SUT Journal of Mathematics
Vol. 39, No. 1 (2003), 71–83

Maximal inequalities for a series of continuous local martingales

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(Received February 3, 2003)

Abstract Let $\{X^j = (X_t^j, \mathcal{F}_t), j \geq 1\}$ be a sequence of continuous local martingales and $\{\langle X^j\rangle\}$ the corresponding sequence of their quadratic variation processes and let $H_n(x, y)$, $n = 1, 2, \dots$ be the Hermite polynomials with parametric variable y.

In this paper, we consider the series $\sum_{j=1}^{\infty} H_n^2(X^j, \langle X^j\rangle)$ of the continuous local martingales

$$H_n(X^j, \langle X^j\rangle) = \Big(H_n(X_t^j, \langle X^j\rangle_t), \mathcal{F}_t\Big)_{t\geq 0}, \quad j = 1, 2, \dots ,$$

and its discrete analogue, and obtain some maximal inequalities.

AMS 2000 *Mathematics Subject Classification.* 60G44, 60G42, 60H05.

Key words and phrases. Hermite polynomials, the Burkholder-Davis-Gundy inequalities, the Barlow-Yor inequalities, continuous local martingale, series of martingales and martingale transform.

§1. Introduction

Consider the Hermite polynomials $H_n(x, y)$, $n \geq 1$ with parameter y. As is well-known, for every $n = 1, 2, \dots$

$$H_n(x, y) = \left(\frac{y}{2}\right)^{\frac{n}{2}} h_n\left(\frac{x}{\sqrt{2y}}\right) \quad (y > 0) \tag{1.1}$$

where $h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. More generally, $H_n(x, y)$ can be defined as

$$H_n(x, y) = (-y)^n e^{\frac{x^2}{2y}} \frac{\partial^n}{\partial x^n} e^{-\frac{x^2}{2y}} \quad (n = 1, 2, \dots) \tag{1.2}$$

with $H_0(x,y)=1$.

Now, let $X=(X_t,\mathcal{F}_t)$ be a continuous local martingale with the quadratic variation process $\langle X\rangle$. Then the process (see [9, p.151])

$$H_n(X,\langle X\rangle)=(H_n(X_t,\langle X\rangle_t),\mathcal{F}_t)$$

is a continuous local martingale for every $n=1,2,\dots$ and

$$H_n(X_t,\langle X\rangle_t)=n\int_0^t H_{n-1}(X_s,\langle X\rangle_s)dX_s,\quad n=1,2,\dots. \tag{1.3}$$

For the process $H_n(X,\langle X\rangle)$ $(n=1,2,\dots)$, as an analog of the celebrated Burkholder-Davis-Gundy inequalities

$$c_p\big\|\langle X\rangle_T^{1/2}\big\|_p\le\big\|X_T\big\|_p\quad(1<p<\infty)$$

and

$$\big\|X_T\big\|_p\le C_p\big\|\langle X\rangle_T^{1/2}\big\|_p\quad(1\le p<\infty)$$

for all $(\mathcal{F}_t)$-stopping times T, where c_p and C_p are some positive constants depending only on p, E.Carlen and P.Krée obtained in [3] L^p–estimates (see also [11]):

$$c_{p,n}\left\|\langle X\rangle_T^{n/2}\right\|_p\le\|H_n(X_T,\langle X\rangle_T)\|_p\le C_{p,n}\left\|\langle X\rangle_T^{n/2}\right\|_p \tag{1.4}$$

with some positive constants $c_{p,n}$ and $C_{p,n}$ depending only on n and p for all stopping times T, where the right side holds for $p\ge1$ and the left side for $p>1$. In the present paper, we shall investigate the L^p–norm for the series $\sum_{j=1}^{\infty}H_n^2(X^j,\langle X^j\rangle)$, where $\{X^j=(X_t^j,(\mathcal{F}_t)),j\ge1\}$ is a sequence of continuous local martingales with their quadratic variation processes $\langle X^j\rangle,j\ge1$. For simplicity, we denote $H_n(t,j)\equiv H_n(X_t^j,\langle X^j\rangle_t)$ and $H_n(j)=(H_n(t,j),\mathcal{F}_t)$ for $n,\ j=1,2,\dots$.

Throughout this paper, we shall work with a filtered complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ with the usual conditions. Let C stand for some positive constant depending only on the subscripts and its value may be different in different appearance, and this assumption is also adaptable to c. Denote by $\mathbb{R}$ the set of real numbers.

Our main theorem is the following

Theorem 1.1. *Let $\{X^j,j\ge1\}$ be a sequence of continuous local martingales with their quadratic variation processes $\langle X^j\rangle,j\ge1$ and let $0<p<\infty$. Then* *the inequalities*

(1.5)
$$c_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p \le \Big\|\Big(\sup_{t\ge 0}\sum_{j=1}^{\infty}H_n^2(t,j)\Big)^{1/2}\Big\|_p \le C_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p$$

hold for all $n \ge 1$, where $c_{n,p}$ and $C_{n,p}$ are some positive constants depending only on n and p.

§2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

Lemma 2.1. *Let A and B be two continuous, $(\mathcal{F}_t)$–adapted, increasing processes, with $A_0 = 0$ and $B_0 = 0$, and let there exist some constants $\alpha, \beta > 0$ such that*

$$E\left[(A_T^\beta - A_S^\beta)^\alpha\right] \le C_{\alpha,\beta}\,\|B_T\|_\infty^{\alpha\beta}\,P(S < T)$$

holds for all couples (S,T) of stopping times S, T with $S \le T$. Then, for any $0 < p < \infty$, we have

$$E\left[A_\infty^p\right] \le C_{p,\alpha,\beta}E\left[B_\infty^p\right].$$

The proof of the lemma above can be found in [5]. By using the lemma, S. D. Jacka and M. Yor proved in [5] (Theorem 10 and Theorem 11) (see also [8]) that the inequalities

(2.1)
$$c_p\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty\Big)^{1/2}\Big\|_p \le \Big\|\Big(\sup_{t\ge 0}\sum_{j=1}^{\infty}(X_t^j)^2\Big)^{1/2}\Big\|_p \le C_p\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty\Big)^{1/2}\Big\|_p$$

hold for all $0 < p < \infty$ and all sequences $\{X^j\}$ of continuous local martingales with their quadratic variation processes $\{\langle X^j\rangle\}$, and furthermore, they gave also estimates on the constants c_p and C_p. In fact, more generally we have

Lemma 2.2. *Under the conditions of Theorem* 1.1*, we have*

(2.2)
$$c_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p \le \Big\|\Big(\sup_{t\ge 0}\sum_{j=1}^{\infty}(X_t^j)^{2n}\Big)^{1/2}\Big\|_p \le C_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p$$

for all $n \ge 1$.

Proof. Let

$$M_t = \left(\sum_{j=1}^{\infty}(X_t^j)^{2n}\right)^{1/2} \qquad \text{and} \qquad N_t = \left(\sum_{j=1}^{\infty}\langle X^j\rangle_t^n\right)^{1/2}.$$

For any pair (S,T) of stopping times with $S \le T$, we have

$$\begin{aligned} E\left[(M_T^*)^2 - (M_S^*)^2\right] &= E\bigg[\sup_{0\le t\le T}\sum_{j=1}^{\infty}(X_t^j)^{2n} - \sup_{0\le t\le S}\sum_{j=1}^{\infty}(X_t^j)^{2n}\bigg] \\ &\le E\bigg[\sup_{S\le t\le T}\sum_{j=1}^{\infty}(X_t^j)^{2n}1_{\{S<T\}}\bigg] \\ &\le E\bigg[\sum_{j=1}^{\infty}\Big(\sup_{S\le t\le T}|X_t^j|1_{\{S<T\}}\Big)^{2n}\bigg] \\ &\le E\bigg[\sum_{j=1}^{\infty}\Big(\sup_{0\le t<\infty}|X_{(t+S)\wedge T}^j|1_{\{S<T\}}\Big)^{2n}\bigg]. \end{aligned}$$

Noting that $\{X^j_{(t+S)\wedge T}1_{\{S<T\}}, \mathcal{F}_{(t+S)}\}$ is a continuous local martingale, we get

$$\begin{aligned} E\left[(M_T^*)^2 - (M_S^*)^2\right] &\le C_n E\left[\sum_{j=1}^{\infty}\langle X^j\rangle_T^n 1_{\{S<T\}}\right] \\ &\le C_n\left\|\sum_{j=1}^{\infty}\langle X^j\rangle_T^n\right\|_{\infty} P(S<T) \\ &= C_n\left\|N_T\right\|_{\infty}^2 P(S<T). \end{aligned}$$

It follows from Lemma 2.1 with $\alpha = 1$ and $\beta = 2$ that the right inequality in (2.2). Similarly, one can give the left inequality in (2.2). This completes the proof. □

From the proof of the lemma, we also have for all $0 < p < \infty$

$$\left\|\sum_{j=1}^{\infty}(X^j)^{*2n}\right\|_p \le C_{n,p}\left\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\right\|_p,$$

which yields

$$c_{n,p}\left\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\right\|_p \le \left\|\Big(\sum_{j=1}^{\infty}(X^j)^{*2n}\Big)^{1/2}\right\|_p \le C_{n,p}\left\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\right\|_p.$$

Now, let $X = (X_t, \mathcal{F}_t)_{t\geq 0}$ be a continuous local martingale with quadratic variation process $\langle X\rangle_t$. From (1.1) and the property of Hermite polynomials, we have

$$H_n(X_t, \langle X\rangle_t) = \sum_{i=0}^{[n/2]} C_n^{(i)} X_t^{n-2i} \langle X\rangle_t^i \tag{2.3}$$

for all $n \geq 0$, where $[x]$ stands for the integer part of x and

$$C_n^{(i)} = (-1)^i \frac{n!}{(n-2i)!i!2^i}.$$

On the other hand, it is also known that $\{H_n(X, \langle X\rangle), n \geq 2\}$ satisfies the following identity

$$\begin{aligned} &H_n(X_t, \langle X\rangle_t) H_{n-2}(X_t, \langle X\rangle_t) \\ &\quad = \frac{n}{n-1} H_{n-1}^2(X_t, \langle X\rangle_t) - \sum_{k=1}^{n} \frac{(n-2)!}{(n-k)!} H_{n-k}^2(X_t, \langle X\rangle_t) \langle X\rangle_t^{k-1}. \end{aligned} \tag{2.4}$$

This is proved in [3] by applying the Kailath-Segall identity

$$H_n(X_t, \langle X\rangle_t) = X_t H_{n-1}(X_t, \langle X\rangle_t) - (n-1)\langle X\rangle_t H_{n-2}(X_t, \langle X\rangle_t).$$

In fact, we may obtain (2.4) by applying the representation (2.3). Thus, from (2.4) we get

$$(n-2)!\langle X\rangle_t^{n-1} \leq H_{n-1}^2(X_t, \langle X\rangle_t) - H_n(X_t, \langle X\rangle_t) H_{n-2}(X_t, \langle X\rangle_t).$$

Integrating both sides of the inequality above on $[0, t]$ with respect to the measure $d\langle X\rangle_t$, we get

$$(n-2)!\langle X\rangle_t^n \leq \frac{1}{n}\big\langle H_n(X_t, \langle X\rangle_t)\big\rangle_t - n\int_0^t H_n(X_s, \langle X\rangle_s) H_{n-2}(X_s, \langle X\rangle_s) d\langle X\rangle_s \tag{2.5}$$

for all $n \geq 2$, since

$$\big\langle H_n(X_t, \langle X\rangle_t)\big\rangle_t = n^2 \int_0^t H_{n-1}^2(X_s, \langle X\rangle_s) d\langle X\rangle_s$$

from (1.3).

Proposition 2.1. *Under the conditions of Theorem* 1.1, *we have*

$$\bigg\| \Big(\sup_{t\geq 0} \sum_{j=1}^{\infty} H_{n-i}^{\frac{2n}{n-i}}(t, j)\Big)^{1/2} \bigg\|_p \leq C_{n,p} \bigg\| \Big(\sum_{j=1}^{\infty} \langle X^j\rangle_\infty^n\Big)^{1/2} \bigg\|_p \tag{2.6}$$

for all $0 \leq i < n$ *and all* $0 < p < \infty$.

Proof. Let $0 \le i < n$, $n \ge 2$ and $0 < p < \infty$.

From (2.3) and the inequality

$$\left(\sum_{i=1}^{m} a_i\right)^r \le m^{r-1}\sum_{i=1}^{m} a_i^r \qquad (a_i \ge 0, r \ge 1),$$

we have

$$H_{n-i}^{\frac{2n}{n-i}}(t,j) \le (n-i)^{\frac{2n}{n-i}-1}\sum_{k=0}^{[\frac{n-i}{2}]} |C_{n-i}^{(k)}|^{\frac{2n}{n-i}} (X_t^j)^{\frac{2n(n-i-2k)}{n-i}} \langle X^j\rangle_t^{\frac{2kn}{n-i}} \tag{2.7}$$

for all $j \ge 1$.

On the other hand, when $1 \le k < \frac{n-i}{2}$, by applying the Hölder inequality with exponents $s = \frac{n-i}{n-i-2k}$ and $r = \frac{n-i}{2k}$ and then applying Lemma 2.2 we get

$$\begin{aligned}
&\left\|\left(\sup_{t\ge 0}\sum_{j=1}^{\infty}(X_t^j)^{\frac{2n(n-i-2k)}{n-i}}\langle X^j\rangle_t^{\frac{2kn}{n-i}}\right)^{1/2}\right\|_p \\
&\qquad \le \left\|\left(\sup_{t\ge 0}\sum_{j=1}^{\infty}(X_t^j)^{2n}\right)^{1/2}\right\|_p^{\frac{n-i-2k}{n-i}}\left\|\left(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\right)^{1/2}\right\|_p^{\frac{2k}{n-i}} \\
&\qquad \le C_{n,p}\left\|\left(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\right)^{1/2}\right\|_p
\end{aligned}$$

for all $0 < p < \infty$.

Clearly, the inequality above is also true if $k = \frac{n-i}{2}$.

Combining these with (2.7) and Lemma 2.2, we obtain for $1 \le p < \infty$

$$\begin{aligned}
&\left\|\left(\sup_{t\ge}\sum_{j=1}^{\infty}H_{n-i}^{\frac{2n}{n-i}}(t,j)\right)^{1/2}\right\|_p \le (n-i)^{\frac{2n}{n-i}-1}\left\|\left(\sup_{t\ge 0}\sum_{j=1}^{\infty}(X_t^j)^{2n}\right)^{1/2}\right\|_p + \\
&\quad (n-i)^{\frac{2n}{n-i}-1}\sum_{k=1}^{[\frac{n-i}{2}]}|C_{n-i}^{(k)}|^{\frac{2n}{n-i}}\left\|\left(\sup_{t\ge 0}\sum_{j=1}^{\infty}(X_t^j)^{\frac{2n(n-i-2k)}{n-i}}\langle X^j\rangle_t^{\frac{2kn}{n-i}}\right)^{1/2}\right\|_p \\
&\qquad\qquad \le C_{n,p}\left\|\left(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\right)^{1/2}\right\|_p
\end{aligned}$$

and for $0 < p < 1$

$$\Big\|\Big(\sup_{t\geq}\sum_{j=1}^{\infty} H_{n-i}^{\frac{2n}{n-i}}(t,j)\Big)^{1/2}\Big\|_p^p \leq (n-i)^{p(\frac{2n}{n-i}-1)}\Big\|\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}(X_t^j)^{2n}\Big)^{1/2}\Big\|_p^p +$$
$$(n-i)^{p(\frac{2n}{n-i}-1)}\sum_{k=1}^{[\frac{n-i}{2}]}|C_{n-i}^{(k)}|^{\frac{2np}{n-i}}\Big\|\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}(X_t^j)^{\frac{2n(n-i-2k)}{n-i}}\langle X^j\rangle_t^{\frac{2kn}{n-i}}\Big)^{1/2}\Big\|_p^p$$
$$\leq C_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p^p.$$

This completes the proof. □

Proof of Theorem 1.1

Let $0 < p < \infty$ and $n \geq 2$.

The right inequality in (1.5) follows from Proposition 2.1 with $i = 0$.

Now, let us prove the left inequality in (1.5). By (2.5) and the Cauchy-Schwarz inequality we have

$$\Big((n-2)!\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2} \leq \frac{1}{\sqrt{n}}\Big(\sum_{j=1}^{\infty}\langle H_n(j)\rangle_\infty\Big)^{1/2}$$
$$+\sqrt{n}\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}H_n^2(t,j)\Big)^{1/4}\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}H_{n-2}^2(t,j)\langle X^j\rangle_t^2\Big)^{1/4}. \tag{2.8}$$

On the other hand, for $n > 2$, from (2.6) we have

$$\Big\|\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}H_{n-2}^{\frac{2n}{n-2}}(t,j)\Big)^{1/2}\Big\|_p \leq C_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p.$$

It follows that

$$\Big\|\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}H_{n-2}^2(t,j)\langle X^j\rangle_t^2\Big)^{1/2}\Big\|_p$$
$$\leq \Big\|\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}H_{n-2}^{\frac{2n}{n-2}}(t,j)\Big)^{1/2}\Big\|_p^{(n-2)/n}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p^{2/n}$$
$$\leq C_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p$$

by applying the Hölder inequality with exponents $s = \frac{n}{n-2}$ and $r = \frac{n}{2}$. Clearly, the inequality above is also valid for $n = 2$.

Combining these with (2.8) and (2.2), we get for $0 < p < 1$

$$\left(\sqrt{(n-2)!}\right)^p \left\| \Big(\sum_{j=1}^{\infty} \langle X^j \rangle_\infty^n\Big)^{1/2} \right\|_p^p \le c_{n,p} \left\| \Big(\sup_{t\ge 0} \sum_{j=1}^{\infty} H_n^2(t,j)\Big)^{1/2} \right\|_p^p + C_{n,p} \left\| \Big(\sup_{t\ge 0} \sum_{j=1}^{\infty} H_n^2(t,j)\Big)^{1/2} \right\|_p^{p/2} \left\| \Big(\sum_{j=1}^{\infty} \langle X^j \rangle_\infty^n\Big)^{1/2} \right\|_p^{p/2}$$

and for $1 \le p < \infty$

$$\sqrt{(n-2)!} \left\| \Big(\sum_{j=1}^{\infty} \langle X^j \rangle_\infty^n\Big)^{1/2} \right\|_p \le c_{n,p} \left\| \Big(\sup_{t\ge 0} \sum_{j=1}^{\infty} H_n^2(t,j)\Big)^{1/2} \right\|_p + C_{n,p} \left\| \Big(\sup_{t\ge 0} \sum_{j=1}^{\infty} H_n^2(t,j)\Big)^{1/2} \right\|_p^{1/2} \left\| \Big(\sum_{j=1}^{\infty} \langle X^j \rangle_\infty^n\Big)^{1/2} \right\|_p^{1/2}.$$

Solving these quadratic inequalities above, we obtain the left inequality in (1.5). This completes the proof of Theorem 1.1. □

As is well-known, for any continuous semimartingale X the Meyer–Tanaka formula

$$|X_t - x| - |X_0 - x| = \int_0^t \operatorname{sgn}(X_s - x) dX_s + \mathcal{L}_t^x(X)$$

may be considered as a definition of the local time $\{\mathcal{L}_t^x(X), t \ge 0\}$ of X at $x \in \mathbb{R}$. In particular, if X is a continuous local martingale, then $\mathcal{L}_t^x(X)$ has a continuous version in both variables. Here, we shall use such a version of local time.

The fundamental formula of occupation density for a continuous semimartingale is

$$\int_0^t \Phi(X_s) d\langle X \rangle_s = \int_{-\infty}^{\infty} \Phi(x) \mathcal{L}_t^x(X) dx$$

for all bounded, Borel functions $\Phi : \mathbb{R} \to \mathbb{R}$, which gives

$$\langle X \rangle_\infty \le 2X_\infty^* \mathcal{L}_\infty^*(X). \tag{2.9}$$

For any continuous local martingale X, M.T. Barlow and M. Yor obtained in [2] the well-known inequalities (the Barlow-Yor inequalities)

$$c_p \left\| \langle X \rangle_\infty^{1/2} \right\|_p \le \|\mathcal{L}_\infty^*(X)\|_p \le C_p \left\| \langle X \rangle_\infty^{1/2} \right\| \quad (0 < p < \infty), \tag{2.10}$$

where $\mathcal{L}_t^*(X) = \sup_{x\in\mathbb{R}} \mathcal{L}_t^x(X)$. It follows that

(2.11)

$$c_{n,p} \left\| \Big(\sum_{j=1}^{\infty} \langle X^j \rangle_\infty^n\Big)^{1/2} \right\|_p \le \left\| \Big(\sum_{j=1}^{\infty} \mathcal{L}_\infty^{*2n}(X^j)\Big)^{1/2} \right\|_p \le C_{n,p} \left\| \Big(\sum_{j=1}^{\infty} \langle X^j \rangle_\infty^n\Big)^{1/2} \right\|_p$$

for all $n \geq 1$. Indeed, the right inequality in (2.11) follows from Lemma 2.2 and (2.9), and the left inequality (2.11) can be proved by applying Lemma 2.1 and the Barlow-Yor inequalities (2.10).

Corollary 2.1. *Let $\{\mathcal{L}_t^x(n, X^j)\}$ be the local time of $H_n(j)$ at $x \in \mathbb{R}$. Then under the condition of Theorem* 1.1*, we have*

(2.12)
$$c_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p \leq \Big\|\Big(\sum_{j=1}^{\infty}\mathcal{L}_\infty^{*2n}(n, X^j)\Big)^{1/2}\Big\|_p \leq C_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle X^j\rangle_\infty^n\Big)^{1/2}\Big\|_p$$

for all $n \geq 1$.

Now, let $B = (B_t)_{t\geq 0}$ be a d-dimensional Brownian motion and let $N^j = (N_t^j)$ be a predictable process on $\mathbb{R}^d$ satisfying

$$E\left[\left(\int_0^\infty |N_s^j|^2\, ds\right)^2\right] < \infty$$

for every $j = 1, 2, 3, \cdots$, where $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^d$. Denote for every $j = 1, 2, 3, \ldots$

$$M_t^j \equiv \int_0^t N_s^j \cdot dB_s \qquad \text{and} \qquad \langle M^j\rangle_\infty \equiv \int_0^\infty |N_t^j|^2 dt.$$

Then the following corollary extends the result in [1].

Corollary 2.2. *Let $0 < p < \infty$ and let M^j $(j = 1, 2, 3, \ldots)$ be defined as above. Then the inequalities*

$$c_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle M^j\rangle_\infty^n\Big)^{1/2}\Big\|_p \leq \Big\|\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}H_n^2(M_t^j, \langle M^j\rangle_t)\Big)^{1/2}\Big\|_p$$

and

$$\Big\|\Big(\sup_{t\geq 0}\sum_{j=1}^{\infty}H_n^2(M_t^j, \langle M^j\rangle_t)\Big)^{1/2}\Big\|_p \leq C_{n,p}\Big\|\Big(\sum_{j=1}^{\infty}\langle M^j\rangle_\infty^n\Big)^{1/2}\Big\|_p$$

hold for all $n \geq 1$.

§3. A discrete analogue

In this section, we consider the discrete analogue of $H_n(X, \langle X\rangle)$.

Let $f = (f_n, \mathcal{F}_n)$ be a martingale with its difference $d = (d_k)$ and $f_0 = d_0 = 0$. Define the iteration $I^{(m)}(f) = (I_n^{(m)}(f), (\mathcal{F}_n))\,(m \geq 0)$ of martingale transforms inductively by

$$I_n^{(m)}(f) = m\sum_{k=0}^{n} I_{k-1}^{(m-1)}(f)d_k \qquad \text{and} \qquad I_{-1}^{(m)} = 0 \qquad (m \geq 0) \tag{3.1}$$

with

$$I_n^{(0)}(f) = 1 \quad \text{and} \quad I_n^{(1)}(f) = f_n \qquad \text{for } n = 0, 1, 2, \dots,$$

which are the discrete analogue of the iterated stochastic integrals. It is clear that the identity (3.1) is equivalent to

$$I_n^{(m)}(f) - I_{n-1}^{(m)}(f) = mI_{n-1}^{(m-1)}(f)d_n \qquad \text{and} \qquad I_{-1}^{(m)} = 0 \qquad (m \geq 0).$$

The next lemma is the discrete analogue of Lemma 2.1 with $\beta = \alpha = 1$.

Lemma 3.1. *Let A and B be two non-negative, $(\mathcal{F}_n)$–adapted, increasing random sequence with $A_0 = 0$ and $B_0 = 0$. If*

$$E\left[A_\infty - A_{T-1}\right] \leq CE\left[B_\infty 1_{\{T<\infty\}}\right]$$

holds for all stopping times T, then, for any $1 \leq p < \infty$, we have

$$E\left[A_\infty^p\right] \leq c_p E\left[B_\infty^p\right].$$

For the proof of the lemma, see [6] or Remark 1 in [7, p.87]. By using the lemma above, similar to the proof of Lemma 2.2, we can give the following.

Lemma 3.2. *Let $\{f^j = (f_n^j, \mathcal{F}_n),\ j = 1, 2, \dots\}$ be a sequence of martingales with their differences $\{d(j) = (d_{k,j}),\ j = 1, 2, \dots\}$ and $1 \leq p < \infty$. Then the inequality*

$$\left\|\sup_{n\geq 0}\sum_{j=1}^{\infty}|f_n^j|^m\right\|_p \leq C_{m,p}\left\|\sum_{j=1}^{\infty}S^m(f^j)\right\|_p \tag{3.2}$$

holds for all $m \geq 1$, where

$$S_n^2(f^j) = \sum_{k=0}^{n} d_{k,j}^2 \qquad \textit{and} \qquad S^2(f^j) = S_\infty^2(f^j).$$

Theorem 3.1. *Let $1 \leq p < \infty$ and $m \geq 1$. Then the inequality*

$$\left\|\sup_{n\geq 0}\sum_{j=1}^{\infty}\left(I_n^{(m-i)}(f^j)\right)^{m/(m-i)}\right\|_p \leq C_{m,p}\left\|\sum_{j=1}^{\infty}S^m(f^j)\right\|_p \tag{3.3}$$

holds for all $0 \leq i < m$.

Proof. Let $m \geq 1$, $0 \leq i < m$ and $1 \leq p < \infty$. From the definition of $I^{(m)}(f^j)$, we see that there are some constants $C_k \geq 0,\ k = 0, 1, \dots, m-i$ such that

$$I_n^{(m-i)}(f^j) \leq \sum_{k=0}^{m-i} C_k |f_n^j|^{m-i-k} S_n^k(f^j)$$

and so

$$\left(I_n^{(m-i)}(f^j)\right)^{\frac{m}{m-i}} \leq (m-i)^{\frac{m}{m-i}-1} \sum_{k=0}^{m-i} (C_k)^{\frac{m}{m-i}} |f_n^j|^{\frac{m(m-i-k)}{m-i}} S_n^{\frac{mk}{m-i}}(f^j) \tag{3.4}$$

for all j.

On the other hand, for all $1 \leq k < m-i$ by applying the Hölder inequality with exponents $s = \frac{m-i}{m-i-k}$ and $r = \frac{m-i}{k}$ and Lemma 3.2, we get

$$\begin{aligned}\left\| \sup_{n\geq 0} \sum_{j=1}^{\infty} |f_n^j|^{\frac{m(m-i-k)}{m-i}} S_n^{\frac{km}{m-i}}(f^j) \right\|_p &\leq \left\| \sup_{n\geq 0} \sum_{j=1}^{\infty} |f_n^j|^m \right\|_p^{\frac{m-i-k}{m-i}} \left\| \sum_{j=1}^{\infty} S^m(f^j) \right\|_p^{\frac{k}{m-i}} \\ &\leq C_{m,p} \left\| \sum_{j=1}^{\infty} S^m(f^j) \right\|_p .\end{aligned}$$

It follows from (3.4) that

$$\begin{aligned}\left\| \sup_{n\geq 0} \sum_{j=1}^{\infty} \left(I_n^{(m-i)}(f^j)\right)^{\frac{m}{m-i}} \right\|_p &\leq (m-i)^{\frac{m}{m-i}-1} \left\| \sup_{n\geq 0} \sum_{j=1}^{\infty} |f_n^j|^m \right\|_p + \\ (m-i)^{\frac{m}{m-i}-1} &\sum_{k=1}^{m-i} (C_k)^{\frac{m}{m-i}} \left\| \sup_{n\geq 0} \sum_{j=1}^{\infty} |f_n^j|^{\frac{m(m-i-k)}{m-i}} S_n^{\frac{km}{m-i}}(f^j) \right\|_p \\ &\leq C_{m,p} \left\| \sum_{j=1}^{\infty} S^m(f^j) \right\|_p .\end{aligned}$$

This completes the proof. □

Corollary 3.1. *Under the conditions of Theorem* 3.1*, we have*

$$\left\| \sup_{n\geq 0} \sum_{j=1}^{\infty} \left(I_n^{(m)}(f^j)\right) \right\|_p \leq C_{m,p} \left\| \sum_{j=1}^{\infty} S^m(f^j) \right\|_p$$

for all $m \geq 1$.

Now, as usual, denote

$$s_n^2(f) = \sum_{k=1}^{n} E\left[(f_k - f_{k-1})^2 \mid \mathcal{F}_{k-1}\right] \qquad \text{and} \qquad s(f) = s_\infty(f)$$

for a martingale $f = (f_n, \mathcal{F}_n)$ with $f_0 = 0$. Then we have

Corollary 3.2. *Under the conditions of Theorem* 3.1*, the inequalities*

$$\left\|\sum_{j=1}^{\infty} s\left(I^{(m)}(f^j)\right)\right\|_p \leq C_{m,p}\left\|\sum_{j=1}^{\infty} S^m(f^j)\right\|_p^{(m-1)/m}\left\|\sum_{j=1}^{\infty} s^m(f^j)\right\|_p^{1/m} \tag{3.5}$$

holds for all $1 \leq p < \infty$ *and* $m = 1, 2, 3, \ldots$.

Proof. Let $m \geq 1$ and $1 \leq p < \infty$.

Observe that $I_k^{(m)}(f^j)$ is $\mathcal{F}_k$–measurable for every $j \geq 1$, we have

$$\begin{aligned} s_n\left(I^{(m)}(f^j)\right) &= \left(\sum_{k=1}^{n} E\left[\left(I_k^{(m)}(f^j) - I_{k-1}^{(m)}(f^j)\right)^2 \,\middle|\, \mathcal{F}_{k-1}\right]\right)^{1/2} \\ &= \left(\sum_{k=1}^{n} E\left[\left(I_{k-1}^{(m-1)}(f^j)\right)^2 d_{k,j}^2 \,\middle|\, \mathcal{F}_{k-1}\right]\right)^{1/2} \\ &= \left(\sum_{k=1}^{n} \left(I_{k-1}^{(m-1)}(f^j)\right)^2 E\left[d_{k,j}^2 \,\middle|\, \mathcal{F}_{k-1}\right]\right)^{1/2} \\ &\leq \sup_{0\leq k\leq n} I_k^{(m-1)}(f^j) s_n(f^j), \end{aligned}$$

which gives (3.5) by applying the Hölder inequality with exponents $r = m$ and $s = m/(m-1)$ and Theorem 3.1. □

Acknowledgement

The first author would like to thank Professor N. Kazamaki for his guidance and kindness on the study of martingales and related fields. The authors wish also to thank an anonymous earnest referee for a careful reading of the manuscript and some helpful comments.

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