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Moser's construction of time-dependent Hamiltonian function which defines a Hamiltonian map on $\mathbb{R}^{2n}$

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Abstract. We will show that if symplectomorphisms on $\mathbb{R}^{2n}$ admit the generating function with the integrability condition, then these symplectomorphisms are Hamiltonian maps. This is an extension of results of J.Moser in [M].

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§1. Introduction

Let $\varphi : (\xi, \xi') \mapsto (\eta, \eta')$ be a symplectomorphism defined on $\mathbb{R}^{2n}$. Here $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ is a vector, similarly for ξ', η and η'. If φ is the time-1 map of the flow defined a time-dependent Hamiltonian system, this symplectomorphism φ is called a *Hamiltonian map* (for the detail about the Hamiltonian map, see [HZ] and [MS]).

It is an important problem in symplectic geometry to find conditions for a symplectomorphism to be a Hamiltonian map (see [MS]). And if a given symplectomorphism turns out to be a Hamiltonian map, we would like to construct a Hamiltonian function of the Hamiltonian map. However, a little is known about how to construct a Hamiltonian function which defines a Hamiltonian map.

In the present paper, we consider the symplectomorphisms on $\mathbb{R}^{2n}$ which admit a generating function. A generating function is defined as follows.

Definition 1.1. *Let $\varphi : (\xi, \xi') \mapsto (\eta, \eta')$ be a symplectomorphism defined on $\mathbb{R}^{2n}$. If there exists a smooth function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\xi, \eta) \mapsto h(\xi, \eta)$ such* *that*

$$\frac{\partial h}{\partial \xi_i} = -\xi_i', \quad \frac{\partial h}{\partial \eta_i} = \eta_i' \qquad (i = 1, \cdots, n), \tag{1.1}$$

then h is called a generating function for φ.

In the case of $\mathbb{R}^2$, J.Moser showed in [M] the symplectomorphism which admits a generating function h is a Hamiltonian map, provided $\dfrac{\partial^2 h}{\partial \xi \partial \eta} \neq 0$. We prove that Moser's result can be extended to the case of $\mathbb{R}^{2n}$, if the generating function h satisfies further the integrability condition. Here is our main result.

Theorem 1.2. *Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} : (\xi, \xi') \mapsto (\eta, \eta')$ be a symplectomorphism which admits a generating function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\xi, \eta) \mapsto h(\xi, \eta)$. Suppose h satisfies following conditions.*

$$(\textit{Legendre condition}) \quad \det\left(\frac{\partial^2 h}{\partial \xi_i \partial \eta_j}\right) \neq 0, \tag{1.2}$$

$$(\textit{integrability condition}) \quad \frac{\partial^2 h}{\partial \xi_i \partial \eta_j} = \frac{\partial^2 h}{\partial \xi_j \partial \eta_i} \quad (i, j = 1, \cdots, n). \tag{1.3}$$

Then φ is a Hamiltonian map.

The paper is organized as follows. In Section 2 we deal with the variational problem whose extremal curves are segments. This is one of key steps of our construction. In Section 3 we explain what is Moser's construction. Section 4 is devoted to the proof of the main theorem. The final Section 5 is concluding remark.

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§2. Functional with segments as extremal curves

Let $F = F(t, x, p)$ be a smooth function of $2n + 1$ variables $(t, x_1, \cdots x_n, p_1, \cdots, p_n)$, $x = x(t) : [0, 1] \to \mathbb{R}^n$ be a smooth curve which satisfies

$$x(0) = \xi, \quad x(1) = \eta \qquad (\xi, \eta \in \mathbb{R}^n).$$

It is well-known (see [AM]) that the curve $x(t)$ is the extremal for the functional

$$\int_0^1 F(t, x(t), \dot{x}(t))\, dt \tag{2.1}$$

if and only if it satisfies the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial F}{\partial p_i} - \frac{\partial F}{\partial x_i} = 0 \quad (i = 1, \cdots, n). \tag{2.2}$$

From now on, we require that the extremal curves of (2.1) are segments

$$x(t) = \xi + t(\eta - \xi) \tag{2.3}$$

for any ξ, $\eta \in \mathbb{R}^n$. Denote by $S = S(\xi, \eta)$ the extremal integral, i.e.

$$S(\xi, \eta) = \int_0^1 F(t, \xi + t(\eta - \xi), \eta - \xi)\, dt. \tag{2.4}$$

Then the Lagrangian function F satisfies following two propositions.

Proposition 2.1. *We define the Euler-Lagrange operator as*

$$\mathcal{E}_i = \left(\partial_t + \sum_{k=1}^{n} p_k \partial_{x_k}\right)\partial_{p_i} - \partial_{x_i} \quad (i = 1, \cdots, n).$$

Then

$$(\mathcal{E}_i F)(t, x, p) = 0. \tag{2.5}$$

Proof. If $x(t) = \xi + t(\eta - \xi)$ is the extremal of (2.1) we can compute the Euler-Lagrange equation (2.2) as follows.

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial F}{\partial p_i}(t, \xi + t(\eta - \xi), \eta - \xi) - \frac{\partial F}{\partial x_i}(t, \xi + t(\eta - \xi), \eta - \xi) \\ &= \frac{\partial^2 F}{\partial t \partial p_i} + \sum_{k=1}^{n} (\eta_k - \xi_k) \frac{\partial^2 F}{\partial x_k \partial p_i} - \frac{\partial F}{\partial x_i}. \end{aligned}$$

Therefore,

$$(\mathcal{E}_i F)(t, \xi + t(\eta - \xi), \eta - \xi) = 0.$$

□

Proposition 2.2. *For any* ξ, $\eta \in \mathbb{R}^n$,

$$\begin{cases} \dfrac{\partial S}{\partial \xi_i} = -F_{p_i}(0, \xi, \eta - \xi) \\ \dfrac{\partial S}{\partial \eta_i} = F_{p_i}(1, \eta, \eta - \xi) \end{cases} \quad (i = 1, \cdots, n). \tag{2.6}$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial \xi_i} S(\xi,\eta) &= \int_0^1 \frac{\partial}{\partial \xi_i} F(t, \xi + t(\eta-\xi), \eta - \xi)\, dt \\ &= \int_0^1 \left\{ (1-t)\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial p_i} \right\} dt\end{aligned}$$

Applying the Euler-Lagrange equation (2.2), we get

$$\begin{aligned}&= \int_0^1 \left\{ -F_{p_i} + (1-t)\frac{d}{dt} F_{p_i} \right\} dt \\ &= \int_0^1 \frac{d}{dt} \left\{ (1-t) F_{p_i}(t, \xi + t(\eta-\xi), \eta-\xi) \right\} dt \\ &= -F_{p_i}(0, \xi, \eta - \xi).\end{aligned}$$

The second equation can be proved similarly. □

Next, we consider the variational problem for the Hamiltonian system. Let $H = H(t,x,y)$, $t \in [0,1]$ be a time-dependent smooth Hamiltonian function on $\mathbb{R}^{2n}$ endowed with a coordinates x, y. Consider the Hamiltonian system

$$\begin{cases} \dot{x} = H_y \\ \dot{y} = -H_x \end{cases}$$

which satisfies boundary conditions;

$$(x(0), y(0)) = (\xi, \xi'), \quad (x(1), y(1)) = (\eta, \eta') \qquad ((\xi,\xi'),\, (\eta,\eta') \in \mathbb{R}^{2n}).$$

If the Hamiltonian H satisfies the Legendre condition

$$\det\left(\frac{\partial^2 H}{\partial y_i y_j}\right) \neq 0, \tag{2.7}$$

one can introduce the variables $p_i\,(i = 1, \cdots, n)$ by the Legendre transformation

$$p_i = H_{y_i}(t,x,y) \quad (i = 1, \cdots, n). \tag{2.8}$$

Defined the Lagrangian $F(t,x,p)$ as follows.

$$F(t,x,p) = y \cdot p - H(t,x,y). \tag{2.9}$$

Then the Hamiltonian system becomes the Euler-Lagrange equation of the variational problem

$$\int_0^1 F(t, x(t), \dot{x}(t))\, dt,$$

and the Legendre condition (2.7) for H becomes the one for F ;

$$\det\left(\frac{\partial^2 F}{\partial p_i \partial p_j}\right) \neq 0.$$

Proposition 2.3. *If extremal curves of the above variational problem are the segments* (2.3) *then*

$$F_{p_i}(0, \xi, \eta - \xi) = \xi_i', \quad F_{p_i}(1, \eta, \eta - \xi) = \eta_i' \qquad (i = 1, \cdots, n). \tag{2.10}$$

Proof. Differentiation (2.9) with respect to p_i yields

$$\begin{aligned} \frac{\partial}{\partial p_i} F(t, x, p) &= y_i + \sum_{j=1}^{n} \frac{\partial y_j}{\partial p_i} p_j - \sum_{j=1}^{n} \frac{\partial H}{\partial y_j} \frac{\partial y_j}{\partial p_i} \\ &= y_i + \frac{\partial y}{\partial p_i} \cdot p - p \cdot \frac{\partial y}{\partial p_i} \\ &= y_i \end{aligned}$$

Therefore,

$$F_{p_i}(t, x(t), \dot{x}(t)) = y_i(t).$$

Setting $t = 0, 1$ we get the required statement. □

§3. Moser's construction of time-dependent Hamiltonian function

In this section, we consider the condition for a symplectomorphism which admits a generating function to be a Hamiltonian map. Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} : (\xi, \xi') \mapsto (\eta, \eta')$ be the symplectomorphism and $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\xi, \eta) \mapsto h(\xi, \eta)$ be the generating function for φ.

In the previous section, we discussed the necessary condition for φ to be a Hamiltonian map. From this point of view, we will construct the Lagrangian F which satisfies the following properties.

(3.1)

$$\begin{cases} (i)\ \mathcal{E}_i F(t, x, p) = 0 \qquad (i = 1, \cdots, n), \\ (ii)\ F_{p_i}(0, \xi, \eta - \xi) = -h_{\xi_i}, \quad F_{p_i}(1, \eta, \eta - \xi) = h_{\eta_i} \qquad (i = 1, \cdots, n), \\ (iii)\ \det\left(\dfrac{\partial^2 F}{\partial p_i \partial p_j}\right) \neq 0. \end{cases}$$

Indeed from (*iii*), we can obtain the Hamiltonian system by the inverse of the Legendre transformation (2.8). Denote by $\varphi_H : (\xi, \xi') \mapsto (\eta, \eta')$ the corresponding Hamiltonian map (for the detail about the relation between the Hamiltonian system and the Lagrangian system, see [AM] Chapter 3). On the other hand, form (*i*), segments (2.3) are extremal curves of (2.1). Thus F satisfies (2.10) in Proposition 2.3. Combining (2.10) with (*ii*), we get

$$\frac{\partial h}{\partial \xi_i} = -\xi_i', \quad \frac{\partial h}{\partial \eta_i} = \eta_i' \qquad (i = 1, \cdots, n).$$

This shows that h is the generating function for φ_H, hence $\varphi = \varphi_H$. In particular, φ is a Hamiltonian map.

Here, in order to construct F with above conditions, we suppose that h satisfies the following assumption;

Assumption. *The generating function h satisfies the Legendre condition* (1.2) *and the integrability condition* (1.3)*.*

According to [M], we set the Lagrangian F as follows.

(3.2)
$$F(t,x,p) = F_0(t,x,p) + \sum_{k=1}^{n} p_k\{t h_{\eta_k}(x,x) - (1-t)h_{\xi_k}(x,x)\} + h(x,x)$$

(3.3)
$$F_0(t,x,p) = -\sum_{i,j=1}^{n}\left\{\int_0^1\int_0^1 h_{\xi_i\eta_j}(x-uvtp, x+uv(1-t)p)v\,du\,dv\right\}p_ip_j$$

Next section, we shall prove that this F satisfies the condition (3.1), provided h satisfies the above assumption.

Remark 3.1. In [M], J. Moser discussed the case of $\mathbb{R}^2$. In this case by differentiation the Euler-Lagrange equation with respect to p, Lagrangian F satisfies

$$(\partial_t + p\partial_x)F_{pp}(t,x,p) = 0 \tag{3.4}$$

i.e.

$$F_{pp}(t,x,p) = G(x-tp,p)$$

for some arbitrary function $G(x,p)$. And in order that F satisfies (*iii*) of the condition (3.1), he set

$$G(x,p) = -h_{\xi\eta}(x,x+p)$$

i.e.

$$F_{pp}(t,x,p) = -h_{\xi\eta}(x-tp, x+(1-t)p).$$

Then, one have

$$F(t,x,p) = -\int_0^p h_{\xi\eta}(x-tq, x+(1-t)q)\,dq + C(t,x,p)$$

for some arbitrary function $C(t,x,p)$. Finally, he set

$$C(t,x,p) = p\{th_\eta(x,x) - (1-t)h_\xi(x,x)\} + h(x,x)$$

for some technical reason. Note that above F satisfies the Euler-Lagrange equation and the Legendre condition.

Similarly in our case of $\mathbb{R}^{2n}$, we consider F to satisfy

$$F_{p_ip_j}(t,x,p) = -h_{\xi_i\eta_j}(x-tp, x+(1-t)p) \qquad (i,j=1,\cdots,n). \tag{3.5}$$

However, the partial differential equation of F corresponding to (3.4) is

$$\left(\partial_t + \sum_{k=1}^n p_k\partial_{x_k}\right)F_{p_ip_j} = F_{x_ip_j} - F_{x_jp_i} \qquad (i,j=1,\cdots,n),$$

which is different of (3.4). So we constructed F from (3.5) in order to satisfy (i) of the condition (3.1).

§4. Proof of Theorem 1.2

Let us begin with proofs of several lemmas.

Lemma 4.1. *The function $F_0(t,x,p)$ defined by* (3.3) *satisfies*

$$\frac{\partial F_0}{\partial p_i} = -\sum_{j=1}^n\left\{\int_0^1 h_{\xi_i\eta_j}(x-utp, x+u(1-t)p)\,du\right\}p_j. \tag{4.1}$$

Proof. Applying the integrability condition (1.3), we get

$$h_{\xi_j\eta_i} = h_{\xi_i\eta_j}, \quad h_{\xi_k\eta_j\eta_i} = h_{\xi_i\eta_j\eta_k}.$$

Thus

$$
\begin{aligned}
&\frac{\partial}{\partial p_i} F_0 \\
&= - \int_0^1 \int_0^1 \Bigg\{ 2 \sum_{j=1}^n h_{\xi_i \eta_j}(x - uvtp, x + uv(1-t)p) v p_j \\
&\quad + \sum_{j,k=1}^n \{ -uvt h_{\xi_i \eta_j \xi_k}(x - uvtp, x + uv(1-t)p) \\
&\quad + uv(1-t) h_{\xi_i \eta_j \eta_k}(x - uvtp, x + uv(1-t)p) \} v p_j p_k \Bigg\} \, du \, dv \\
&= - \int_0^1 \int_0^1 \sum_{j=1}^n \Bigg\{ 2v \cdot h_{\xi_i \eta_j}(x - uvtp, x + uv(1-t)p) \\
&\quad + v^2 \cdot \sum_{k=1}^n \{ -utp_k h_{\xi_i \eta_j \xi_k}(x - uvtp, x + uv(1-t)p) \\
&\quad + uv(1-t) p_k h_{\xi_i \eta_j \eta_k}(x - uvtp, x + uv(1-t)p) \} \Bigg\} p_j \, dv \, dv \\
&= - \int_0^1 \sum_{j=1}^n \Bigg\{ \int_0^1 \frac{\partial}{\partial v} \Big(v^2 h_{\xi_i \eta_j}(x - uvtp, x + uv(1-t)p) \Big) \, dv \Bigg\} p_j \, du \\
&= - \int_0^1 \sum_{j=1}^n h_{\xi_i \eta_j}(x - utp, x + u(1-t)p) p_j \, du.
\end{aligned}
$$

□

Lemma 4.2.

$$
(\mathcal{E}_i F)(t, x, p) = 0 \quad (i = 1, \cdots, n).
$$

Proof. From Lemma 4.1,

$$
\begin{aligned}
\frac{\partial}{\partial t} \frac{\partial F_0}{\partial p_i} &= \int_0^1 \sum_{j=1}^n \Bigg\{ \sum_{k=1}^n u p_k \{ h_{\xi_i \eta_j \xi_k}(x - utp, x + u(1-t)p) \\
&\quad + h_{\xi_i \eta_j \eta_k}(x - utp, x + u(1-t)p) \} \Bigg\} p_j \, du \\
&= \int_0^1 \sum_{j,k=1}^n u p_j p_k \{ h_{\xi_i \eta_j \xi_k}(x - utp, x + u(1-t)p) \\
&\quad + h_{\xi_i \eta_j \eta_k}(x - utp, x + u(1-t)p) \} \, du,
\end{aligned}
$$

$$\frac{\partial}{\partial x_k}\frac{\partial F_0}{\partial p_i} = -\int_0^1 \sum_{j=1}^n p_j\{h_{\xi_i\eta_j\xi_k}(x-utp, x+u(1-t)p) \\ + h_{\xi_i\eta_j\eta_k}(x-utp, x+u(1-t)p)\}\, du.$$

Thus

$$\left(\partial_t + \sum_{k=1}^n p_k\partial_{x_k}\right)\partial_{p_i}F_0 = -\sum_{j,k=1}^n \left\{\int_0^1 (1-u)\{h_{\xi_i\eta_j\xi_k}(x-utp, x+u(1-t)p) \\ + h_{\xi_i\eta_j\eta_k}(x-utp, x+u(1-t)p)\}\, du\right\} p_j p_k.$$

On the other hand,

$$\frac{\partial}{\partial x_i}F_0 = -\sum_{j,k=1}^n \left\{\int_0^1\int_0^1 v\{h_{\xi_i\eta_j\xi_k}(x-uvtp, x+uv(1-t)p) \\ + h_{\xi_i\eta_j\eta_k}(x-uvtp, x+uv(1-t)p)\}\, du\, dv\right\} p_k p_j.$$

Now note that

$$\int_0^1\int_0^1 f(uv)v\, dv\, du = \int_0^1 (1-u)f(u)\, du$$

for arbitrary continuous function f of one variable. Applying this, we get

$$\int_0^1\int_0^1 v\{h_{\xi_i\eta_j\xi_k}(x-uvtp, x+uv(1-t)p) \\ + h_{\xi_i\eta_j\eta_k}(x-uvtp, x+uv(1-t)p)\}\, du\, dv \\ = \int_0^1 (1-u)\{h_{\xi_i\eta_j\xi_k}(x-utp, x+u(1-t)p) \\ + h_{\xi_i\eta_j\eta_k}(x-utp, x+u(1-t)p)\}\, du.$$

Hence

$$\left(\partial_t + \sum_{k=1}^n p_k\partial_{x_k}\right)\partial_{p_i}F_0 = \frac{\partial}{\partial x_i}F_0.$$

Therefore,

$$(\mathcal{E}_i F_0)(t, x, p) = 0.$$

Here setting

$$C(t, x, p) = \sum_{k=1}^n p_k\{th_{\eta_k}(x, x) - (1-t)h_{\xi_k}(x, x)\} + h(x, x)$$

then

$$
\begin{aligned}
&\left(\partial_t + \sum_{k=1}^{n} p_k \partial_{x_k}\right)\partial_{p_i} C(t,x,p) \\
=&(h_{\xi_i}(x,x) + h_{\eta_i}(x,x)) \\
&+ \sum_{k=1}^{n} p_k \{t(h_{\eta_i\xi_k}(x,x) + h_{\eta_i\eta_k}(x,x)) - (1-t)(h_{\xi_i\xi_k}(x,x) + h_{\xi_i\eta_k}(x,x))\} \\
=&\partial_{x_i} C(t,x,p).
\end{aligned}
$$

Hence,

$$(\mathcal{E}_i C)(t,x,p) = 0.$$

Therefore

$$(\mathcal{E}_i F)(t,x,p) = (\mathcal{E}_i F_0)(t,x,p) + (\mathcal{E}_i C)(t,x,p) = 0$$

for all $i = 0, \cdots, n$. □

Lemma 4.3.

$$F_{p_i}(0,\xi,\eta-\xi) = -h_{\xi_i}, \quad F_{p_i}(1,\eta,\eta-\xi) = h_{\eta_i} \qquad (i = 1,\cdots,n).$$

Proof. Setting $t = 0$ at (4.1), we get

$$
\begin{aligned}
\frac{\partial F_0}{\partial p_i}(0,x,p) &= -\int_0^1 \sum_{j=1}^{n} h_{\xi_i\eta_j}(x, x+up) p_j \, du \\
&= -\int_0^1 \frac{\partial}{\partial u}\left\{h_{\xi_i}(x,x+up)\right\} du \\
&= -h_{\xi_i}(x,x+p) + h_{\xi_i}(x,x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
F_{p_i}(0,\xi,\eta-\xi) = &\frac{\partial F_0}{\partial p_i}(0,\xi,\eta-\xi) - h_{\xi_i}(\xi,\xi) \\
&= -h_{\xi_i}(\xi,\eta) + h_{\xi_i}(\xi,\xi) - h_{\xi_i}(\xi,\xi) = -h_{\xi_i}(\xi,\eta).
\end{aligned}
$$

The second equation can be proved similarly. □

Lemma 4.4.

$$\det\left(\frac{\partial^2 F}{\partial p_i \partial p_j}\right) \neq 0.$$

Proof. Differentiation (4.1) with respect to p_j $(j = 1, \cdots, n)$, we get

$$\begin{aligned}
&\frac{\partial^2 F}{\partial p_i \partial p_j}(t,x,p) \\
&= -\int_0^1 \Big\{ h_{\xi_i\eta_j}(x-utp, x+u(1-t)p) + \sum_{k=1}^n \{-uth_{\xi_i\eta_k\xi_j}(x-utp, x+u(1-t)p) \\
&\quad + u(1-t)h_{\xi_i\eta_k\eta_j}(x-utp, x+u(1-t)p)\}p_k \Big\}\, du.
\end{aligned}$$

Applying again the integrability condition (1.3), we get $h_{\xi_i\eta_k\xi_j} = h_{\xi_i\eta_j\xi_k}$. Thus

$$\begin{aligned}
&= -\int_0^1 \Big\{ h_{\xi_i\eta_j}(x-utp, x+u(1-t)p) \\
&\qquad + u\sum_{k=1}^n \{-tp_k h_{\xi_i\eta_j\xi_k}(x-utp, x+u(1-t)p) \\
&\quad + (1-t)p_k h_{\xi_i\eta_j\eta_k}(x-utp, x+u(1-t)p)\} \Big\}\, du \\
&= -\int_0^1 \frac{\partial}{\partial u}\Big\{ u h_{\xi_i\eta_j}(x-utp, x+u(1-t)p) \Big\}\, du \\
&= -\, h_{\xi_i\eta_j}(x-tp, x+(1-t)p).
\end{aligned}$$

From Legendre condition (1.2), we obtain the required statement. □

Consequently, the Lagrangian $F = F(t,p,x)$ defined by (3.2), (3.3) satisfies the condition (3.1). As discussed in the previous section, the Hamiltonian $H = H(t,x,y)$ is obtained by the Legendre transformation

$$y_i = F_{p_i}(t,x,p), \quad H(t,x,y) = y \cdot p - F(t,x,p).$$

And the Hamiltonian map defined by the the Hamiltonian system for H coincides with φ. In particular, φ is a Hamiltonian map. This completes the proof of Theorem 1.2.

§5. Conclusion

We would like to mention a relation between the generating function h and the extremal integral S defined by (2.4). According to Proposition 2.2, it turns out that we constructed the Lagrangian F so that S satisfies

$$dS = dh.$$

As a matter of fact, it is easy to verify that the more strong condition holds; $S = h$.

Since our result is to obtain the Hamiltonian function H of concrete from the generating function h, we can treat several applications of this.

Indeed suppose that h satisfies further the following assumption;

$$(\textit{periodicity condition}) \quad h(\xi + z, \eta + z) = h(\xi, \eta) \qquad (z \in \mathbb{Z}^n),$$

then the Hamiltonian function obtained in the main theorem also has the periodicity with respect to x;

$$H(t, x + z, y) = H(t, x, y) \qquad (z \in \mathbb{Z}^n).$$

And then our main result can be extended to the case of twist mappings on the cotangent bundle $T^*\mathbb{T}^n$ of the n-torus. This subject and its relation to Hofer geometry will be treated in [OS].

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