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The Basis Number of The Lexicographic Product of Different Ladders

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Abstract. The basis number of a graph G is defined to be the least integer d such that there is a basis $\mathcal{B}$ of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. We investigate the basis number of the lexicographic product of two circular ladders, two Möbius ladders, a circular ladder and a Möbius ladder, a Möbius ladder and a circular ladder, a ladder and a circular ladder, a circular ladder and a ladder, a Möbius ladder and a ladder, a ladder and a Möbius ladder, and two ladders.

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§1. Introduction

For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in G generate a subspace called the cycle space of G denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r \qquad (1)$$

where r is the number of connected components.

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a $d-$fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. The basis number $b(G)$ of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The basis number was introduced by Schmeichel [12] in 1981, but already in 1937 MacLane [11] gave a criterion for a graph to be planar. In fact, MacLane proved that a graph is planar if and only if its basis number is less than or equal to 2.

In 1990, Hulsurkar [7] studied the graph structure of Weyl groups and proved that most of the graphs $\Gamma(W)$ are non planar ($b(\Gamma(W)) \geq 3$, (the exact basis number for those graphs is still unknown) which plays an important role in studying the modular representation on the semi-simple Lie algebra and Chevalley groups[13]. The basis number of certain classes of non planar graphs plays an important role in studying the graphs $\Gamma(W)$ where $\Gamma(W)$ is the graph defined for Weyl groups which is compatible with the partial order introduced earlier for the proof of Verma's conjecture on Weyl's dimension polynomial [8].

In 1981, Schmeichel [12] proved that $b(K_n) = 3$ whenever $n \geq 5$ and $b(K_{n,m}) \leq 4$ for each n and m. In 1982, Bank and Schmeichel [5] proved that $b(Q_n) = 4$ whenever $n \geq 7$. Many papers appeared to investigate the basis number of certain graphs, especially the graph products, see [1], [3], [4], [9] and [10].

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The cartesian product $G^* = G \times H$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$. The lexicographic product $G^* = G[H]$ has the vertex set $V(G^*) = V(G) \times V(H)$ and the edge set $E(G^*) = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$.

The required basis of $\mathcal{C}(G)$ is a basis that is $b(G)$-fold. Let G and H be two graphs and $\varphi : G \rightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\mathcal{B}' = \{\varphi(c) | c \in \mathcal{B}\}$ is called the corresponding (required) basis of $\mathcal{B}$ in H. A graph is called perfect matching if the degree of each vertex is 1.

Throughout this work $f_B(e)$ stands for the number of cycles in B containing e where $B \subset \mathcal{C}(G)$, $\mathcal{B}_G$ stands for the required basis of G, and $\lfloor x \rfloor$ stands for the greatest integer less than or equal to x. Finally, if B is a set of cycles, then $E(B) = \cup_{c \in B} E(c)$.

§2. Main Results

In this section, we compute the basis number of the lexicographic product of two circular ladders, two Möbius ladders, a circular ladder and a Möbius ladder, a Möbius ladder and a circular ladder, a ladder and a circular ladder, a circular ladder and a ladder, a Möbius ladder and a ladder, a ladder and a Möbius ladder, and two ladders. Actually we show, under some restrictions on their orders, the basis number is 4. Let $u_1, u_2, ..., u_{2m}$ and $a_1, a_2, \ldots, a_{2n}$

be sets of vertices. For any edge $e = ab$ one can easily see that $e[N_{2m}]$ is isomorphic to $K_{2m,2m}$ where N_{2m} is a null graph with $2m$ vertices. Moreover, $e[P_{2m}]$ is decomposable into $e[N_{2m}] \cup (a \times P_{2m}) \cup (b \times P_{2m})$, where P_{2m} is a path of order $2m$. Let

$$\mathcal{A}_{ab} = \{(a, u_j)(b, u_l)(a, u_{j+1})(b, u_{l+1})(a, u_j) : 1 \leq j, l \leq 2m - 1\}.$$

Then $\mathcal{A}_{ab}$ is the Schemeichel's 4-fold basis of $\mathcal{C}(e[N_{2m}])$ (see Theorem 2.4 in [11]). Moreover, (1) if $e = (a, u_1)(b, u_{2m})$ or $e = (a, u_{2m})(b, u_1)$ or $e = (a, u_1)(b, u_1)$ or $e = (a, u_{2m})(b, u_{2m})$, then $f_{\mathcal{A}_{ab}}(e) = 1$. (2) If $e = (a, u_1)(b, u_l)$ or $(a, u_j)(b, u_1)$ or $(a, u_{2m})(b, u_l)$ or $(a, u_j)(b, u_{2m})$, then $f_{\mathcal{A}_{ab}}(e) \leq 2$. (3) if $e \in E(e[N_{2m}])$ and is not of the above form, then $f_{\mathcal{A}_{ab}}(e) \leq 4$. Also, for any edge $e = ab$, set

$$\begin{aligned}
\mathcal{D}_a &= \{\mathcal{D}_a^{(j)} = (a, u_j)(a, u_{j+1})(b, u_{2m})(a, u_j) : 1 \leq j \leq 2m - 1\},\\
\mathcal{D}_b &= \{\mathcal{D}_b^{(j)} = (b, u_j)(b, u_{j+1})(a, u_{2m})(b, u_j) : 1 \leq j \leq 2m - 1\},
\end{aligned}$$

$$\mathcal{D}_{ab} = \mathcal{D}_a \cup \mathcal{D}_b.$$

Lemma 2.1. *For any edge $e = ab$, $\mathcal{B}_{ab} = \mathcal{A}_{ab} \cup \mathcal{D}_{ab}$ is a linearly independent set of cycles.*

Proof. Since $\mathcal{A}_{ab}$ is a basis of $\mathcal{C}(e[N_{2m}])$, $\mathcal{A}_{ab}$ is a linearly dependent set of cycles. Since each cycle $\mathcal{D}_a^{(j)}$ contains $(a, u_j)(a, u_{j+1})$ which is not in any other cycle of $\mathcal{D}_a \cup \mathcal{A}_{ab}$, $\mathcal{D}_a \cup \mathcal{A}_{ab}$ is linearly independent. Similarly, each cycle $\mathcal{D}_b^{(j)}$ contains $(b, u_j)(b, u_{j+1})$ which is not in any other cycle of $\mathcal{D}_a \cup \mathcal{D}_b \cup \mathcal{A}_{ab}$. Therefore, $\mathcal{B}_{ab}$ is a linearly independent set of cycles. The proof is complete.

Lemma 2.2. *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and $E(A) \cap E(B)$ induces a forest in G (we allow the possibility that $E(A) \cap E(B) = \phi$). Then $A \cup B$ is linearly independent.*

Proof. Assume that $A \cup B$ is linearly independent. Then there are $C_1, C_2, \ldots, C_l \in A$ and $D_1, D_2, \ldots, D_t \in B$ such that $\sum_1^l C_i = \sum_1^t D_i$ (mod 2). Hence,

$$E(C_1 \oplus C_2 \oplus \cdots \oplus C_l) = E(D_1 \oplus D_2 \oplus \cdots \oplus D_t) \subseteq E(A) \cap E(B),$$

where $\oplus$ is the ring sum. So $C_1 \oplus C_2 \oplus \cdots \oplus C_l$ and $D_1 \oplus D_2 \oplus \cdots \oplus D_t$ are subsets of a forest which contradicts the fact that any linear combination of cycles of linearly independent set of cycles is a cycle or an edge disjoint union of cycles. The proof is complete.

Lemma 2.3. $(\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}})\cup(\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}})$ *is a linearly independent set.*

Proof. By Lemma 2.1, $\mathcal{B}_{a_{n-i}a_{n+i+1}}$ is a linearly independent set of cycles for each $i=1,2,\ldots,n-1$. It is easy to see that the subgraph consisting of $\{a_{n-i}a_{n+i+1}; i=1,2,\ldots,n-1\}$ is a perfect matching. Thus, $E(\mathcal{B}_{a_{n-i}a_{n+i+1}})\cap E(\mathcal{B}_{a_{n-r}a_{n+r+1}})=\phi$ for each $i\neq r$. So, $\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}$ is a linearly independent set. We now proceed using induction on n to show that $\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}$ is linearly independent. By Lemma 2.1, $\mathcal{B}_{a_ia_{i+1}}$ is linearly independent for each $i=1,2,\ldots,2n-1$. Thus, the result is true for $n=2$. Assume that $n>2$ and the result is true for smaller values of n (i,e. $\cup_{i=1}^{2n-3}\mathcal{B}_{a_ia_{i+1}}$ is linearly independent). Since $E(\mathcal{B}_{a_{2n-2}a_{2n-1}})\cap E(\mathcal{B}_{a_{2n-1}a_{2n}})=\{(a_{2n-1,}u_j)(a_{2n-1,}u_{j+1})|j=1,2,\ldots 2m-1\}$, which is an edge set of a path, then, by lemma 2.2, $\mathcal{B}_{a_{2n-2}a_{2n-1}}\cup\mathcal{B}_{a_{2n-1}a_{2n}}$ is linearly independent. Similarly, $E(\cup_{i=1}^{2n-3}\mathcal{B}_{a_ia_{i+1}})\cap E(\mathcal{B}_{a_{2n-2}a_{2n-1}}\cup\mathcal{B}_{a_{2n-1}a_{2n}})=\{(a_{2n-2,}u_j)(a_{2n-2,}u_{j+1})|j=1,2,\ldots 2m-1\}$ which is an edge set of a path. Thus, by lemma 2.2, $\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}$ is linearly independent. Finally, note that $E(\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}})\cap E(\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}})=\{(a_{n-i,}u_j)(a_{n-i,}u_{j+1})|i=1,2,\ldots,n-1; j=1,2,\ldots 2m-1\}\cup\{(a_{n+i+1,}u_j)(a_{n+i+1,}u_{j+1})|i=1,2,\ldots,n-1; j=1,2,\ldots 2m-1\}$ which forms edges of a forest. Thus, by Lemma 2.2, $(\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}})\cup(\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}})$ is a linearly independent set. The proof is complete.

Now, for $i=1,2,\ldots,2n$ and $j=1,2,\ldots,m-1$, we set

$$\mathcal{F}^{(a_i)}_{m-j,m+j+1}=(a_i,u_{m-j})(a_i,u_{m+j+1})(a_i,u_{m+j})(a_i,u_{m-j+1})(a_i,u_{m-j}),$$

for $i=1,2,\ldots,2n-1, 1<s\leq 2m-1$ and $1\leq t<2m-1$, set

$$\begin{aligned}\mathcal{G}^{(a_i)}_{1,s} &= (a_i,u_1)(a_i,u_s)(a_{i+1},u_1)(a_i,u_{2m})(a_i,u_1),\\ \mathcal{G}^{(a_i)}_{2m,t} &= (a_i,u_{t+1})(a_i,u_{2m})(a_{i+1},u_1)(a_i,u_{t+1}).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{G}^{(a_{2n})}_{1,s} &= (a_{2n},u_1)(a_{2n},u_s)(a_1,u_1)(a_{2n},u_{2m})(a_{2n},u_1),\\ \mathcal{G}^{(a_{2n})}_{2m,t} &= (a_{2n},u_{t+1})(a_{2n},u_{2m})(a_1,u_1)(a_{2n},u_{t+1}),\end{aligned}$$

$$\mathcal{F}^{(a_i)}=\cup_{j=1}^{m-1}\mathcal{F}^{(a_i)}_{m-j,m+j+1} \text{ and } \mathcal{G}^{(a_i)}_{s,t}=\mathcal{G}^{(a_i)}_{1,s}\cup\mathcal{G}^{(a_i)}_{2m,t}.$$

Lemma 2.4. $(\cup_{i=1}^{2n}\mathcal{F}^{(a_i)})\cup(\cup_{i=1}^{2n}\mathcal{G}^{(a_i)}_{s,t})$ *is a linearly independent set of cycles whenever* $3\leq s\leq 2m-1$, $1\leq t\leq 2m-3$ *and* $|s-t|\neq 1$.

Proof. Since

$$E(\mathcal{G}_{1,s}^{(a_i)}) \cap E(\mathcal{G}_{2m,t}^{(a_i)}) = \begin{cases} (a_i, u_{2m})(a_{i+1}, u_1), & \text{if } i \neq 2n \\ (a_{2n}, u_{2m})(a_1, u_1), & \text{if } i = 2n, \end{cases}$$

which is an edge, and $E(\mathcal{G}_{s,t}^{(a_i)}) \cap E(\mathcal{G}_{s,t}^{(a_r)}) = \phi$ for each $i \neq r$, by lemma 2.2 we have $\cup_{i=1}^{2n} \mathcal{G}_{s,t}^{(a_i)}$ is linearly independent. We now prove that $\cup_{i=1}^{2n} \mathcal{F}^{(a_i)}$ is linearly independent. Since $E(\mathcal{F}^{(a_i)}) \cap E(\mathcal{F}^{(a_l)}) = \phi$ for each $i \neq l$, it suffices to show that $\mathcal{F}^{(a_i)}$ is linearly independent for each $i = 1, 2, \ldots, 2n$. To show that, we proceed by induction on m. If $m = 2$, then $\mathcal{F}^{(a_i)}$ consists only of one cycle $\mathcal{F}_{1,4}^{(a_i)}$ which is linearly independent. Assume that $m > 2$ and the result is true for smaller values of m. It is easy to see that $\mathcal{F}_{1,2m}^{(a_i)}$ contains $(a_i, u_m)(a_i, u_{2m})$ which is not in $\cup_{j=1}^{m-2} \mathcal{F}_{m-j,m+j+1}^{(a_i)}$. Thus, by the inductive step $\mathcal{F}^{(a_i)}$ is linearly independent for each $i = 1, 2, \ldots, 2n$.

$$E(\cup_{i=1}^{2n} \mathcal{G}_{s,t}^{(a_i)}) \cap E(\cup_{i=1}^{2n} \mathcal{F}^{(a_i)}) = \{(a_i, u_{2m})(a_i, u_1) | i = 1, 2, \ldots, 2n\}$$

which is an edge set of a perfect matching. Thus, by Lemma 2.2, $(\cup_{i=1}^{2n} \mathcal{F}^{(a_i)}) \cup (\cup_{i=1}^{2n} \mathcal{G}_{s,t}^{(a_i)})$ is linearly independent. The proof is complete.

A circular ladder graph, CL_m, is visualized as a two concentric m-cycles in which each of the m pairs of the corresponding vertices is joined by an edge (i,e; if we assume the two concentric cycles are $u_1u_2 \ldots u_mu_1$ and $v_1v_2 \ldots v_mv_1$, then $E(CL_m) = E(u_1u_2 \ldots u_mu_1) \cup E(v_1v_2 \ldots v_mv_1) \cup \{\ u_iv_i : 1 \leq i \leq m\})$. For simplicity, we identify the vertices of CL_m as follows: $u_{m+1} = v_m, u_{m+2} = v_{m-1}, \ldots, u_{2m} = v_1$. Throughout this work CL_m will be taken as a cycle $u_1u_2 \ldots u_mu_{m+1} \ldots u_{2m}u_1$, in addition to the following edge set $\{u_{m-j}u_{m+j+1} : j = 1, 2, 3, \ldots, m-2\} \cup \{u_1u_m, u_{m+1}u_{2m}\}$. Similarly, CL_n will be the circular ladder with a vertex set $\{a_1, a_2, \ldots, a_{2n}\}$ and an edge set is as defined above. Now we can look at $CL_n[CL_m]$ as a graph that consists of $3n$ copies of $K_{2m,2m}$ in addition to $2n$ copies of CL_m. Note that each copy of $K_{2m,2m}$ is isomorphic to $e[N_{2m}]$ where $e \in E(CL_n)$ and each copy of CL_m is isomorphic to $a_i \times CL_m$.

Noting that $|E(CL_n[CL_m])| = 12m^2n + 6mn$ and $|V(CL_n[CL_m])| = 4mn$ and by the aid of equation (1), we have the following result.

Lemma 2.5. *dim* $\mathcal{C}(CL_n[CL_m]) = 12m^2n + 2mn + 1$.

Note 2.1. *For each* $n \geq 3$, $b(CL_n) = 2$.

Theorem 2.1. *For each* $n, m \geq 3$, *we have* $b(CL_n[CL_m]) \leq 4$. *Moreover, the equality holds for (*$n \geq 4$ *and* $m \geq 3$*) and (*$n = 3$ *and* $m \geq 7$*).*

Proof. To prove that $b(CL_n[CL_m]) \leq 4$, it suffices to exhibit a 4-fold basis. Define $\mathcal{B}(CL_n[CL_m]) = (\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup (\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}) \cup \mathcal{B}_{a_1a_n} \cup \mathcal{B}_{a_{n+1}a_{2n}} \cup (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup (\cup_{i=1}^{2n}\mathcal{G}_{m,m}^{(a_i)}) \cup \mathcal{B}_{u_{1,CL_n}}$ where $\mathcal{B}_{u_{1,CL_n}}$ is the corresponding required basis of $\mathcal{B}_{CL_n}$ in $CL_n \times u_1$. By Lemmas 2.3, 2.4 and being $\mathcal{B}_{CL_n}$ a basis, we have each of $(\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup (\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}), \mathcal{B}_{a_1a_n}, \mathcal{B}_{a_{n+1}a_{2n}}, (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup (\cup_{i=1}^{2n}\mathcal{G}_{m,m}^{(a_i)})$ and $\mathcal{B}_{u_{1,CL_n}}$ is linearly independent. By an argument similar to that in Lemma 2.3, we can show that

$$(\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup (\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}) \cup \mathcal{B}_{a_1a_n} \cup \mathcal{B}_{a_{n+1}a_{2n}}$$

is linearly independent. For simplicity, let $B_e = \mathcal{B}_{a_ia_j}$ where $e = a_ia_j (i < j)$, and $A = \mathcal{B}_{u_{1,CL_n}}$. Also, let $\mathcal{B}^* = (\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup (\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}) \cup \mathcal{B}_{a_1a_n} \cup \mathcal{B}_{a_{n+1}a_{2n}}$. We now show that $\mathcal{B}^* \cup \mathcal{B}_{u_{1,CL_n}} = (\cup_{e\in E(CL_n)}B_e) \cup A$ is linearly independent. Suppose that $(\cup_{e\in E(CL_n)}B_e) \cup A$ is linearly dependent. Then there exist $C_1, C_2, \ldots, C_p \in A$ and $D_{e,1}, D_{e,2}, \ldots, D_{e,q_e} \in B_e$ such that $\sum_{k=1}^{p} C_k = \sum_{e\in M\subseteq E(CL_n)} \sum_{k=1}^{q_e} D_{e,k} \pmod 2$. Let $f_0 = (a_{i_0}, u_1)(a_{j_0}, u_1)$ be an edge which occurs in $\sum_{k=1}^{p} C_k \pmod 2$ and let $e_0 = a_{i_0}a_{j_0}$. Since $f_0 \notin E(B_e)$ for each $e \neq e_0$, f_0 must occur in $\sum_{k=1}^{q_{e_0}} D_{e_0,k} \pmod 2$. Since the number of those edges in $E(B_{e_0})$ which join $\{a_{i_0}\} \times \{u_1, u_2, \ldots, u_{2m}\}$ and $\{a_{j_0}\} \times \{u_1, u_2, \ldots, u_{2m}\}$ and occur in $\sum_{k=1}^{q_{e_0}} D_{e_0,k} \pmod 2$ is even, there exists another edge $f \in E(B_{e_0})$. Thus f remains in $\sum_{e\in M\subseteq E(CL_n)} \sum_{k=1}^{q_e} D_{e,k} \pmod 2$. But $f \notin E(A)$ and so f cannot belong to $\sum_{k=1}^{p} C_k \pmod 2$. This is a contradiction. Now any linear combination of cycles of $(\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup (\cup_{i=1}^{2n}\mathcal{G}_{m,m}^{(a_i)})$ must contain at least one edge of $(a_i, u_{m-j})(a_i, u_{m+j+1})$,$(a_i, u_1)(a_i, u_m)$ and $(a_i, u_{m+1})(a_i, u_{2m})$ which is not in any cycle of $(\cup_{e\in E(CL_n)}B_e) \cup A$ for some i, j. Thus, $\mathcal{B}(CL_n[CL_m])$ is linearly independent. Let $e' = ab$. Since

$$\begin{aligned} |B_{e'}| &= |\mathcal{B}_{ab}| = |\mathcal{A}_{ab}| + |\mathcal{D}_{ab}| \\ &= (2m-1)^2 + 2(2m-1) \\ &= 4m^2 - 1, \end{aligned}$$

and

$$|\mathcal{F}^{(a_i)}| + |\mathcal{G}_{s,t}^{(a_i)}| = m + 1,$$

hence

$$\begin{aligned} |\mathcal{B}(CL_n[CL_m])| &= |\cup_{e\in E(CL_n)} B_e| + |\cup_{i=1}^{2n} \mathcal{F}^{(a_i)}| + |\cup_{i=1}^{2n} \mathcal{G}_{m,m}^{(a_i)}| + |\mathcal{B}_{u_{1,CL_n}}| \\ &= \sum_{i=1}^{3n}(4m^2-1) + \sum_{i=1}^{2n}(m+1) + \dim\ \mathcal{C}(CL_n) \\ &= 3n(4m^2-1) + 2n(m+1) + (n+1) \\ &= 12m^2n + 2mn + 1 \\ &= \dim\ \mathcal{C}(CL_n[CL_m]). \end{aligned}$$

Thus, $\mathcal{B}(CL_n[CL_m])$ is a basis of $\mathcal{C}(CL_n[CL_m])$. We now show that $\mathcal{B}(CL_n[CL_m])$ is of fold 4. For simplicity, assume that $\mathcal{F} = (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)})$ and $\mathcal{G} = (\cup_{i=1}^{2n}\mathcal{G}_{m,m}^{(a_i)})$. Let $e \in E(CL_n[CL_m])$. (1) If $e \in E(CL_n \times u_1)$, then $f_{\mathcal{B}^*}(e) = 1, f_{\mathcal{F}}(e) = 0, f_{\mathcal{G}}(e) = 0, f_A(e) \leq 2$. (2) If $e = (a_i, u_j)(a_i, u_{j+1})$, then $f_{\mathcal{B}^*}(e) \leq 3, f_{\mathcal{F}}(e) \leq 1, f_{\mathcal{G}}(e) = 0, f_A(e) = 0$. (3) If $e = (a_i, u_{m-j})(a_i, u_{m+j+1})$, then $f_{\mathcal{B}^*}(e) = 0, f_{\mathcal{F}}(e) \leq 2, f_{\mathcal{G}}(e) = 1, f_A(e) = 0$. (4) if $e = (a_i, u_1)(a_i, u_m)$, or $(a_i, u_{m+1})(a_i, u_{2m})$ then $f_{\mathcal{B}^*}(e) = 0, f_{\mathcal{F}}(e) = 0, f_{\mathcal{G}}(e) \leq 2, f_A(e) = 0$. (5) If $e = (a_i, u_m)(a_{i+1}, u_1)$ or $(a_i, u_{m+1})(a_{i+1}, u_1)$, then $f_{\mathcal{B}^*}(e) \leq 2, f_{\mathcal{F}}(e) = 0, f_{\mathcal{G}}(e) = 0, f_A(e) = 0$. (6) If $e = (a_i, u_{2m})(a_{i+1}, u_1)$, then $f_{\mathcal{B}^*}(e) = 2, f_{\mathcal{F}}(e) = 1, f_{\mathcal{G}}(e) = 1, f_A(e) = 0$. (7) If $e = (a_i, u_{2m})(a_{i+1}, u_{2m})$ then $f_{\mathcal{B}^*}(e) = 3, f_{\mathcal{F}}(e) = 1, f_{\mathcal{G}}(e) = 0, f_A(e) = 0$. (8) If $e \in E(CL_n[CL_m])$ and is not of any form above, then $f_{\mathcal{B}^*}(e) \leq 4, f_{\mathcal{F}}(e) = 0, f_{\mathcal{G}}(e) = 0, f_A(e) = 0$.

On the other hand, to show that $b(CL_n[CL_m]) \geq 4$ for any n, m as they were stated in the theorem, we have to exclude any possibility for the cycle space $\mathcal{C}(CL_n[CL_m])$ to have a 3-fold basis for any n, m as they are stated in the theorem.. To this end, suppose that $\mathcal{B}$ is a 3-fold basis of the cycle space, then we have the following three cases:

Case 1. Suppose that $\mathcal{B}$ consists only of 3-cycles. We now consider two subcases:

Subcase1. $n \geq 4$. Then $|\mathcal{B}| \leq 18mn$ because any 3-cycle must contain an edge of $a_i \times CL_m$ for some $1 \leq i \leq 2n$ and each edge is of fold at most 3. This is equivalent to the inequality that $12m^2n + 2mn + 1 \leq 18mn$ which implies that $12m^2n + 1 \leq 16mn$ and so $12m \leq 16$. Thus $m \leq 1$. This is a contradiction.

Subcase2. $n = 3$. Then $|\mathcal{B}| \leq 3(18m + 8m^2)$ because each edge of $CL_3[CL_m]$ is of fold at most 3 and if C is a 3-cycle not containing an edge of $a_i \times CL_m$ for $1 \leq i \leq 6$, then it contains an edge of $\{(a_1, u_j)(a_3, u_k), (a_4, u_j)(a_6, u_k) | 1 \leq j, k \leq 2m\}$. This is equivalent to the inequality $36m^2 + 6m + 1 \leq 54m + 24m^2$ which implies that $12m^2 + 1 \leq 48m$ and so $m \leq 4 - 1/(12m)$. Thus $m < 4$. This is a contradiction.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equals to 4. Then $4|\mathcal{B}| \leq 3|E(CL_n[CL_m])|$ because the length of each cycle of $\mathcal{B}$ greater than or equal to 4 and each edge is of fold at most 3. Thus, $4(12m^2n + 2mn + 1) \leq 3(12m^2n + 6mn)$ which is equivalent to $12m^2n + 4 \leq 10mn$. which has no solution. This is a contradiction.

Case 3. Suppose that $\mathcal{B}$ consists of s 3-cycles and t cycles of length greater than or equal to 4. Then $t \leq \lfloor (3(12m^2n + 6mn) - 3s)/4 \rfloor$ because the length of each cycle of s is 3 and each cycle of t is at least 4 and the fold of each edge is at most 3. Hence, $|\mathcal{B}| = s + t \leq s + \lfloor (3(12m^2n + 6mn) - 3s)/4 \rfloor$ this implies that $4(12m^2n + 2mn + 1) \leq 4s + 3(12m^2n + 6mn) - 3s$. By simplifying the inequality we have $12m^2n + 4 \leq s + 10mn$. To this end, we consider two subcases:

Subcase1. $n \geq 4$. Then as in Subcase 1 of Case 1 $s \leq 18mn$, thus$12m^2n+4 \leq 28mn$ and so $12m \leq 28$. Therefore, $m \leq 2$. This is a contradiction.

Subcase2. $n = 3$. Then as in Subcase 2 of Case 1 $s \leq 54m + 24m^2$, thus $36m^2 + 4 \leq 24m^2 + 84m$ which implies that $m \leq 7 - 4/(12m)$. Thus $m < 7$. This is a contradiction. The proof the theorem is complete.

The Möbius ladder ML_m is obtained by deleting from the circular ladder CL_m two of its parallel curved edge and replacing them with two edges that cross-match their endpoints

Note 2.2. $b(ML_n) = \begin{cases} 2, & \textit{if } n = 2 \\ 3, & \textit{if } n \geq 3. \end{cases}$

Theorem 2.2. *For each $n, m \geq 2$, we have $b(ML_n[ML_m]) \leq 4$. Moreover, the equality holds for $n \geq 3$ and $m \geq 3$.*

Proof. By the definition of the Möbius ladder, one may assume that ML_m and ML_n are obtained from the circular ladders CL_m and CL_n by deleting u_mu_1 and $u_{2m}u_{m+1}$ from CL_m and a_na_1 and $a_{2n}a_{n+1}$ from CL_n and replacing them by $u_{m+1}u_1$ and $u_{2m}u_m$, and $a_{n+1}a_1$ and $a_{2n}a_n$. Thus, $ML_n[ML_m]$ is obtained from $CL_n[CL_m]$ by deleting all the following edges $\{(a_i, u_1)(a_i, u_m), (a_i, u_{m+1})(a_i, u_{2m})|1 \leq i \leq 2n\} \cup E(\mathcal{A}_{a_1a_n}) \cup E(\mathcal{A}_{a_{n+1}a_{2n}})$ and replacing them by $\{(a_i, u_1)(a_i, u_{m+1}), (a_i,\ u_m)(a_i, u_{2m})|1 \leq i \leq 2n\} \cup E(\mathcal{A}_{a_1a_{n+1}}) \cup E(\mathcal{A}_{a_na_{2n}})$. Define $\mathcal{B}(ML_n[ML_m]) = (\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup (\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}) \cup (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup (\cup_{i=1}^{2n}\mathcal{G}_{m+1,m-1}^{(a_i)}) \cup \mathcal{B}_{a_1a_{n+1}} \cup \mathcal{B}_{a_na_{2n}} \cup \mathcal{B}_{u_{1,ML_n}}$, where $\mathcal{B}_{u_{1,ML_n}}$ is the corresponding required basis of $\mathcal{B}_{ML_n}$ in the copy $ML_n \times u_1$. Using the same argument as in the proof of Theorem 2.1 taking into account $b(ML_n) \leq 3$, we can prove that $\mathcal{B}(ML_n[ML_m])$ is a 4-fold basis.
On the other hand, to show that $b(ML_n[ML_m]) \geq 4$ for any m, n as they were stated in the Theorem, we argue more or less as in the proof of the three cases in Theorem 2.1, replacing CL_n and CL_m by ML_n and ML_m, respectively, and taking only Subcase 1 of Case 1 and of Case 3 for $n \geq 3$. The proof is complete.

Now, $CL_n[ML_m]$ is obtained from $CL_n[CL_m]$ by replacing $\{(a_i, u_1)(a_i, u_m), (a_i, u_{m+1})(a_i, u_{2m})|1 \leq i \leq 2n\}$ with $\{(a_i, u_1)(a_i, u_{m+1}), (a_i, u_m)(a_i, u_{2m})|1 \leq i \leq 2n\}$. Similarly, $ML_n[CL_m]$ is obtained from $CL_n[CL_m]$ by replacing $E(\mathcal{A}_{a_1a_n}) \cup E(\mathcal{A}_{a_{n+1}a_{2n}})$ with $E(\mathcal{A}_{a_1a_{n+1}}) \cup E(\mathcal{A}_{a_na_{2n}})$.

Theorem 2.3. *For each $n \geq 3$ and $m \geq 2$, we have $b(CL_n[ML_m]) \leq 4$. Moreover, the equality holds for ($n \geq 4$ and $m \geq 3$) and ($n = 3$ and $m \geq 7$). Also, for each $n \geq 2$ and $m \geq 3$, we have $b(ML_n[CL_m]) \leq 4$. Moreover, the equality holds for $n \geq 3$ and $m \geq 3$.*

Proof. Define $\mathcal{B}(CL_n[ML_m]) = (\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup (\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}) \cup (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup(\cup_{i=1}^{2n}\mathcal{G}_{m+1,m-1}^{(a_i)}) \cup \mathcal{B}_{a_1a_n} \cup \mathcal{B}_{a_{n+1}a_{2n}} \cup \mathcal{B}_{u_{1,CL_n}}$ and $\mathcal{B}(ML_n[CL_m]) = (\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup(\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}) \cup (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup (\cup_{i=1}^{2n}\mathcal{G}_{m,m}^{(a_i)}) \cup \mathcal{B}_{a_1a_{n+1}} \cup \mathcal{B}_{a_na_{2n}} \cup \mathcal{B}_{u_{1,ML_n}}$. By the same argument as in the above Theorems with the above replacements, we can prove that $\mathcal{B}(CL_n[ML_m])$ and $\mathcal{B}(ML_n[CL_m])$ are the required basis. The proof is complete.

The ladder, L_m, is obtained by deleting from the circular ladder CL_m two of its parallel curved edges. Let, L_m be obtained from CL_m by deleting u_1u_m and $u_{m+1}u_{2m}$. Also, let L_n be obtained from CL_n by deleting a_1a_n and $a_{n+1}a_{2n}$. Then, $L_n[CL_m]$ is a subgraph of $CL_n[CL_m]$ obtained by deleting $E(\mathcal{A}_{a_1a_n}) \cup E(\mathcal{A}_{a_{n+1}a_{2n}})$.

Lemma 2.6. *$dim\ \mathcal{C}(L_n[CL_m]) = dim\ \mathcal{C}(L_n[ML_m]) = 12m^2n - 8m^2 + 2mn + 1$ and $dim\ \mathcal{C}(CL_n[L_m]) = dim\ \mathcal{C}(ML_n[L_m]) = 12m^2n + 2mn - 4n + 1$.*

Theorem 2.4. *For each $n \geq 2$ and $m \geq 3$, we have $b(L_n[CL_m]) \leq 4$. Moreover, the equality holds for $n \geq 2$ and $m \geq 4$.*

Proof. Define, $\mathcal{B}(L_n[CL_m]) = (\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}})\cup(\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}})\cup(\cup_{i=1}^{2n}\mathcal{G}_{m,m}^{(a_i)}) \cup (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup \mathcal{B}_{u_{1,L_n}}$. Note that $\mathcal{B}(L_n[CL_m]) \subset \mathcal{B}(CL_n[CL_m])$. Thus, $\mathcal{B}(L_n[CL_m\ ])$ is a linearly independent set of fold 4. Since $|\mathcal{B}(L_n[CL_m])| = \dim\ \mathcal{C}(L_n[CL_m\ ])$, $\mathcal{B}(L_n[CL_m])$ is a 4-fold basis of $\mathcal{C}(L_n[CL_m])$. On the other hand, to show that $b(L_n[CL_m]) \geq 4$ for all $n \geq 2$ and $m \geq 4$, we suppose that $\mathcal{B}$ is a 3-fold basis of the cycle space $\mathcal{C}(L_n[CL_m])$, then we argue more or less as in the proof of Theorem 2.1 taking into account that in Cases 1 and 3 we deal only with one subcase for $n \geq 2$. The proof is complete.

Observe that $CL_n[L_m]$ is a subgraph of $CL_n[CL_m]$ obtained by deleting $\{(a_i, u_1)(a_i, u_m),\ (a_i, u_{m+1})(a_i, u_{2m}) | 1 \leq i \leq 2n\}$.

Theorem 2.5. *For each $n \geq 3$ and $m \geq 2$, we have $b(CL_n[L_m]) \leq 4$. Moreover, the equality holds for ($n \geq 4$ and $m \geq 3$) and ($n = 3$ and $m \geq 7$).*

Proof. Let $\mathcal{B}(CL_n[L_m]) = (\cup_{i=1}^{2n-1}\mathcal{B}_{a_ia_{i+1}}) \cup (\cup_{i=1}^{n-1}\mathcal{B}_{a_{n-i}a_{n+i+1}}) \cup \mathcal{B}_{a_1a_{n+1}} \cup \mathcal{B}_{a_{n+1}a_{2n}} \cup (\cup_{i=1}^{2n}\mathcal{F}^{(a_i)}) \cup \mathcal{B}_{u_{1,CL_n}}$. Since $\mathcal{B}(CL_n[L_m]) \subset \mathcal{B}(CL_n[CL_m])$ and $\mathcal{B}(CL_n[CL_m])$ is a linearly independent set of fold 4, and $|\mathcal{B}(L_n[CL_m])| = \dim\ \mathcal{C}(CL_n[L_m])$, $\mathcal{B}(CL_n[L_m])$ is a 4-fold basis of $\mathcal{C}(CL_n[L_m])$. On the other hand, the inequality $b(CL_n[L_m]) \geq 3$ follows by the same argument of the three cases of Theorem 2.1. We omit the detail. The proof is complete.

Now, we consider $L_n[ML_m]$ and $ML_n[L_m]$ as subgraphs obtained from $L_n[CL_m]$ and $CL_n[L_m]$ by deleting $\{(a_i, u_1)(a_i, u_m), (a_i, u_{m+1})(a_i, u_{2m}) | 1 \leq i \leq 2n\}$ and $E(\mathcal{A}_{a_1a_n}) \cup E(\mathcal{A}_{a_{n+1}a_{2n}})$ and replacing them with the following sets

$\{(a_i, u_1)(a_i, u_{m+1}), (a_i, u_m)(a_i, u_{2m}) | 1 \leq i \leq 2n\}$ and $E(\mathcal{A}_{a_1 a_{n+1}}) \cup E(\mathcal{A}_{a_n a_{2n}})$, respectively.

Theorem 2.6. *For each $n, m \geq 2$, we have $b(ML_n[L_m])$, $b(L_n[ML_m]) \leq 4$. Moreover, $b(ML_n[L_m]) = 4$ whenever $n \geq 3$ and $m \geq 4$ and $b(L_n[ML_m]) = 4$ whenever $n \geq 2$ and $m \geq 4$.*

Proof. Following the above replacements of edges, we can repeat the proof of Theorems 2.4 and 2.5 to show that the following sets of cycles are bases and satisfy the fold stated in the theorem.

$\mathcal{B}(ML_n[L_m]) = (\cup_{i=1}^{2n-1} \mathcal{B}_{a_i a_{i+1}}) \cup (\cup_{i=1}^{n-1} \mathcal{B}_{a_{n-i} a_{n+i+1}}) \cup \mathcal{B}_{a_1 a_{m+1}} \cup \mathcal{B}_{a_m a_{2m}} \cup (\cup_{i=1}^{2n} \mathcal{F}^{(a_i)}) \cup \mathcal{B}_{u_1, L_n}$

$\mathcal{B}(L_n[ML_m]) = (\cup_{i=1}^{2n-1} \mathcal{B}_{a_i a_{i+1}}) \cup (\cup_{i=1}^{n-1} \mathcal{B}_{a_{n-i} a_{n+i+1}}) \cup (\cup_{i=1}^{2n} \mathcal{G}_{m+1, m-2}^{(a_i)}) \cup (\cup_{i=1}^{2n} \mathcal{F}^{(a_i)}) \cup \mathcal{B}_{u_1, L_n}$. The proof is complete.

Finally, $L_n[L_m]$ is obtained by deleting the following set of edges $\{(a_i, u_1)(a_i, u_m), (a_i, u_{m+1})(a_i, u_{2m}) | 1 \leq i \leq 2n\}$ from $L_n[CL_m]$.

Theorem 2.7. *For each $n, m \geq 2$, we have $b(L_n[L_m]) \leq 4$. Moreover, the equality holds for $n \geq 2$ and $m \geq 4$.*

Proof. Let $\mathcal{B}(L_n[L_m]) = (\cup_{i=1}^{2n-1} \mathcal{B}_{a_i a_{i+1}}) \cup (\cup_{i=1}^{n-1} \mathcal{B}_{a_{n-i} a_{n+i+1}}) \cup (\cup_{i=1}^{2n} \mathcal{F}^{(a_i)}) \cup \mathcal{B}_{u_1, L_n}$. Note that $\mathcal{B}(L_n[L_m]) \subset \mathcal{B}(L_n[CL_m])$. Thus, $\mathcal{B}(L_n[L_m])$ is a linearly independent set of fold 4. It is easy to see that $|\mathcal{B}(L_n[CL_m])| = \dim\ \mathcal{C}(CL_n[L_m])$, then $\mathcal{B}(L_n[L_m])$ is a basis of $\mathcal{C}(L_n[L_m])$. On the other hand, to show that $b(CL_n[L_m]) \geq 4$, we use argument similar to that in Theorem 2.1 taking into account that in Case 1 and Case 3 we deal only with one subcase for $n \geq 2$.

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References

[1] A.A. Ali, *The basis number of the direct products of paths and cycles*, Ars Combin. 27 (1989), 155-163.

[2] A.A. Ali, *The basis number of complete multipartite graphs*, Ars Combin. 28 (1989), 41-49.

[3] A.A. Al-Rhayyel and M.M. Jaradat, *On the basis number of the direct product of some graphs*, Bull. Cal. Math. Soc. 88 (1996), no. 6, 509-516.

[4] M. Y. Alzoubi and M.M. Jaradat, *The basis number of the composition of theta graphs with stars and wheels*, Acta Math. Hungar. 103 (2004), no. 4, 201-209.

[5] J.A. Banks and E.F. Schmeichel, *The basis number of n-cube*, J. Combin. Theory Ser. B 33 (1982), no. 2, 95-100.

[6] J.A. Bondy and U.S. Murty, *"Graph theory with applications"*, American Elsevier Publishing Co. Inc., New York, 1976.

[7] S. G. Hulsurkar, *Non-planarity of graphs on Weyl groups*, Jour. Math. Phy. Sci., 24 (1990), no. 6, 363-367.

[8] S. G. Hulsurkar, *A proof of Verma's conjecture on Weyl's dimension polynomial*, Inventions Math., 27 (1974), 45-52.

[9] M.M. Jaradat, *On the basis number of the direct product of graphs,* Australasian Journal of Combinatorics 27 (2003), 293-306.

[10] M.M. Jaradat, *The basis number of the direct product of a theta graph and a path,* to appear in Ars Combin.

[11] S. MacLane, *A combinatorial condition for planar graphs,* Fundamenta Math. 28 (1937), 22-32.

[12] E.F. Schmeichel, *The basis number of a graph*, J. Combin. Theory Ser. B 30 (1981), no. 2, 123-129.

[13] D.N. Verma, *A Role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, "Lie groups and their representations"*, Proceeding of the 1971 Summer School on representation Theory (Ed. I. M. Gel'fand). Budapest: Akademiai Kiado.

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