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Equations of Specific Cubic Surfaces and Automorphisms

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Abstract. If a cubic surface has a non-trivial automorphism, it specifies the structure of the cubic surface. When the automorphism group is big enough the isomorphic class is uniquely determined. In this paper, we provide equations of specific cubic surfaces and generators of automorphism groups explicitly.

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§1. Introduction

There are many ways to parameterize a family of non-singular cubic surfaces [Cay], [N], [S]. In [H], we used the moduli of configurations of six points in a general position on the projective plane to parameterize non-singular cubic surfaces. For a configuration of six points, we determined the automorphism group of the cubic surface associated with the configuration as a subgroup of the Weyl group $W(E_6)$ of type E_6. The largest order of automorphism groups is 648. The surface defined by the equation;

$$y_1^3 + y_2^3 + y_3^3 + y_4^3 = 0$$

is a cubic surface with an automorphism group of order 648. If a cubic surface has an automorphism group of order 648 it is isomorphic to the surface above. Also for $n = 8, 108, 120$, the isomorphic class of a cubic surface with an automorphism group of order n is unique.

Here we provide an equation of such cubic surface and generators of the automorphism group as a subgroup of the projective general linear group of the projective space.

In §2, for a configuration of six points, we take a birational map from the projective plane onto a cubic surface in the projective space in concrete form so that the equation of the image is acquired.

In §3, we proceed in the following way. Let S be a cubic surface with $\mathrm{Aut}(S) \neq \{id_S\}$. We may assume the equation of S is one derived in §2. An automorphism of S preserves the configuration of lines on S. By virtue of the above rational map and the equation of S, we derive a projective transformation preserving the equation of S from an automorphism of S.

In §4, we provide an equation of a cubic surface with a maximal automorphism group and generators of the automorphism group as projective linear transformations.

§2. Equations of Cubic Surfaces

We shall fix an algebraically closed field of characteristic $= 0$ as a ground field. We follow the notation in [H]. Put

$$U = \{((a_1 : a_2 : a_3), (b_1 : b_2 : b_3)) | gh \prod_i a_i b_i \prod_{i<j} (a_i - a_j)(b_i - b_j) \prod_{i<j} (a_i b_j - a_j b_i) \neq 0\},$$

where $g = a_1(b_2 - b_3) + a_2(b_3 - b_1) + a_3(b_1 - b_2)$ and $h = a_1b_1(a_2b_3 - a_3b_2) + a_2b_2(a_3b_1 - a_1b_3) + a_3b_3(a_1b_2 - a_2b_1)$, $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1)$ and $P_4 = (1 : 1 : 1)$. Then for a pair $P = (P_5, P_6) \in \mathbb{P}^2 \times \mathbb{P}^2$, the six points P_i $(i = 1, \ldots, 6)$ are in a general position if and only if $P \in U$.

Let l_{ij} be the line passing through P_i and P_j. Among fifteen such lines, we need the following lines and equations.

l_{12}: $f_{12} = x_3 = 0$
l_{14}: $f_{14} = -x_2 + x_3 = 0$
l_{16}: $f_{16} = -b_3x_2 + b_2x_3 = 0$
l_{23}: $f_{23} = x_1 = 0$
l_{25}: $f_{25} = a_3x_1 - a_1x_3 = 0$
l_{34}: $f_{34} = -x_1 + x_2 = 0$
l_{36}: $f_{36} = -b_2x_1 + b_1x_2 = 0$
l_{45}: $f_{45} = (-a_2 + a_3)x_1 + (a_1 - a_3)x_2 + (-a_1 + a_2)x_3 = 0$
l_{56}: $f_{56} = (-a_3b_2 + a_2b_3)x_1 + (a_3b_1 - a_1b_3)x_2 + (-a_2b_1 + a_1b_2)x_3 = 0.$

We also put
$y_1 = f_{12}f_{34}f_{56}$
$y_2 = f_{23}f_{45}f_{16}$
$y_3 = f_{12}f_{36}f_{45}$
$y_4 = f_{16}f_{25}f_{34}$
$y_5 = f_{14}f_{25}f_{36}$
$y_6 = f_{14}f_{23}f_{56}.$

Lemma 2.1. 1. *y_1, y_2, y_3, y_4 are linearly independent.*

2. $(a_1-a_2)b_3y_5 = (a_1-a_3)b_2y_1 + a_3(b_1-b_2)y_2 + (-a_3b_2+a_1b_3)y_3 + (-a_2b_1 + a_3b_1 + a_1b_2 - a_3b_2)y_4$

3. $(-a_1+a_2)b_3y_6 = (-a_1b_2+a_3b_2+a_1b_3-a_2b_3)y_1 + (-a_3b_1+a_3b_2+a_1b_3 - a_2b_3)y_2 + (a_3b_2-a_2b_3)y_3 + (a_2b_1 - a_3b_1 - a_1b_2 + a_3b_2 + a_1b_3 - a_2b_3)y_4$

Proof. y_1 and y_3 are linearly independent and y_1 and y_3 form a linear system Λ_1 with a fixed component l_{12}. y_2 and y_4 are linearly independent and y_2 and y_4 form a linear system Λ_2 with a fixed component l_{16}. Therefore $\Lambda_1 \cap \Lambda_2 = \{0\}$ so that y_1, y_2, y_3, y_4 are linearly independent.

It is straightforward to verify the relations (2) and (3). $\square$

Since y_1, y_2, y_3, y_4 are linearly independent,

$$\Psi_P = (y_1 : y_2 : y_3 : y_4) : \mathbb{P}^2 \cdots \to S_P \subset \mathbb{P}^3$$

is a birational mapping onto the closed image S_P. By the trivial relation $y_1y_2y_5 = y_3y_4y_6$, we derive the equation of S_P.

Proposition 2.2. *The equation of the cubic surface S_P is*

$F_P = (a_1-a_3)b_2y_1^2y_2 + a_3(b_1-b_2)y_1y_2^2 + (-a_3b_2+a_1b_3)y_1y_2y_3 + (-a_2b_1+a_3b_1+a_1b_2-a_3b_2)y_1y_2y_4 + (-a_1b_2+a_3b_2+a_1b_3-a_2b_3)y_1y_3y_4 + (-a_3b_1+a_3b_2+a_1b_3-a_2b_3)y_2y_3y_4 + (a_3b_2-a_2b_3)y_3^2y_4 + (a_2b_1-a_3b_1-a_1b_2+a_3b_2+a_1b_3-a_2b_3)y_3y_4^2 = 0.$

§3. Cubic Surface with a non-trivial automorphism

Let $E_i = \Psi_P(P_i)$ $(i = 1, \ldots, 6)$ be the exceptional curves. Let F_j be the proper transform of the non-singular conic passing through all P_i but P_j, and L_{ij} be the proper transform of the line l_{ij}. Then the set $\Gamma = \{E_i, F_j, L_{ij}\}$ is all lines on the cubic surface S_P. The automorphism group $G = \operatorname{Aut}\Gamma$ of the configuration Γ is isomorphic to the Weyl group $W(E_6)$ of type E_6 [M]. In [H], we defined an action of G on U [H, Proposition 3.1] and we derived that the automorphism group $\operatorname{Aut} S_P$ is isomorphic to the stabilizer subgroup G_P of the point P in U [H, Theorem 3.2].

Now let S be a cubic surface with a non-trivial automorphism. Then $S \cong S_P$ for some $P \in U$. By [H, Theorem 5.1], $G_P \cap (4A_1) \neq \emptyset$ where $(4A_1)$ is the set of all elements in G which belong to the conjugacy class $4A_1$ [Car].

Put $g = s(1,1,1,1,1)r[72] = ((a_3b_2 : a_2b_3 : a_3b_3), (a_3b_1 : a_1b_3 : a_3b_3))$, then $g \in (4A_1)$ (see [H, 6.4(2)]). Therefore we may assume $P \in U_g$, that is $g \in \operatorname{Aut} S_P$, where U_g is the set of invariant locus of g in U.

Lemma 3.1. *The action of g on $\Gamma = \{E_i, F_j, L_{ij}\}$ is as follows*

$$\begin{aligned}
&g(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,\\
&\qquad L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})\\
&= (F_6, F_5, F_4, F_3, F_2, F_1, E_6, E_5, E_4, E_3, E_2, E_1,\\
&\qquad L_{56}, L_{46}, L_{36}, L_{26}, L_{16}, L_{45}, L_{35}, L_{25}, L_{15}, L_{34}, L_{24}, L_{14}, L_{23}, L_{13}, L_{12})
\end{aligned}$$

Proof. Straightforward. Tedious but simple. □

If there is a linear transformation T such that $T(L) = g(L)$ for all $L \in \Gamma$ then T preserves S_P and the restriction of T on S_P coincide with g. Seeking for such a linear transformation, it is enough to only take specific five points in a general position each of which is an intersection of two lines in Γ.

Here we take $Q_1 = L_{12} \cap L_{34}$, $Q_2 = L_{34} \cap L_{56}$, $Q_3 = L_{12} \cap L_{56}$, $Q_4 = L_{16} \cap L_{23}$, $Q_5 = L_{15} \cap L_{24}$. Then by Lemma 3.1, $Q'_1 = g(Q_1) = L_{34} \cap L_{56}$, $Q'_2 = g(Q_2) = L_{12} \cap L_{34}$, $Q'_3 = g(Q_3) = L_{12} \cap L_{56}$, $Q'_4 = g(Q_4) = L_{16} \cap L_{45}$, $Q'_5 = g(Q_5) = L_{26} \cap L_{35}$.

Put $P = (P_5, P_6) = ((1 : a : b), (1 : c : bc)) \in U_g$.

Lemma 3.2. $Q_1 = (0 : 1 : 0 : 0)$, $Q_2 = (0 : c - ac : 1 - 2c + ac : 0)$, $Q_3 = (0 : a-b-c+2bc-abc : 0 : b(1-2c+ac))$, $Q_4 = ((1-a)b : 0 : 1-2ab : 0)$, $Q_5 = ((1-a)bc : (1-a)bc : a-bc : (-1+a)bc)$, $Q'_1 = (0 : (1-a)c : 1-2c+ac : 0)$, $Q'_2 = (0 : 1 : 0 : 0)$, $Q'_3 = (0 : a - b - c + 2bc - abc : 0 : b(1 - 2c + ac))$, $Q'_4 = (1 : 0 : 0 : 0)$, $Q'_5 = (1 : 1 : -1 : -1)$.

Proof. $Q_1 = \Psi_P(l_{12} \cap l_{34}) = \Psi_P(1 : 1 : 0) = (f_{12}f_{34}f_{56} : f_{23}f_{45}f_{16} : f_{12}f_{36}f_{45} : f_{16}f_{25}f_{34})(1 : 1 : 0) = (0 : (1-a)(c-cb) : 0 : 0) = (0 : 1 : 0 : 0)$. The others are obtained similarly. □

Let T_P be the linear transformation with $T_P(Q_i) = Q'_i$ $(i = 1, \ldots, 5)$.

Lemma 3.3.

$$T_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1-2b+ab}{(1-a)b} & \frac{1-2c+ac}{(1-a)c} & -1 & \frac{-a+b+c-2bc+abc}{(1-a)bc} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 3.4. *If S is a non-singular cubic surface with a non-trivial automorphism then S is isomorphic to S_P for some $P \in U$ where $P = ((1 : a : b), (1 : c : bc))$ and S_P is the cubic surface defined by the equation;*

$F_P = (1-b)cy_1^2y_2 + b(1-c)y_1y_2^2 + (-a+b+c-bc)y_1y_2y_4 + (-1+2b-ab)cy_1y_3y_4 + (-1+2c-ac)by_2y_3y_4 + (1-a)bcy_3^2y_4 + (a-b-c+2bc-abc)y_3y_4^2 = 0.$

The linear transformation T_P in Lemma 3.3 *preserves the equation $F_P = 0$ and induces a non-trivial automorphism of S_P.*

§4. Cubic Surface with maximal automorphism group

Let S be a non-singular cubic surface. If the order of the automorphism group of S is 648 or 120 or 108 or 8 the isomorphic class of S is unique [H]. Here we provide an equation of S and generators of $\operatorname{Aut} S$. Since the case of order 648 is well known, we treat only the other three cases.

4.1. The case of order 120

Take the point

$$P = ((1 : \frac{-1+\sqrt{5}}{2} : \frac{3-\sqrt{5}}{2}), (1 : \frac{3-\sqrt{5}}{2} : \frac{-1+\sqrt{5}}{2})).$$

Then the equation of the cubic surface S_P is

$$F_P = y_1^2 y_2 + y_1 y_2^2 + 2y_1 y_2 y_3 - y_3^2 y_4 + y_3 y_4^2 = 0.$$

The automorphism group $G_P = \operatorname{Aut} S_P$ is of order 120 and generated by the following four elements g_1, g_2, g_3, g_4.

$$g_1 = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & -1 & -2 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

The action of g_i on the configuration Γ is as follows.

$g_1(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,$
$L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})$
$= (F_6, F_4, F_5, F_2, F_3, F_1, E_6, E_4, E_5, E_2, E_3, E_1,$

$L_{46}, L_{56}, L_{26}, L_{36}, L_{16}, L_{45}, L_{24}, L_{34}, L_{14}, L_{25}, L_{35}, L_{15}, L_{23}, L_{12}, L_{13})$
$g_2(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,$
$L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})$
$= (F_6, F_3, F_2, F_5, F_4, F_1, E_6, E_3, E_2, E_5, E_4, E_1,$
$L_{36}, L_{26}, L_{56}, L_{46}, L_{16}, L_{23}, L_{35}, L_{34}, L_{13}, L_{25}, L_{24}, L_{12}, L_{45}, L_{15}, L_{14})$
$g_3(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,$
$L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})$
$= (F_5, F_6, F_4, F_3, F_1, F_2, E_5, E_6, E_4, E_3, E_1, E_2,$
$L_{56}, L_{45}, L_{35}, L_{15}, L_{25}, L_{46}, L_{36}, L_{16}, L_{26}, L_{34}, L_{14}, L_{24}, L_{13}, L_{23}, L_{12})$
$g_4(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,$
$L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})$
$= (F_4, F_6, F_5, F_1, F_3, F_2, E_4, E_6, E_5, E_1, E_3, E_2,$
$L_{46}, L_{45}, L_{14}, L_{34}, L_{24}, L_{56}, L_{16}, L_{36}, L_{26}, L_{15}, L_{35}, L_{25}, L_{13}, L_{12}, L_{23})$

4.2. The case of order 108

Take the point

$$P = ((1 : \frac{\sqrt{3}-\sqrt{-1}}{2} : \frac{\sqrt{-1}(\sqrt{3}-2-\sqrt{-1})}{2}), (1 : \frac{\sqrt{-1}(\sqrt{3}-2-\sqrt{-1})}{2} : \frac{2-\sqrt{-1}-\sqrt{3}}{2})).$$

Then the equation of the cubic surface S_P is
$F_P = y_1^2y_2 + y_1y_2^2 - (\frac{\sqrt{3}-1}{2} + \frac{(\sqrt{3}+1)\sqrt{-1}}{2})y_1y_2y_3 - (\frac{\sqrt{3}-1}{2} - \frac{(\sqrt{3}+1)\sqrt{-1}}{2})y_1y_2y_4 + \frac{(\sqrt{3}+1)(1+\sqrt{-1})}{2}y_3^2y_4 + \frac{(\sqrt{3}+1)(1-\sqrt{-1})}{2}y_3y_4^2 = 0.$

The automorphism group $G_P = \operatorname{Aut} S_P$ is of order 108 and generated by the following two elements g_1, g_2.

$$g_1 = \begin{pmatrix} \frac{-2+\sqrt{3}+\sqrt{-1}}{2} & \sqrt{-1} & \frac{\sqrt{3}+\sqrt{-1}}{2} & \frac{-1-\sqrt{-3}}{2} \\ \frac{(\sqrt{3}-1)(1-\sqrt{-1})}{2} & \frac{-1+(2-\sqrt{3})\sqrt{-1}}{2} & \frac{(\sqrt{3}-1)(1-\sqrt{-1})}{2} & \frac{-1+(2-\sqrt{3})\sqrt{-1}}{2} \\ \frac{2-\sqrt{3}-\sqrt{-1}}{2} & \frac{1-(2-\sqrt{3})\sqrt{-1}}{2} & \frac{-\sqrt{3}-\sqrt{-1}}{2} & \frac{(\sqrt{3}-1)(\sqrt{-1}-1)}{2} \\ \frac{(\sqrt{3}-1)(\sqrt{-1}-1)}{2} & \frac{-1-(2-\sqrt{3})\sqrt{-1}}{2} & \frac{(\sqrt{3}-1)(\sqrt{-1}-1)}{2} & \frac{-1-(2-\sqrt{3})\sqrt{-1}}{2} \end{pmatrix},$$

$$g_2 = \begin{pmatrix} \frac{-1-(2+\sqrt{3})\sqrt{-1}}{2} & \frac{-1-(2+\sqrt{3})\sqrt{-1}}{2} & \frac{(\sqrt{3}+1)(\sqrt{-1}-\sqrt{3})}{2} & 1+\sqrt{3} \\ -1 & \frac{\sqrt{3}-\sqrt{-1}}{2} & \frac{\sqrt{3}-\sqrt{-1}}{2} & -1 \\ \frac{-\sqrt{3}+\sqrt{-1}}{2} & \frac{-\sqrt{3}+\sqrt{-1}}{2} & 1+\sqrt{-1} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The action of g_i on the configuration Γ is as follows.
$g_1(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,$

$$\begin{aligned}&\qquad L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})\\ &= (L_{12}, F_4, L_{23}, E_4, L_{25}, L_{26}, L_{14}, F_2, L_{34}, E_2, L_{45}, L_{46},\\ &\qquad E_1, L_{56}, F_1, L_{36}, L_{35}, E_3, L_{24}, E_5, E_6, F_3, L_{16}, L_{15}, F_5, F_6, L_{13})\\ &g_2(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,\\ &\qquad L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})\\ &= (E_3, L_{45}, L_{14}, L_{15}, E_2, E_6, L_{26}, F_1, F_5, F_4, L_{36}, L_{23},\\ &\qquad L_{13}, L_{35}, L_{34}, F_6, F_2, E_4, E_5, L_{12}, L_{16}, E_1, L_{25}, L_{56}, L_{24}, L_{46}, F_3)\end{aligned}$$

4.3. The case of order 8

Take the point

$$P = ((1 : 1 - \sqrt{-1} : -\sqrt{-1}), (1 : -\sqrt{-1} - \frac{1+\sqrt{-1}}{\sqrt{2}} : (1-\sqrt{-1})(1+\frac{1}{\sqrt{2}}))).$$

Then the equation of the cubic surface S_P is

$F_P = (1-(1+\sqrt{2})\sqrt{-1})y_1^2y_2 + \frac{(\sqrt{2}+1)(1-\sqrt{-1})}{\sqrt{2}}y_1y_2^2 + (2+\sqrt{2}-(1+\sqrt{2})\sqrt{-1})y_1y_2y_3 - (1+\sqrt{2})\sqrt{-1}y_1y_2y_4 + (2\sqrt{-1} + \frac{1+3\sqrt{-1}}{\sqrt{2}})y_1y_3y_4 + (2+\sqrt{2})\sqrt{-1}y_2y_3y_4 + (-1 + 2\sqrt{-1} - \frac{1-3\sqrt{-1}}{\sqrt{2}})y_3^2y_4 + (1 + 2\sqrt{-1} + \frac{1+3\sqrt{-1}}{\sqrt{2}})y_3y_4^2 = 0.$

The automorphism group $G_P = \operatorname{Aut} S_P$ is of order 8 and generated by the following element g.

$$g = \begin{pmatrix} -1 & \frac{-1-(1+\sqrt{2})\sqrt{-1}}{2} & \frac{-(1+\sqrt{2})(1+\sqrt{-1})}{\sqrt{2}} & \frac{-1-\sqrt{2}+\sqrt{-1}}{2} \\ \sqrt{-1} & 0 & 0 & \frac{1+\sqrt{-1}}{\sqrt{2}} \\ -\sqrt{-1} & 0 & 0 & -\sqrt{-1} \\ 1 & \frac{1+\sqrt{-1}}{\sqrt{2}} & \frac{(1+\sqrt{2})(1+\sqrt{-1})}{\sqrt{2}} & \frac{1-\sqrt{-1}}{\sqrt{2}} \end{pmatrix}$$

The action of g on the configuration Γ is as follows.

$$\begin{aligned}&g(E_1, E_2, E_3, E_4, E_5, E_6, F_1, F_2, F_3, F_4, F_5, F_6,\\ &\qquad L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{23}, L_{24}, L_{25}, L_{26}, L_{34}, L_{35}, L_{36}, L_{45}, L_{46}, L_{56})\\ &= (L_{34}, L_{24}, L_{23}, E_5, E_1, E_6, F_2, F_3, F_4, L_{16}, L_{56}, L_{15},\\ &\qquad E_4, E_3, L_{25}, L_{12}, L_{26}, E_2, L_{35}, L_{13}, L_{36}, L_{45}, L_{14}, L_{46}, F_6, F_1, F_5).\end{aligned}$$

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