). Then from $i(\partial B_r W_0^{n_j}) \ge \dim(W_\infty \cap E_{n_j}) + k$ and the compatibility of $\{E_n\}_n$, we have $Tokyo J. Math (to appear). A \cap (V_\infty \cap E_{n_j}) \neq \emptyset$ for every $A \in \Sigma_j^1$. Therefore we obtain $0 \le c_j^1 \le \cdots \le c_j^k \le -\varepsilon < 0$. This is a contradiction. Next if the case (ii) of (Φ3) holds, since Φ has only finitely many critical points in $\Phi^{-1}((-\infty, 0])$, then there exist an $\varepsilon_0 > 0$ and some $r_0 > 0$ such that $K([-\varepsilon_0, 0]) \cap B_{2r_0}E = \{0\}$ and $r \geq r_0$, where $r > 0$ is a constant satisfying (2.6). Then by (Φ1) and (ii) of (Φ3), there exist $n_0 \in \mathbb{N}$ and $d > 0$ such that

$$\|\nabla \Phi_n(u)\| \geq d \quad \text{for } {}^\forall u \in \Phi_n^{-1}([-\varepsilon_0, 0]) \cap B_{2r_0}E \setminus B_{r_0/2}E$$

for every $n \geq n_0$. Therefore by the standard deformation argument, there exist $(\varepsilon_0 \geq)\, \varepsilon > 0$ and $\eta_n \in C([0,1] \times E_n, E_n)$ for every $n \geq n_0$ satisfying (a) $\sim$ (e); (a) $\eta_n(0, u) = u$ for every $u \in E_n$, (b) $\Phi_n(\eta_n(1, u)) \leq -\varepsilon$ for every $u \in \partial B_{r_0} W_0^n$ and (c) $\eta_n(t, u)$ is odd in u for every $t \in [0, 1]$. Moreover, we have $i(\eta_n(1, \partial B_{r_0} W_0^n)) \geq i(\partial B_{r_0} W_0^n) = \dim W_0^n$ by using (iii) and (v) in Proposition 17. This yields the same contradiction as in the case (i) of (Φ3). Hence we obtain $M > 0$, that is $\inf_{V_\infty} \Phi < 0$.

Because $\{E_n\}_n$ is compatible with the decomposition $V_\infty \oplus W_\infty$, c_j^l is well defined for $1 \leq l \leq k$ and

$$c_j^l \in [-M, -\varepsilon] \quad (1 \leq l \leq k), \tag{2.17}$$

where $\varepsilon > 0$ is some constant independ of j from above argument and (Φ3). By the standard argument (cf. Theorem 6.1 in [13]), it is shown that $c_j^l \in [-M, -\varepsilon]$ $(1 \leq l \leq k)$ are critical values of Φ_{n_j}, since Φ_{n_j} satisfies $(PS)_c$ for every $c \in \mathbb{R}$. Then, by taking a subsequence if necessary, we may assume that there exist $c^l \in [-M, -\varepsilon]$ $(1 \leq l \leq k)$ such that $c_j^l \to c^l$ as $(j \to \infty)$ for $1 \leq l \leq k$. We note $c^1 \leq c^2 \leq \cdots \leq c^k$. Then we have $K_{c^l} \neq \emptyset$ for $1 \leq l \leq k$ since (Φ1) holds and c_j^l is a critical values of Φ_{n_j}. Suppose that $1 \leq l \leq k$ and $m(l) =: m \in \mathbb{N} \cup \{0\}$ the largest integer such that $l + m \leq k$ and $c := c^l = \cdots = c^{l+m}$. There exists some $\delta_0 > 0$ such that

$$K([c - \delta_0, c + \delta_0]) \setminus K_c = \emptyset$$

since c is an isolated critical value. Let $K_c := \{\pm u_1, \cdots, \pm u_p\}$ with $u_i \neq u_j$ for every $i \neq j$. Next we choose weakly open convex neighborhoods N_q of u_q $(1 \leq q \leq p)$ such that

$$\operatorname{dist}(\overline{N_q}, -\overline{N_q}) > 0 \text{ and } \operatorname{dist}(\overline{N_{q'}}, \overline{N_q} \cup -\overline{N_q}) > 0 \ (q \neq q'). \tag{2.18}$$

Indeed, we shall show (2.18) in l^2 because E is a separable Hilbert space. So let $u_i := (u_n^i)_{n=1}^\infty \in l^2$ $(1 \leq i \leq p)$. Since $u_i \neq u_j$ for every $i \neq j$, there exists some $N \in \mathbb{N}$ such that $u_i^N \neq u_j^N$ in $\mathbb{R}^N$ $(i \neq j)$, where $u_i^N := (u_n^i)_{n=1}^N$ $(1 \leq i \leq p)$. Therefore there exists some $\delta > 0$ such that

$$\operatorname{dist}(\overline{B_q}, -\overline{B_q}) > 0 \text{ and } \operatorname{dist}(\overline{B_{q'}}, \overline{B_q} \cup -\overline{B_q}) > 0 \ (q \neq q'),$$

where $B_q := B_\delta(u_q^N) := \{v \in \mathbb{R}^N \,|\, \operatorname{dist}(u_q^N, v) < \delta\}$. Hence we can choose $N_q := \{v \in l^2 \,|\, \operatorname{dist}(u_q^N, v^N) < \delta\}$.

Next we choose $\delta_1 > 0$ satisfying

$$\operatorname{dist}\left((N_q)_{\delta_1}, (\pm N_{q'})_{\delta_1}\right) > 0 \ (q \neq q') \text{ and } \operatorname{dist}\left((N_q)_{\delta_1}, (-N_q)_{\delta_1}\right) > 0,$$

where $\pm N_q := N_q \cup (-N_q)$ and $(N_q)_{\delta_1} := \{u \in E \mid \operatorname{dist}(u, N_q) \leq \delta_1\}$. We set $N := (\pm N_1) \cup \cdots \cup (\pm N_p)$. Then we can easily see that $i((N)_{\delta_1}) = 1$. On the other hand, the assumption $(\Phi 1)$, there exists some $b > 0$ and $n_1 \in \mathbb{N}$ such that

$$\|\nabla \Phi_n(u)\| \geq b \quad \text{if } u \in \Phi_n^{-1}([c - \delta_0, c + \delta_0]) \setminus N$$

for every $n \geq n_1$. Indeed, we assume that for every $n \in \mathbb{N}$ there exists some $u_n \in \Phi_n^{-1}([c-\delta_0, c+\delta_0]) \setminus N$ such that $\|\nabla \Phi_n(u_n)\| \leq 1/n$. Then we have some subsequence $\{u_{n_j}\}$ of $\{u_n\}$ being a $(WPS)^*_{c'}$ sequence ($c' \in [c-\delta_0, c+\delta_0]$). By the assumption $(\Phi 1)$, $\{u_{n_j}\}$ has a subsequence weakly convergent to a critical point u_0 of Φ with $\Phi(u_0) = c'$. On the other hand, since N is a weakly open neighborhood of K_c, we obtain $u_0 \notin N$, that is $u_0 \notin K_c$ and $c \neq c'$. This is a contradiction because c is the unique critical value in $[c - \delta_0, c + \delta_0]$.

Then we put $\varepsilon_1 := \min\{\delta_0/3, \delta_1 b/12, b/6\}$, by the standard deformation argument, for every $n \geq n_1$, there exists $\eta_n \in C([0,1] \times E_n, E_n)$ satisfying (a) $\sim$ (e):

(a) $\eta_n(t, u)$ is non-increasing in t for every $u \in E_n$,

(b) $\eta_n(t, u) = u$ for $t \in [0,1]$, $u \notin \Phi_n^{-1}([c-3\varepsilon_1, c+3\varepsilon_1])$ and also for $t \in [0,1]$, $u \in N$,

(c) $\eta_n(1, \overline{D \setminus (N)_{\delta_1}}) \subset \Phi^{c-\varepsilon_1}$ if $D \subset E_n$ satisfies $D \subset \Phi_n^{c+\varepsilon_1}$,

(d) $\eta_n(t, u)$ is odd in u for every $t \in [0,1]$,

(e) $\eta_n(t, \cdot)$ is a homeomorphism from E_n to E_n for every $t \in [0,1]$

Because c_j^l and c_j^{l+m} are convergent to c, there exists some $n_2 \geq n_1$ such that $c_j^l, c_j^{l+m} \in [c - \varepsilon_1/2, c + \varepsilon_1/2]$ for every j satisfying $n_j \geq n_2$. Now we fix j such that $n_j \geq n_2$. By the definition of c_j^{l+m}, there exists some $D \in \Sigma_j^{l+m}$ such that $\max_D \Phi_{n_j} \leq c+\varepsilon_1$. Using above (c), we obtain $\eta_{n_j}(1, \overline{D \setminus (N)_{\delta_1}}) \subset \Phi^{c-\varepsilon_1}$. Hence

$$i\left(\eta_{n_j}(1, \overline{D \setminus (N)_{\delta_1}})\right) \leq \dim(E_{n_j} \cap W_\infty) + (l-1)$$

holds, by the definition of c_j^l and $c - \varepsilon_1 < c_j^l$. Therefore we obtain

$$\begin{aligned}
\dim(E_{n_j} \cap W_\infty) + l + m &\leq i(D) \leq i(\overline{D \setminus (N)_{\delta_1}}) + i((N)_{\delta_1}) \\
&\leq i\left(\eta_{n_j}(1, \overline{D \setminus (N)_{\delta_1}})\right) + 1 \\
&\leq \dim(E_{n_j} \cap W_\infty) + (l-1) + 1.
\end{aligned}$$

Hence we have $m = 0$. This yields $-M \leq c^1 < c^2 < \cdots < c^k \leq -\varepsilon$. Therefore Φ has at least k pairs of non-trivial critical points in $\Phi^{-1}((-\infty, 0])$. ∎

§3. Applications

Let us return to the nonlinear wave equation (WE):

$$\text{(WE)}\qquad \begin{cases} \Box u(x,t) = h(x,t,u(x,t)), & (0 < x < \pi,\ t \in \mathbb{R}), \\ u(0,t) = u(\pi,t) = 0, & (t \in \mathbb{R}), \\ u(x,t+2\pi) = u(x,t), & (0 < x < \pi,\ t \in \mathbb{R}). \end{cases}$$

The nonlinear term $h\colon [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}$ is assumed to satisfy the following conditions (h1) to (h3).

(h1) h is continuous and $h(x,t+2\pi,\xi) = h(x,t,\xi)$ $((x,t,\xi) \in [0,\pi] \times \mathbb{R}^2)$.

(h2) h is non-decreasing in ξ and $h(x,t,\xi) \neq 0$ $(\xi \neq 0)$.

(h3) There exist constants $b_0 \geq 0$, $b > 0$ that satisfy the following properties:
$g_0(x,t,\xi) := h(x,t,\xi) - b_0\xi = o(|\xi|)$ as $\xi \to 0$ uniformly in (x,t),
$g(x,t,\xi) := h(x,t,\xi) - b\xi = o(|\xi|)$ as $|\xi| \to \infty$ uniformly in (x,t).

Let $Q := (0,\pi) \times (0,2\pi)$ and define

$$b_0^+ := \min\{\, \lambda \mid \lambda \in \sigma(\Box),\ b_0 < \lambda \,\},\ b_0^- := \max\{\, \lambda \mid \lambda \in \sigma(\Box),\ \lambda < b_0 \,\},$$

where $\Box$ (D'Alembertian) means the self-adjoint operator in $L^2(Q)$ obtained as the closure of $\partial^2/\partial t^2 - \partial^2/\partial x^2$ with domain $\{\, u \in C^2([0,\pi]\times\mathbb{R}) \mid u(x,t+2\pi) = u(x,t),\ u(0,t) = u(\pi,t) = 0 \,\}$.

Theorem 20 *Assume that the non-linear term h of the equation* (WE) *satisfies the conditions* (h1) $\sim$ (h3) *and let b_0, g_0, b and g be as in* (h3). *Set $G(x,t,\xi) := \int_0^\xi g(x,t,s)\,ds$, $G_0(x,t,\xi) := \int_0^\xi g_0(x,t,s)\,ds$ and consider the following conditions:*

(C1) *g is bounded, and $G(x,t,\xi) \to +\infty$ (as $|\xi| \to \infty$) uniformly in (x,t),*

(C2) *the following condition* (a1) *or* (a2) *holds for some constants $0 < \alpha \leq \beta < 1$ satisfying $\beta - \frac{\alpha}{2} < \frac{1}{2}$, $c_1, c_2 > 0$, and $d_1, d_2 \geq 0$:*

(a1) $|g(x,t,\xi)| \leq c_1|\xi|^\beta + d_1,\ G(x,t,\xi) \geq c_2|\xi|^{\alpha+1} - d_2|\xi|,$
(a2) $|g(x,t,\xi)| \leq c_1|\xi|^\beta + d_1,\ G(x,t,\xi) \leq -c_2|\xi|^{\alpha+1} + d_2|\xi|.$

(C3) *There exists a $\delta > 0$ such that $G_0(x,t,\xi) \geq 0$ if $|\xi| \leq \delta$,*

(C4) *There exists a $\delta > 0$ such that $G_0(x,t,\xi) \leq 0$ if $|\xi| \leq \delta$.*

Then (WE) *has a non-trivial weak solution in each of the following cases* (A1) *to* (A4)*:*

(A1) $b_0 \notin \sigma(\Box)$, $b \notin \sigma(\Box)$ *and* $b \notin [b_0^-, b_0^+)$*;*

(A2) $b_0 \in \sigma(\Box)$, $b \notin \sigma(\Box)$, *and one of the following conditions hold:*

> (1) $b \notin [b_0, b_0^+)$ *and* (C3)*;*
> (2) $b \notin [b_0^-, b_0)$ *and* (C4)*;*

(A3) $b_0 \notin \sigma(\Box)$, $b \in \sigma(\Box)$, *and one of the following conditions holds:*

> (1) $b \notin [b_0^-, b_0]$ *and* (C1) *or* (a1) *of* (C2)*;*
> (2) $b \notin [b_0, b_0^+]$ *and* (a2) *of* (C2)*;*

(A4) $b_0 \in \sigma(\Box)$, $b \in \sigma(\Box)$, *and one of the following conditions holds:*

> (1) (C3), $b_0 \neq b$ *and* (C1) *or* (a1) *of* (C2)*;*
> (2) (C3), $b_0^+ \neq b$ *and* (a2) *of* (C2)*;*
> (3) (C4), $b_0^- \neq b$ *and* (C1) *or* (a1) *of* (C2)*;*
> (4) (C4), $b_0 \neq b$ *and* (a2) *of* (C2)*;*

Remark 21 In the nonresonant case ($b \notin \sigma(\Box)$), Theorem 20 is contained in the results of [2] and [12].

In the resonant case ($b \in \sigma(\Box)$), the condition imposed on b_0 and b is more general than the result of [12]. In [12], provided the case (1) of (A4), the existence of a weak nontrivial solution to (WE) is proved only under the condition of $b_0 < b$. On the other hand, Theorem 20 shows that the condition $b_0 > b$ in the case (1) of (A4) also implies the existence of a nontrivial solution.

We note that $b_0 = 0$ yields condition (C3) by the assumption (h2) and $h(x,t,\xi) = g_0(x,t,\xi)$.

Although the difficult part of the proof of Theorem 20 relies on Theorem 14 in Section 2, the fundamental plan of proof is almost parallel to that in [12]. So, we only give the proof for the case of assumption (1) of (A3). First we recall the variational setting for the existence of a weak solution to (WE).

By the Fourier series expansion, every real-valued $u \in L^2(Q)$ can be written as

$$u(x,t) = \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} u_{kj} \sin jx \, e^{ikt}$$

with $\overline{u_{kj}} = u_{-kj}$ for all j, k. Using this expansion, we set

$$\|u\|_E := \left(\pi^2 \sum_{j \neq |k|} |j^2 - k^2| |u_{kj}|^2 + \pi^2 \sum_{j=|k|} |u_{kj}|^2 \right)^{1/2}$$

and we define the space E by

$$E := \{\, u \in L^2(Q) \mid \|u\|_E < \infty \,\}.$$

Then E is a Hilbert space with the inner-product

$$\langle u, v\rangle_E := \pi^2 \sum_{j\neq|k|} |j^2 - k^2| u_{kj}\overline{v_{kj}} + \pi^2 \sum_{j=|k|} u_{kj}\overline{v_{kj}},$$

where u_{kj}, v_{kj} are Fourier coefficients of u, v respectively. E has an orthogonal decomposition $E = E^+ \oplus E^0 \oplus E^-$ where

$$E^+ := \left\{ u \in E \;:\; u(x,t) = \sum_{j^2>k^2} u_{kj} \sin jx e^{ikt} \right\},$$

$$E^0 := \left\{ u \in E \;:\; u(x,t) = \sum_{j^2=k^2} u_{kj} \sin jx e^{ikt} \right\},$$

and

$$E^- := \left\{ u \in E \;:\; u(x,t) = \sum_{j^2<k^2} u_{kj} \sin jx e^{ikt} \right\}.$$

The orthogonal projections onto E^-, E^0 and E^+ are designated by P^-, P^0 and P^+, respectively.

It is well known that the inclusions $E^\pm \hookrightarrow L^2(Q)$ are compact and E^0 is a closed subspace of $L^2(Q)$.

For each $n \in \mathbb{N}$, we set

$$E_n := \operatorname{span}\{\sin jx \sin kt, \sin jx \cos kt \;:\; 0 < j \le n, |k| \le n\}.$$

Then $\{E_n\}_n$ is an increasing sequence of finite dimensional subspace of E with $\cup_{n=1}^\infty E_n$ being dense in E. Let us note that this sequence is compatible with the decomposition $E = E^- \oplus E^0 \oplus E^+$, i.e., the orthogonal projection onto E_n commutes with P^-, P^0 and P^+ for every n.

Consider the functional Φ defined on E by

$$\Phi(u) := \frac{1}{2}\int_Q (u_x^2 - u_t^2)dxdt - \int_Q H(x,t,u)\,dxdt \tag{3.1}$$

$$= \frac{1}{2}(\|P^+u\|^2 - \|P^-u\|^2) - \Psi(u), \tag{3.2}$$

where $H(x,t,\xi) := \int_0^\xi h(x,t,s)\,ds$, $\Psi(u) := \int_Q H(x,t,u)\,dxdt$. Under the conditions (h1) to (h3), it is clear that $\Phi(u)$ is a C^1 class functional on E with $\langle \nabla\Phi(u), v\rangle_E = \langle (P^+ - P^-)u, v\rangle_E - \int_Q h(x,t,u(x,t))v(x,t)\,dxdt$.

It is well known that a critical point of Φ is a weak solution to (WE).

The proof for the case (1) of (A3)

We shall show that the functional Φ defined by (3.2) satisfies the assumptions $(\Phi 1) \sim (\Phi 4)$ in Theorem 14 and the dimension condition (2.14) or (2.15) in Remark 15 holds.

In [2], it is shown that Φ satisfies condition (ii) of $(\Phi 3)$ with respect to $(V_0, W_0) = (X_0^+, X_0^-)$, where

$$X_0^+ := \left\{ w \in E \;:\; w(x,t) = \sum_{j^2-k^2>b_0} u_{kj} \sin jx e^{ikt} \right\},$$

$$X_0^- := \left\{ w \in E \;:\; w(x,t) = \sum_{j^2-k^2<b_0} u_{kj} \sin jx e^{ikt} \right\}.$$

Moreover, in [12], it is shown that Φ satisfies $(\Phi 1)$, $(\Phi 2)$ and $(\Phi 4)$ with respect to $(V_\infty, W_\infty) = (X^+, X^- \oplus X^0)$, where

$$X^+ := \left\{ u \in E \;:\; u(x,t) = \sum_{j^2-k^2>b} u_{kj} \sin jx e^{ikt} \right\},$$

$$X^0 := \left\{ u \in E \;:\; u(x,t) = \sum_{j^2-k^2=b} u_{kj} \sin jx e^{ikt} \right\},$$

$$X^- := \left\{ u \in E \;:\; u(x,t) = \sum_{j^2-k^2<b} u_{kj} \sin jx e^{ikt} \right\}.$$

Finally we check the dimension condition. First we suppose that $b < b_0^-$. Let $E(\lambda)$ be the eigenspace of $\lambda \in \sigma(\Box)$. Then the definition of X^+, X_0^+ and the assumption $b < b_0^-$ imply $X_0^+ \oplus E(b_0^-) \subset X^+$. Moreover $E(b_0^-) \subset E_n$ for large $n \in \mathbb{N}$. Therefore, if n is large enough, we obtain

$$E(b_0^-) \oplus \left(E_n \cap X_0^+\right) = E_n \cap \left(X_0^+ \oplus E(b_0^-)\right) \subset E_n \cap X^+.$$

Hence

$$\liminf_{n\to\infty} [\dim(E_n \cap V_\infty) - \dim(E_n \cap V_0)] \geq \dim E(b_0^-) > 0.$$

Next we assume $b > b_0$. Then we similarly have $X_0^- \subset X^0 \oplus X^-$ and $E(b) \subset E_n$ for large $n \in \mathbb{N}$. Therefore, if n is large enough, we obtain

$$\left(E_n \cap X_0^-\right) \oplus E(b) = E_n \cap \left(X_0^- \oplus X^0\right) \subset E_n \cap \left(X^- \oplus X^0\right).$$

This implies

$$\liminf_{n\to\infty}[\dim(E_n \cap W_\infty) - \dim(E_n \cap W_0)] \geq \dim E(b) > 0.$$

■

If the nonlinear term h is odd in ξ, then we obtain the following result by applying Theorem 19 in Section 2. We omit the proof here because it is easy to check the assumptions in Theorem 19 for Φ defined by (3.1). Indeed, we can prove the boundness of Palais–Smale sequences of Φ_n defined on each finite dimension subspace E_n by the same argument as in [2, Proposition 2.6] and [12, Proposition 18].

Theorem 22 *Assume that the non-linear term h of the equation* (WE) *satisfies the conditions* (h1) $\sim$ (h3) *and let b_0, g_0, b and g be as in* (h3). *Moreover we suppose that $h(x,t,\xi)$ is odd in ξ. Then* (WE) *has at least k pairs of weak solutions in each of the following cases* (A1) *to* (A4)*:*

(A1) *$b_0 \notin \sigma(\Box)$, $b \notin \sigma(\Box)$ and $b \notin [b_0^-, b_0^+)$ with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_5$ (if $b \geq b_0^+$);*

(A2) *$b_0 \in \sigma(\Box)$, $b \notin \sigma(\Box)$, and one of the following conditions hold:*

- (1) *$b \notin [b_0, b_0^+)$ and* (C3) *with $k = \sharp K_3$ (if $b < b_0$), $k = \sharp K_5$ (if $b_0^+ \leq b$);*
- (2) *$b \notin [b_0^-, b_0)$ and* (C4) *with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_6$ (if $b_0 \leq b$);*

(A3) *$b_0 \notin \sigma(\Box)$, $b \in \sigma(\Box)$, and one of the following conditions holds:*

- (1) *$b \notin [b_0^-, b_0]$ and* (C1) *or* (a1) of (C2)
  with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_7$ (if $b_0 < b$);
- (2) *$b \notin [b_0, b_0^+]$ and* (a2) of (C2)
  with $k = \sharp K_2$ (if $b < b_0$), $k = \sharp K_5$ (if $b_0^+ < b$);

(A4) *$b_0 \in \sigma(\Box)$, $b \in \sigma(\Box)$, and one of the following conditions holds:*

- (1) (C3), *$b_0 \neq b$ and* (C1) *or* (a1) of (C2)
  with $k = \sharp K_7$ (if $b_0 < b$), $k = \sharp K_3$ (if $b < b_0$);
- (2) (C3), *$b_0^+ \neq b$ and* (a2) of (C2)
  with $k = \sharp K_4$ (if $b < b_0^+$), $k = \sharp K_5$ (if $b_0^+ < b$);
- (3) (C4), *$b_0^- \neq b$ and* (C1) *or* (a1) of (C2)
  with $k = \sharp K_1$ (if $b < b_0^-$), $k = \sharp K_8$ (if $b_0^- < b$);

(4) (C4), $b_0 \neq b$ *and* (a2) of (C2)
with $k = \sharp K_2$ *(if* $b < b_0$*)*, $k = \sharp K_6$ *(if* $b_0 < b$*)*;

where

$$\begin{aligned}
K_1 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b < j^2 - k^2 < b_0\right\},\\
K_2 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b \leq j^2 - k^2 < b_0\right\},\\
K_3 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b < j^2 - k^2 \leq b_0\right\},\\
K_4 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b \leq j^2 - k^2 \leq b_0\right\},\\
K_5 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b_0 < j^2 - k^2 < b\right\},\\
K_6 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b_0 \leq j^2 - k^2 < b\right\},\\
K_7 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b_0 < j^2 - k^2 \leq b\right\},\\
K_8 &:= \left\{(j,k) \in \mathbb{N} \times \mathbb{Z} \,|\, b_0 \leq j^2 - k^2 \leq b\right\}.
\end{aligned}$$

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