

Warped product submanifolds in Sasakian space forms

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(Received July 8, 2002; Revised November 6, 2002)

Abstract. Recently, Chen established a general sharp inequality for warped products in real space forms. As applications, he obtained obstructions to minimal isometric immersions of warped products into real space forms. In the present paper, we obtain sharp inequalities for warped products isometrically immersed in Sasakian space forms. Some applications are derived.

AMS 2000 Mathematics Subject Classification. 53C40, 53B25, 53C25.

Key words and phrases. Warped products, mean curvature, Sasakian space form, C -totally real submanifold.

§1. Introduction

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1 . The warped product of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ (see, for instance, [4]).

It is well-known that the notion of warped products plays some important role in Differential Geometry as well as in Physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [3].

Let $x : M_1 \times_f M_2 \rightarrow \widetilde{M}(c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold $\widetilde{M}(c)$ with constant sectional curvature c . We denote by h the second fundamental form of x and $H_i = \frac{1}{n_i} \text{trace } h_i$,

*This paper was written while the second author has visited Yamagata University, Faculty of Education, supported by a JSPS research fellowship. He would like to express his hearty thanks for the hospitality he received during this visit.

where trace h_i is the trace of h restricted to M_i and $n_i = \dim M_i$ ($i = 1, 2$). We call H_i ($i = 1, 2$) the partial mean curvature vectors.

The immersion x is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields X and Z tangent to M_1 and M_2 respectively.

In [4], Chen established the following sharp relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$.

Theorem 1.1. *Let x be an isometric immersion of an n -dimensional warped product $M_1 \times_f M_2$ into an m -dimensional Riemannian manifold $\widetilde{M}(c)$ of constant holomorphic sectional curvature c . Then:*

$$(1.1) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 c,$$

where $n_i = \dim M_i$, $i = 1, 2$, and Δ is the Laplacian operator of M_1 .

Moreover, the equality case of (1.1) holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_i , $i = 1, 2$, are the partial mean curvature vectors.

As applications, the author obtained necessary conditions for a warped product to admit a minimal isometric immersion in a Euclidean space or in a real space form (see [4]). Examples of submanifolds satisfying the equality case of (1.1) are given.

§2. Preliminaries

A $(2m + 1)$ -dimensional Riemannian manifold (\widetilde{M}, g) is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\widetilde{M}$, a vector field ξ and a 1-form η , satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \\ (\widetilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \quad \widetilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields X, Y on \widetilde{M} , where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g .

A plane section π in $T_p \widetilde{M}$ is called a ϕ -section if it is spanned by X and ϕX , where X is a unit tangent vector orthogonal to ξ . The sectional curvature of a ϕ -section is called a ϕ -sectional curvature. A Sasakian manifold with constant ϕ -sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}(c)$.

The curvature tensor of \tilde{R} of a Sasakian space form $\tilde{M}(c)$ is given by [1]

$$(2.1) \quad \begin{aligned} \tilde{R}(X, Y)Z &= \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\}+ \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned}$$

for any tangent vector fields X, Y, Z on $\tilde{M}(c)$.

As examples of Sasakian space forms we mention \mathbb{R}^{2m+1} and S^{2m+1} , with standard Sasakian structures (see [1], [8]).

Let M be an n -dimensional submanifold in a Sasakian space form $\tilde{M}(c)$ of constant ϕ -sectional curvature c . We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_p M, p \in M$, and ∇ the Riemannian connection of M , respectively. Also, let h be the second fundamental form and R the Riemann curvature tensor of M .

Then the equation of Gauss is given by

$$(2.2) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W)+ \\ &+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned}$$

for any vectors X, Y, Z, W tangent to M .

Let $p \in M$ and $\{e_1, \dots, e_n, \dots, e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p \tilde{M}(c)$, such that e_1, \dots, e_n are tangent to M at p . We denote by H the mean curvature vector, that is

$$(2.3) \quad H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i).$$

Also, we set

$$(2.4) \quad h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m+1\}.$$

and

$$(2.5) \quad \|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).$$

For any tangent vector field X to M , we put $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX , respectively. We denote by

$$(2.6) \quad \|P\|^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j).$$

We recall the following result of Chen for later use.

Lemma [2]. *Let $n \geq 2$ and a_1, \dots, a_n, b real numbers such that*

$$\left(\sum_{i=1}^n a_i \right)^2 = (n-1) \left(\sum_{i=1}^n a_i^2 + b \right)$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = \dots = a_n.$$

§3. C -totally real warped product submanifolds

Chen established a sharp relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$ (see [4]). We prove similar inequalities for warped product submanifolds of a Sasakian space form.

In this section, we investigate C -totally real warped product submanifolds in a Sasakian space form $\widetilde{M}(c)$.

A submanifold M normal to ξ in a Sasakian space form $\widetilde{M}(c)$ is said to be a C -totally real submanifold. It follows that ϕ maps any tangent space of M into the normal space, that is $\phi(T_p M) \subset T_p^\perp M$, for every $p \in M$.

Theorem 3.1. *Let x be a C -totally real isometric immersion of an n -dimensional warped product $M_1 \times_f M_2$ into a $(2m+1)$ -dimensional Sasakian space form $\widetilde{M}(c)$. Then:*

$$(3.1) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c+3}{4},$$

where $n_i = \dim M_i, i = 1, 2$, and Δ is the Laplacian operator of M_1 .

Moreover, the equality case of (3.1) holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors.

Proof. Let $M_1 \times_f M_2$ be a C -totally real warped product submanifold into a Sasakian space form $\widetilde{M}(c)$ of constant ϕ -sectional curvature c .

Since $M_1 \times_f M_2$ is a warped product, it is easily seen that

$$(3.2) \quad \nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z,$$

for any vector fields X, Z tangent to M_1, M_2 , respectively.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$(3.3) \quad K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f} \{(\nabla_X X)f - X^2 f\}.$$

We choose a local orthonormal frame $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$, such that e_1, \dots, e_{n_1} are tangent to M_1 , e_{n_1+1}, \dots, e_n are tangent to M_2 , e_{n+1} is parallel to the mean curvature vector H and $e_{2m+1} = \xi$.

Then, using (3.3), we get

$$(3.4) \quad \frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s),$$

for each $s \in \{n_1 + 1, \dots, n\}$.

From the equation of Gauss, we have

$$(3.5) \quad n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1) \frac{c+3}{4},$$

where τ denotes the scalar curvature of $M_1 \times_f M_2$, that is,

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

We set

$$(3.6) \quad \delta = 2\tau - n(n-1) \frac{c+3}{4} - \frac{n^2}{2} \|H\|^2.$$

Then, (3.5) can be written as

$$(3.7) \quad n^2 \|H\|^2 = 2(\delta + \|h\|^2).$$

With respect to the above orthonormal frame, (3.7) takes the following form:

$$\left(\sum_{i=1}^n h_{ii}^{n+1} \right)^2 = 2 \left\{ \delta + \sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 \right\}.$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^n h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^3 a_i \right)^2 = 2 \left\{ \delta + \sum_{i=1}^3 a_i^2 + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 - \right.$$

$$\left. - \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \right\}.$$

Thus a_1, a_2, a_3 satisfy the Lemma of Chen (for $n = 3$), i.e.

$$\left(\sum_{i=1}^3 a_i \right)^2 = 2 \left(b + \sum_{i=1}^3 a_i^2 \right),$$

with

$$b = \delta + \sum_{1 \leq i \neq j \leq n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h_{ij}^r)^2 - \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1}.$$

Then $2a_1 a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this means

$$(3.8) \quad \sum_{1 \leq j < k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} + \sum_{n_1+1 \leq s < t \leq n} h_{ss}^{n+1} h_{tt}^{n+1} \geq \\ \geq \frac{\delta}{2} + \sum_{1 \leq \alpha < \beta \leq n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{\alpha, \beta=1}^n (h_{\alpha\beta}^r)^2.$$

Equality holds if and only if

$$(3.9) \quad \sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^n h_{tt}^{n+1}.$$

Using again the Gauss equation, we have

$$(3.10) \quad n_2 \frac{\Delta f}{f} = \tau - \sum_{1 \leq j < k \leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1 \leq s < t \leq n} K(e_s \wedge e_t) = \\ = \tau - \frac{n_1(n_1-1)(c+3)}{8} - \sum_{r=n+1}^{2m} \sum_{1 \leq j < k \leq n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) - \\ - \frac{n_2(n_2-1)(c+3)}{8} - \sum_{r=n+1}^{2m} \sum_{n_1+1 \leq s < t \leq n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2).$$

Combining (3.8) and (3.10) and taking account of (3.4), we obtain

$$(3.11) \quad n_2 \frac{\Delta f}{f} \leq \tau - \frac{n(n-1)(c+3)}{8} + n_1 n_2 \frac{c+3}{4} - \frac{\delta}{2}$$

$$\begin{aligned}
 & - \sum_{1 \leq j \leq n_1; n_1+1 \leq t \leq n} (h_{jt}^{n+1})^2 - \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{\alpha, \beta=1}^n (h_{\alpha\beta}^r)^2 + \\
 & + \sum_{r=n+2}^{2m} \sum_{1 \leq j < k \leq n_1} ((h_{jk}^r)^2 - h_{jj}^r h_{kk}^r) + \sum_{r=n+2}^{2m} \sum_{n_1+1 \leq s < t \leq n} ((h_{st}^r)^2 - h_{ss}^r h_{tt}^r) = \\
 & = \tau - \frac{n(n-1)(c+3)}{8} + n_1 n_2 \frac{c+3}{4} - \frac{\delta}{2} - \sum_{r=n+1}^{2m} \sum_{j=1}^{n_1} \sum_{t=n_1+1}^n (h_{jt}^r)^2 - \\
 & - \frac{1}{2} \sum_{r=n+2}^{2m} \left(\sum_{j=1}^{n_1} h_{jj}^r \right)^2 - \frac{1}{2} \sum_{r=n+2}^{2m} \left(\sum_{t=n_1+1}^n h_{tt}^r \right)^2 \leq \\
 & \leq \tau - \frac{n(n-1)(c+3)}{8} + n_1 n_2 \frac{c+3}{4} - \frac{\delta}{2} = \\
 & = \frac{n^2}{4} \|H\|^2 + n_1 n_2 \frac{c+3}{4},
 \end{aligned}$$

which implies the inequality (3.1).

We see that the equality sign of (3.11) holds if and only if

$$(3.12) \quad h_{jt}^r = 0, \quad 1 \leq j \leq n_1, n_1 + 1 \leq t \leq n, n + 1 \leq r \leq 2m,$$

and

$$(3.13) \quad \sum_{i=1}^{n_1} h_{ii}^r = \sum_{t=n_1+1}^n h_{tt}^r = 0, \quad n + 2 \leq r \leq 2m.$$

Obviously (3.12) is equivalent to the mixed totally geodesicness of the warped product $M_1 \times_f M_2$ and (3.9) and (3.13) implies $n_1 H_1 = n_2 H_2$.

The converse statement is straightforward. \square

As applications, we derive certain obstructions to the existence of minimal C -totally real warped product submanifolds in Sasakian space forms.

Corollary 3.2. *Let $M_1 \times_f M_2$ be a warped product whose warping function f is harmonic. Then:*

(i) $M_1 \times_f M_2$ admits no minimal C -totally real immersion into a Sasakian space form $\widetilde{M}(c)$ with $c < -3$.

(ii) Every minimal C -totally real immersion of $M_1 \times_f M_2$ in the standard Sasakian space form \mathbb{R}^{2m+1} is a warped product immersion.

Proof. Assume f is a harmonic function on M_1 and $M_1 \times_f M_2$ admits a minimal C -totally real immersion in a Sasakian space form $\widetilde{M}(c)$. Then, the inequality (3.1) becomes $c \geq -3$.

If $c = -3$, the equality case of (3.1) holds. By Theorem 3.1, it follows that $M_1 \times_f M_2$ is mixed totally geodesic and $H_1 = H_2 = 0$. A well-known result of Nölker [7] implies that the immersion is a warped product immersion. \square

Corollary 3.3. *If the warping function f of a warped product $M_1 \times_f M_2$ is an eigenfunction of the Laplacian on M_1 with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ does not admit a minimal C -totally real immersion in a Sasakian space form $\widetilde{M}(c)$ with $c \leq -3$.*

We give an example of a C -totally real submanifold which satisfies the equality case of (3.1).

Consider $S^5 \subset S^7$ and let n be a unit vector orthogonal to the linear subspace containing S^5 . Let N be any minimal C -totally real surface of S^5 and define the warped product manifold

$$M = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\cos t} N.$$

It is isometrically immersed in S^7 by

$$\psi : M \rightarrow S^7, \quad \psi(t, p) = (\sin t)n + (\cos t)p.$$

This immersion is C -totally real and satisfies the equality case of (3.1) (see also [5]).

§4. Warped product submanifolds tangent to the Reeb vector field ξ

In this section, we investigate warped product submanifolds tangent to the structure vector field ξ in a Sasakian space form $\widetilde{M}(c)$.

We distinguish 2 cases:

- (a) ξ is tangent to M_1 ;
- (b) ξ is tangent to M_2 .

Theorem 4.1. *Let $\widetilde{M}(c)$ be a $(2m + 1)$ -dimensional Sasakian space form and $M_1 \times_f M_2$ an n -dimensional warped product submanifold, such that ξ is tangent to M_1 . Then:*

$$(4.1) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c+3}{4} - \frac{c-1}{4},$$

where $n_i = \dim M_i, i = 1, 2$, and Δ is the Laplacian operator of M_1 .

Moreover, the equality case of (4.1) holds if and only if $M_1 \times_f M_2$ is a mixed totally geodesic submanifold and $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors.

Proof. Let $M_1 \times_f M_2$ be a warped product submanifold of a Sasakian space form $\widetilde{M}(c)$ with constant ϕ -sectional curvature c , such that ξ is tangent to M_1 . It is obvious that M_2 is a C -totally real submanifold of $\widetilde{M}(c)$.

We choose a local orthonormal frame $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_{n_1} = \xi$ are tangent to M_1 , e_{n_1+1}, \dots, e_n are tangent to M_2 and e_{n+1} is parallel to H .

From the equation of Gauss, we have

$$(4.2) \quad n^2 \|H\|^2 = 2\tau + \|h\|^2 - n(n-1)\frac{c+3}{4} - (3\|P\|^2 - 2n+2)\frac{c-1}{4}.$$

We denote

$$(4.3) \quad \delta = 2\tau - n(n-1)\frac{c+3}{4} - (3\|P\|^2 - 2n+2)\frac{c-1}{4} - \frac{n^2}{2}\|H\|^2.$$

Then, (4.2) takes the form

$$(4.4) \quad n^2 \|H\|^2 = 2(\delta + \|h\|^2).$$

We will use the same method as in the proof of Theorem 3.1. We will point-out only the differences.

Using again the Gauss equation, we obtain

$$(4.5) \quad \begin{aligned} n_2 \frac{\Delta f}{f} &= \tau - \sum_{1 \leq j < k \leq n_1} K(e_j \wedge e_k) - \sum_{n_1+1 \leq s < t \leq n} K(e_s \wedge e_t) = \\ &= \tau - \frac{n_1(n_1-1)(c+3)}{8} - \left[3 \sum_{1 \leq j < k \leq n_1-1} g^2(Pe_j, e_k) - n_1 + 1 \right] \frac{c-1}{4} - \\ &- \sum_{r=n+1}^{2m+1} \sum_{1 \leq j < k \leq n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) - \frac{n_2(n_2-1)(c+3)}{8} - \sum_{r=n+1}^{2m+1} \sum_{n_1+1 \leq s < t \leq n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2). \end{aligned}$$

Applying the Lemma of Chen to (4.4) and substituting (4.5), similar computations as in the proof of Theorem 3.1 lead to

$$(4.6) \quad \begin{aligned} n_2 \frac{\Delta f}{f} &\leq \tau - \frac{n(n-1)(c+3)}{8} + n_1 n_2 \frac{c+3}{4} - \frac{\delta}{2} - \\ &- \left[3 \sum_{1 \leq j < k \leq n_1-1} g^2(Pe_j, e_k) - n_1 + 1 \right] \frac{c-1}{4}. \end{aligned}$$

Using (4.3), the inequality (4.6) becomes

$$n_2 \frac{\Delta f}{f} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 \frac{c+3}{4} - n_2 \frac{c-1}{4},$$

i.e. the inequality to prove.

The equality case of the inequality (4.1) is similar to the equality case of (3.1). \square

Assume now that $M_1 \times_f M_2$ is a warped product submanifold of a Sasakian space form $\widetilde{M}(c)$ such that ξ is tangent to M_2 .

If we put $Z = \xi$ in (3.2), it follows that $Xf = 0$, for all vector fields X tangent to M_1 . Thus f is constant and the warped product becomes a Riemannian product.

Proposition 4.2. *Any warped product submanifold $M_1 \times_f M_2$ of a Sasakian space form $\widetilde{M}(c)$ such that ξ is tangent to M_2 is a Riemannian product.*

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