SUT Journal of Mathematics
Vol. 38, No. 2 (2002), 175–184

On edge–magic disconnected graphs

Jaroslav Ivančo and Iveta Lučkaničová

(Received October 29, 2002)

Abstract. A graph G is called edge-magic if it admits a labeling of the vertices and edges by pairwise different integers of $1, 2, \ldots, |V(G)| + |E(G)|$ such that the sum of the label of an edge and the labels of its endpoints is constant independent of the choice of edge. A construction of edge-magic labelings of some disconnected graphs is described. Some edge-magic forests are characterized.

AMS 1991 *Mathematics Subject Classification.* 05C78.

Key words and phrases. Edge-magic graphs, super edge-magic graphs, stars.

§1. Introduction

We consider finite undirected graphs without loops and multiple edges. $V(G)$ and $E(G)$ stand for the vertex set and edge set of a graph G, respectively.

Let G be a graph with p vertices and q edges. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, p+q\}$ is called an *edge-magic total labeling* of G if there exists a constant σ (called the *magic number* of f) such that $f(u)+f(v)+f(uv) = \sigma$ for any edge uv of G. An edge-magic total labeling f is called *super edge-magic* if $f(V(G)) = \{1, 2, \ldots, p\}$ (and so $f(E(G)) = \{p+1, \ldots, p+q\}$). If f is a super edge-magic total labeling of G, then there is an integer μ (clearly, $\mu+p+q = \sigma$) such that

$$\text{(P)} \qquad \{f(x) + f(y) \colon xy \in E(G)\} = \{\mu, \mu + 1, \ldots, \mu + q - 1\}.$$

On the other hand, there exists exactly one extension of a bijection $f \colon V(G) \to \{1, 2, \ldots, p\}$ satisfying (P) to a super edge-magic labeling of G (for any edge xy we put $f(xy) = \mu + p + q - f(x) - f(y)$, see also [6]).

A graph G is called *edge-magic* (*super edge-magic*) if there exists an edge-magic (super edge-magic, respectively) total labeling of G. The concept of edge-magic graphs was introduced by Kotzig and Rosa [8] (under the name of graph with magic valuation). Super edge-magic graphs were introduced by Enomoto, Llado, Nakamigawa and Ringel [2]. More comprehensive information on edge-magic and super edge-magic graphs can be found in [7].

In this paper we describe some constructions of (super) edge-magic total labelings of some disconnected graphs.

§2. Unions of disjoint graphs

A mapping $c\colon V(G)\cup E(G)\to\{1,2,3\}$ is called an *e-m-coloring* of a graph G if $\{c(u),c(v),c(uv)\}=\{1,2,3\}$ for any edge uv of G.

Now, we can prove the following result for a disjoint union of graphs.

Theorem 1. *Let n be an odd positive integer. For $i=1,2,\dots,n$, let G_i, g_i and c_i be an edge-magic graph with p_i vertices and q_i edges, an edge-magic total labeling of G_i with its magic number σ_i and an e-m-coloring of G_i, respectively. Suppose that the following conditions are satisfied*

(1) *there is an integer σ such that $\sigma_i=\sigma$ for all $i=1,2,\dots,n$,*
(2) *if $g_i(x)=g_j(y)$, then $c_i(x)=c_j(y)$, for all $i,j=1,2,\dots,n$, $x\in V(G_i)\cup E(G_i)$ and $y\in V(G_j)\cup E(G_j)$,*
(3) *there is an integer r such that $r=p_1+q_1\ge\cdots\ge p_n+q_n\ge r-1$.*

Then the disjoint union $\cup_{i=1}^n G_i$ is an edge-magic graph.

Moreover, if all g_i are super edge-magic labelings and $p_1=p_2=\cdots=p_n$, then $\cup_{i=1}^n G_i$ is a super edge-magic graph.

Proof. n is an odd integer, so there exists an integer k such that $n=2k+1$. Consider a mapping $\alpha\colon\{1,2,3\}\times\{1,2,\dots,n\}\to\{1,2,\dots,n\}$ defined by

$$\alpha(j,i)=\begin{cases} i+k+1 & \text{for } j=1 \text{ and } i=1,\dots,k,\\ i-k & \text{for } j=1 \text{ and } i=k+1,\dots,n,\\ 1+n-2i & \text{for } j=2 \text{ and } i=1,\dots,k,\\ 1+2n-2i & \text{for } j=2 \text{ and } i=k+1,\dots,n,\\ i & \text{for } j=3 \text{ and } i=1,\dots,n.\end{cases}$$

It is easy to see that $\alpha(1,i)$, $\alpha(2,i)$ and $\alpha(3,i)$ are permutations of $\{1,2,\dots,n\}$. Moreover, $\alpha(1,i)+\alpha(2,i)+\alpha(3,i)=3k+3=3\left\lceil\frac{n}{2}\right\rceil$ for every $i=1,2,\dots,n$.

Without loss of generality we can assume that $c_1(x)=3$ for $x\in V(G_1)\cup E(G_1)$ such that $g_1(x)=r$ (and by (2), $c_i(g_i^{-1}(r))=3$ if $p_i+q_i=r$). Now, consider a mapping f from $V(\cup_{i=1}^n G_i)\cup E(\cup_{i=1}^n G_i)$ into integers given by

$$f(x)=(g_i(x)-1)n+\alpha(c_i(x),i)\quad\text{whenever}\quad x\in V(G_i)\cup E(G_i).$$

According to (2), for every $t\in\{1,2,\dots,r-1\}$ there exists $j\in\{1,2,3\}$ such that $c_i(g_i^{-1}(t))=j$ for all $i=1,2,\dots,n$. As $\alpha(j,i)$ is a permutation, it is not difficult to check that the mapping f uses each integer $1,2,\dots,|V(\cup_{i=1}^n G_i)\cup E(\cup_{i=1}^n G_i)|$ exactly once. Moreover, if $uv\in E(G_i)$, then $f(u)+f(v)+f(uv)=(g_i(u)+g_i(v)+g_i(uv)-3)n+\alpha(c_i(u),i)+\alpha(c_i(v),i)+\alpha(c_i(uv),i)$. Since g_i is an edge-magic total labeling with magic number σ and c_i is an e-m-coloring we have $f(u)+f(v)+f(uv)=(\sigma-3)n+3\left\lceil\frac{n}{2}\right\rceil$. Therefore, the mapping f is an edge-magic total labeling of the graph $\cup_{i=1}^n G_i$.

If all g_i are super edge-magic, then $1 \le f(u) \le (p_i - 1)n + n = |V(\cup_{i=1}^n G_i)|$ for any $u \in V(\cup_{i=1}^n G_i)$. Thus, f is a super edge-magic total labeling, too. $\square$

A caterpillar is a tree with the property that the removal of its pendant vertices leaves a path. Each caterpillar with parts V_1 and V_2 admits a super edge-magic total labeling such that the vertices of V_1 are labeled by $1, \dots, |V_1|$, the vertices of V_2 by $|V_1| + 1, \dots, |V_1| + |V_2|$, the edges by $|V_1| + |V_2| + 1, \dots, 2|V_1| + 2|V_2| - 1$ and its magic number is $3|V_1| + 2|V_2| + 1$ (see [8] or [9]). Then by Theorem 1, we immediately have

Corollary 1. *Let $n \equiv 1 \pmod 2$, p_1 and p_2 be positive integers. For every $i \in \{1, 2, \dots, n\}$, let T_i be a caterpillar having parts with p_1 and p_2 vertices. Then $\cup_{i=1}^n T_i$ is a super edge-magic graph.*

Evidently, an e-m-coloring of G induces a proper (vertex) coloring of G. On the other hand, let $c^* : V(G) \to \{1, 2, 3\}$ be a (proper) 3-coloring of G. Clearly, a mapping $c : V(G) \cup E(G) \to \{1, 2, 3\}$ defined by $c(u) = c^*(u)$ for $u \in V(G)$ and $\{c(uv)\} = \{1, 2, 3\} - \{c^*(u), c^*(v)\}$ for $uv \in E(G)$ is an e-m-coloring of G. So, we immediately obtain: *there exists an e-m-coloring of a graph G if and only if G is 3-colorable.* In [7] there is mentioned that Figueroa-Centeno, Ichishima and Muntaner-Batle [5] prove the following: *if G is a bipartite or tripartite (super) edge-magic graph then nG is (super) edge-magic when n is odd.* By Theorem 1 we obtain an extension of this result.

Corollary 2. *Let G be a 3-colorable graph. Let e be an edge of G such that there is a (super) edge-magic labeling f of G where $f(e) = |V(G)| + |E(G)|$. Then a graph $nG \cup m(G - e)$ is (super) edge-magic for any $n \ge 0$, $m \ge 0$, $1 \le n + m \equiv 1 \pmod 2$.*

In [10] there is proved that nC_k and nP_k are edge-magic when n is an odd integer. A path P_k on k vertices is a caterpillar. Thus, P_k is super edge-magic. A cycle C_k on k vertices is super edge-magic for k odd (see [2]). Moreover, it admits an edge-magic labeling with its maximal value on an edge for all $k \ge 3$ (see [8]). As $C_k - e = P_k$ for any edge e of C_k, then by Corollary 2, we have

Corollary 3. *For nonnegative integers n, m, the following statements hold:*

- *$nC_k \cup mP_k$ is an edge-magic graph when $1 \le n + m \equiv 1 \pmod 2$.*
- *$nC_k \cup mP_k$ is a super edge-magic graph when $1 \le n + m \equiv 1 \pmod 2$ and k is odd.*
- *mP_k is a super edge-magic graph when $m \ge 1$ is odd.*

§3. Unions of two stars

In this part we consider a graph $K_{1,m} \cup K_{1,n}$ for $m \ge 1$, $n \ge 1$. Denote vertices of the graph by $u_{i,j}$, where either $i = 1$ and $j = 0, 1, \dots, m$, or $i = 2$ and $j = 0, 1, \dots, n$, in such a way that its edges are $u_{i,0}u_{i,j}$ for $i \in \{1, 2\}$ and all $j \ge 1$.

In [9] the following assertion is introduced: *If $|E(G)|$ is even, $|V(G)| + |E(G)| \equiv 2 \pmod 4$ and each vertex has odd degree in a graph G, then G is not edge-magic.* Hence, $K_{1,m} \cup K_{1,n}$ is not edge-magic if m and n are both odd. If n is even, then there is an integer t such that $n = 2t$. In this case it is not difficult to check that a mapping f defined by

$$f(u_{i,j}) = \begin{cases} 2+2m+3t & \text{if } i=1 \text{ and } j=0, \\ j & \text{if } i=1 \text{ and } j=1,\dots,m, \\ 1+m+t & \text{if } i=2 \text{ and } j=0, \\ m+j & \text{if } i=2 \text{ and } j=1,\dots,t, \\ 1+m+j & \text{if } i=2 \text{ and } j=t+1,\dots,2t, \end{cases}$$
$$f(u_{i,0}u_{i,j}) = \begin{cases} 2+2m+2t-j & \text{if } i=1 \text{ and } j=1,\dots,m, \\ 3+2m+4t-j & \text{if } i=2 \text{ and } j=1,\dots,t, \\ 2+2m+4t-j & \text{if } i=2 \text{ and } j=t+1,\dots,2t, \end{cases}$$

is an edge-magic total labeling of $K_{1,m} \cup K_{1,2t}$ with magic number $4+4m+5t$. Therefore, we get the following result (see also [5]).

Theorem 2. *$K_{1,m} \cup K_{1,n}$ is an edge-magic graph if and only if mn is even.*

In [5] the authors prove the previous result and also sufficient condition of the next result. However, they only conjecture the necessary condition.

Theorem 3. *$K_{1,m} \cup K_{1,n}$ is a super edge-magic graph if and only if either m is a multiple of $n+1$ or n is a multiple of $m+1$.*

Proof. Let f be a super edge-magic total labeling of $K_{1,m} \cup K_{1,n}$. Assume that central vertices are labeled by l_1 and l_2 (i.e., $f(u_{1,0}) = l_1$ and $f(u_{2,0}) = l_2$). As f satisfies (P), we have

$$\frac{1}{2}(2\mu+m+n-1)(m+n) = \mu + (\mu+1) + \dots + (\mu+m+n-1) =$$
$$\sum_{xy\in E}(f(x)+f(y)) = (m-1)f(u_{1,0}) + (n-1)f(u_{2,0}) + \sum_{z\in V} f(z) =$$
$$(m-1)l_1 + (n-1)l_2 + (1+2+\dots+(m+n+2)) =$$
$$(m-1)l_1 + (n-1)l_2 + \frac{1}{2}(m+n+3)(m+n+2).$$

Hence

$$(*) \qquad \mu(m+n) = 3(m+n+1) + (m-1)l_1 + (n-1)l_2.$$

Clearly, $l_1 + l_2 \notin \{\mu,\dots,\mu+m+n-1\}$ because exactly one endpoint of any edge belongs to $\{u_{1,0}, u_{2,0}\}$. Without loss of generality we can assume that

$l_1+l_2<\mu$ (if $l_1+l_2>\mu+m+n-1$, then we take a super edge-magic labeling g given by $g(u_{i,j})=3+m+n-f(u_{i,j})$). Then $1\in\{l_1,l_2\}$ because an edge xy with endpoint labeled by 1 satisfies $\mu\le f(x)+f(y)=1+f(u_{i,0})<l_1+l_2$ otherwise. Suppose $l_2=1$.

If $l_1=2$, then according to $(*)$ we get

$$\mu(m+n)=3(m+n+1)+2(m-1)+(n-1)=4(m+n)+m.$$

This implies that m is a multiple of $m+n$, a contradiction. Therefore, $l_1>2$. Then, $\mu=l_1+2$ because the vertex labeled 2 must belong to $K_{1,m}$ and by $(*)$ we have $(l_1+2)(m+n)=3(m+n+1)+(m-1)l_1+(n-1)$. Hence, $m=(l_1-2)(n+1)$, which means $m>n$ and m is a multiple of $n+1$.

On the other hand, assume that $m=t(n+1)$. It is not difficult to check that a mapping f given by

$$f(u_{i,j})=\begin{cases}2+t & \text{if } i=1 \text{ and } j=0,\\ \lceil \frac{j}{t}\rceil+j & \text{if } i=1 \text{ and } j=1,\dots,m,\\ 1 & \text{if } i=2 \text{ and } j=0,\\ 1+(j+1)(t+1) & \text{if } i=2 \text{ and } j=1,\dots,n,\end{cases}$$

satisfies (P) for $\mu=t+4$. Thus, $K_{1,m}\cup K_{1,n}$ is super edge-magic. □

§4. Attached graphs

A super edge-magic labeling f of a graph G is said to be *k-interlaced* if for each edge xy either $f(x)\le k<f(y)$ or $f(y)\le k<f(x)$. Clearly, a graph with a k-interlaced labeling is necessarily bipartite and $\{f^{-1}(i)\colon i=1,\dots,k\}$, $\{f^{-1}(i)\colon i=k+1,\dots,|V(G)|\}$ are its parts. Moreover, if f is k-interlaced, then a super edge-magic labeling g, given by $g(x)=1+|V(G)|-f(x)$ for each vertex x, is $(|V(G)|-k)$-interlaced.

Suppose that $v_1,\dots,v_k$ is a subset of vertex set of a graph G_1 and $u_1,\dots,u_k$ is an independent set of a graph G_2. $G_1(v_1,\dots,v_k)\odot G_2(u_1,\dots,u_k)$ denotes the graph obtained by identifying each vertex v_i with a vertex u_i, $i=1,\dots k$. Evidently, $G_1(v_1,\dots,v_k)\odot G_2(u_1,\dots,u_k)$ has $|V(G_1)|+|V(G_2)|-k$ vertices and $|E(G_1)|+|E(G_2)|$ edges.

Theorem 4. *Let g be a super edge-magic labeling of a graph G with the magic number σ_G, f be a k-interlaced super edge-magic labeling of a graph B with the magic number σ_B and let $t=\sigma_G-\sigma_B+|V(B)|+|E(B)|-2|V(G)|+k$. If $0\le t\le|V(G)|-k$, then $G(g^{-1}(t+1),g^{-1}(t+2),\dots,g^{-1}(t+k))\odot B(f^{-1}(1),f^{-1}(2),\dots,f^{-1}(k))$ is a super edge-magic graph.*

Moreover, if a super edge-magic labeling g is k'-interlaced and $t+k\le k'$, then $G(g^{-1}(t+1),g^{-1}(t+2),\dots,g^{-1}(t+k))\odot B(f^{-1}(1),f^{-1}(2),\dots,f^{-1}(k))$ admits a k'-interlaced labeling.

Proof. As $0 \le t \le |V(G)| - k$, $\{g^{-1}(t+i)\colon i = 1, \dots k\} \subseteq V(G)$. Thus a graph $H := G(g^{-1}(t+1), g^{-1}(t+2), \dots, g^{-1}(t+k)) \odot B(f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k))$ can be defined by

$$V(H) = V(G) \cup \{x \in V(B)\colon f(x) > k\} \quad \text{and}$$
$$E(H) = E(G) \cup \{xg^{-1}(t + f(y))\colon xy \in E(B), f(x) > k\}.$$

Consider a mapping h from $V(H)$ to positive integers given by

$$h(x) = \begin{cases} g(x) & \text{for } x \in V(G), \\ f(x) + |V(G)| - k & \text{for } x \notin V(G). \end{cases}$$

Since $\{g(x)+g(y)\colon xy \in E(G)\} = \{\sigma_G - |V(G)| - |E(G)|, \dots, \sigma_G - |V(G)| - 1\}$ and $\{f(x)+f(y)\colon xy \in E(B)\} = \{\sigma_B - |V(B)| - |E(B)|, \dots, \sigma_B - |V(B)| - 1\}$, we get $\{h(x)+h(y)\colon xy \in E(H)\} = \{\sigma_G - |V(G)| - |E(G)|, \dots, \sigma_G - |V(G)| - 1\} \cup \{\sigma_B - |V(B)| - |E(B)| + |V(G)| - k + t, \dots, \sigma_B - |V(B)| - 1 + |V(G)| - k + t\}$. As $\sigma_B - |V(B)| - |E(B)| + |V(G)| - k + t = \sigma_G - |V(G)|$, h satisfies (P). Evidently, h is a bijection into $\{1, \dots, |V(H)|\}$, and so there exists its extension to a super edge-magic labeling of H. Moreover, if g is k'-interlaced and $k+t \le k'$, then the extension of h is k'-interlaced, too. $\square$

$K_{1,k}$ is a caterpillar having parts with 1 and k vertices. So, there exist its 1-interlaced labeling g_k and k-interlaced labeling f_k. We can construct a square of path using induction $P^2_{n+1} = P^2_n(h^{-1}(n-1), h^{-1}(n)) \odot K_{1,2}(f_2^{-1}(1), f_2^{-1}(2))$ and $P^2_2 = K_{1,1}$. Thus, by Theorem 4, we get that P^2_n is a super edge-magic graph (see also [3]). Likewise, $K_{1,n}(g_n^{-1}(1), \dots, g_n^{-1}(1 + n)) \odot K_{1,1+n}(f_{1+n}^{-1}(1), \dots, f_{1+n}^{-1}(1+n))$ is isomorphic to a complete 3-partite graph $K_{1,1,n}$. According to Theorem 4, we immediately obtain that $K_{1,1,n}$ is a super edge-magic graph(see also [1]).

Let $\{u_{j,i}\colon j = 1, 2\ i = 1, \dots, n\}$ and $\{u_{1,i}u_{2,i}\colon i = 1, \dots, n\}$ be the vertex set and edge set of nP_2, respectively. If n is an odd integer and $k := \lceil n/2 \rceil$, then a mapping ψ_n, given by

$$\psi_n(u_{j,i}) = \begin{cases} i & \text{for } j = 1 \text{ and } i = 1, \dots, n, \\ n + k - 1 + i & \text{for } j = 2 \text{ and } i = 1, \dots, k, \\ k - 1 + i & \text{for } j = 2 \text{ and } i = 1 + k, \dots, n, \end{cases}$$

satisfies (P) and so there exists its extension to a super edge-magic labeling of nP_2. Evidently, this extension is n-interlaced with magic number $4n + k + 1$. Moreover, the value $\psi_n(u_{2,k})$ and the sum $\psi_n(u_{1,k}) + \psi_n(u_{2,k})$ are maximal possible. So, a mapping φ_n from $V(nP_2 - u_{2,k})$ into integers, given by $\varphi_n(x) = \psi_n(x)$, satisfies (P), too. Thus, there exists an extension of φ_n to a super edge-magic n-interlaced labeling of $(n-1)P_2 \cup P_1$ with magic number $4n + k - 1$. By Theorem 4, we get

Corollary 4. *Let* m_0 *and* $m_1 \geq m_2 \geq \cdots \geq m_r$ *be positive integers. The union* $K_{1,m_0} \cup 2K_{1,m_1} \cup 2K_{1,m_2} \cup \cdots \cup 2K_{1,m_r}$ *admits a* $(2r+1)$*-interlaced labeling.*

Proof. Put $S_{1+r\pm i} := K_{1,m_i}$ for all $i = 0, 1, \ldots, r$. We show that there is a super edge-magic labeling of $H := \cup_{i=1}^{2r+1} S_i$ such that the label of central vertex of S_i is equal to i and its magic number is $4 + 5r + 2m_0 + 4(m_1 + \cdots + m_r)$. We employ induction on $m = \max\{m_0, m_1, \ldots, m_r\}$.

If $m = 1$, then a graph H is isomorphic to $(2r+1)P_2$ and ψ_{2r+1} is a required labeling with magic number $9r + 6$.

Now suppose that $m > 1$. Let $m_i^* = m_i$ if $m_i < m$, $m_i^* = m_i - 1$ if $m_i = m$, $s = |\{j : m_j = m, 1 \leq j \leq r\}|$ and $t = r - s$. Put $H^* := \cup_{i=1}^{2r+1} S_i^*$, where $S_{1+r\pm i}^* := K_{1,m_i^*}$. By the induction hypothesis there exists a super edge-magic labeling g of H^* such that the label of central vertex of S_i^* is equal to i and its magic number is $4 + 5r + 2m_0^* + 4(m_1^* + \cdots + m_r^*)$. If $m_0 = m$, then H is a graph isomorphic to $H^*(g^{-1}(t+1), \ldots, g^{-1}(t+2s+1)) \odot (2s+1)P_2(\psi_{2s+1}^{-1}(1), \ldots, \psi_{2s+1}^{-1}(2s+1))$. By Theorem 4, H admits a required labeling. If $m_0 < m$, then H is isomorphic to $H^*(g^{-1}(t+1), \ldots, g^{-1}(t+2s+1)) \odot (2sP_2 \cup P_1)(\varphi_{2s+1}^{-1}(1), \ldots, \varphi_{2s+1}^{-1}(2s+1))$ and according to Theorem 4, it admits a required labeling. □

In ([1]) [8] there is proved that kP_2 is (super) edge-magic if and only if k is odd. Figueroa-Centeno, Ichishima and Muntaner-Batle [4] show that $P_3 \cup kP_2$ is super edge-magic for all k. In ([4]) [11] it is shown that kP_3 is (super) edge-magic when k is odd. Yegnanarayanan also conjectures that for all k, kP_3 has an edge-magic total labeling. We conclude this note with a characterization of (super) edge-magic graphs $nP_3 \cup kP_2$.

Theorem 5. *Let* n *and* k *be nonnegative integers such that* $n + k \geq 1$. *Then*

(i) $nP_3 \cup kP_2$ *is edge-magic if and only if either* $n \geq 1$ *or* $n = 0$ *and* k *is odd;*

(ii) $nP_3 \cup kP_2$ *is super edge-magic if and only if it is edge-magic and is different from* $2P_3$.

Proof. If $n + k$ is odd, then by Corollary 4, $nP_3 \cup kP_2$ $(= nK_{1,2} \cup kK_{1,1})$ is super edge-magic. So, next assume that $n + k$ is even. Consider the following cases.

A. $n = 0$. Suppose that f is an edge-magic total labeling of kP_2 with magic number σ. Then

$$k\sigma = \sum_{xy \in E} (f(x) + f(y) + f(xy)) = 1 + \cdots + 3k = \frac{1}{2}(3k+1)3k.$$

Hence, $\sigma = 3(3k+1)/2$. As σ is an integer, k must be odd.

B. $n = 1$. Let $\{v_{0,0}\} \cup \{v_{j,i} \colon j = 1, 2; i = 0, 1, \ldots, k\}$ be the vertex set and let $\{v_{0,0}v_{1,0}\} \cup \{v_{1,i}v_{2,i} \colon i = 0, 1, \ldots, k\}$ be the edge set of $P_3 \cup kP_2$. Consider a bijection ξ_k from the vertex set of $P_3 \cup kP_2$ to $\{1, 2, \ldots, 2k+3\}$ given by

$$\xi_k(v_{j,i}) = \begin{cases} 1 + k + j & \text{for } j \in \{0,1,2\} \text{ and } i = 0, \\ i & \text{for } j = 1 \text{ and } i \in \{1, \ldots, k\}, \end{cases}$$

$$\xi_1(v_{2,1}) = 5,$$

and for $k = 4s \pm 1$, $s \geq 1$, by

$$\xi_{4s-1}(v_{2,i}) = \begin{cases} 1 + 6s + i & \text{for } i \in \{1, \ldots, 2s\} - \{s, s+1\}, \\ 2 + 5s & \text{for } i = s, \\ 1 + 7s & \text{for } i = s+1, \\ 2 + 2s + i & \text{for } i \in \{2s+1, \ldots, 4s-1\} - \{3s\}, \\ 2 + 7s & \text{for } i = 3s, \end{cases}$$

$$\xi_{4s+1}(v_{2,i}) = \begin{cases} 4 + 6s + i & \text{for } i \in \{1, \ldots, 2s+1\} - \{s+1\}, \\ 4 + 5s & \text{for } i = s+1, \\ 3 + 2s + i & \text{for } i \in \{2s+2, \ldots, 4s+1\} - \{3s+1, 3s+2\}, \\ 5 + 5s & \text{for } i = 3s+1, \\ 5 + 7s & \text{for } i = 3s+2. \end{cases}$$

It is not difficult to check that ξ_k satisfies (P) for $\mu = 2 + 3(k+1)/2$. Thus there is an extension of ξ_k to a super edge-magic labeling of $P_3 \cup kP_2$ with magic number $4 + 9(k+1)/2$.

C. $n > 1$, $k > 1$. Put $r := n + k - 1$, $G := P_3 \cup rP_2$ and $t := 1 + \lfloor k/2 \rfloor$. If n is even, then $nP_3 \cup kP_2$ is isomorphic to

$$G(\xi_r^{-1}(t+1), \ldots, \xi_r^{-1}(t+n-1)) \odot (n-1)P_2(\psi_{n-1}^{-1}(1), \ldots, \psi_{n-1}^{-1}(n-1)).$$

If n is odd, then $nP_3 \cup kP_2$ is isomorphic to

$$G(\xi_r^{-1}(t+1), \ldots, \xi_r^{-1}(t+n)) \odot ((n-1)P_2 \cup P_1)(\varphi_n^{-1}(1), \ldots, \varphi_n^{-1}(n)).$$

By Theorem 4, $nP_3 \cup kP_2$ is super edge-magic.

D. $n = 2$, $k = 0$. Theorem 2 and Theorem 3 imply that $2P_3$ is edge-magic but it is not super edge-magic.

E. $n > 2$, $k = 0$. Denote the vertices of nP_3 by $w_{j,i}$, $j \in \{0,1,2\}$, $i \in \{1, \ldots, n\}$, in such a way that its edges are $w_{0,i}w_{1,i}$ and $w_{0,i}w_{2,i}$, $i = 1, \ldots, n$. As n is even, there exists an integer m such that $n = 2m$. If m is even, then define a mapping $\zeta_n \colon V(nP_3) \to \{1, \ldots, 3n\}$ by

$$\zeta_n(w_{j,i}) = \begin{cases} i & \text{if } j = 0,\ 1 \leq i \leq n-1, \\ 2n & \text{if } j = 0,\ i = n, \\ 3n - 2 - 2i + j & \text{if } j > 0,\ 1 \leq i \leq m-1,\ i \equiv 1 \pmod 2, \\ 4n - 2i + j & \text{if } j > 0,\ m+1 \leq i \leq n-1,\ i \equiv 1 \pmod 2, \\ 2n + 1 - 2i + j & \text{if } j > 0,\ 2 \leq i \leq m,\ i \equiv 0 \pmod 2, \\ 3n - 1 - 2i + j & \text{if } j > 0,\ m+2 \leq i \leq n,\ i \equiv 0 \pmod 2. \end{cases}$$

If m is odd, then define ζ_n by

$$\zeta_n(w_{j,i}) = \begin{cases} i & \text{if } j=0,\, 1 \le i \le n-1, \\ 2n & \text{if } j=0,\, i=n, \\ 3n-3+j & \text{if } j>0,\, i=1, \\ 3n-2-2i+j & \text{if } j>0,\, 2 \le i \le m-1,\, i \equiv 0 \pmod 2, \\ 3n-3 & \text{if } j=1,\, i=m+1, \\ 3n & \text{if } j=2,\, i=m+1, \\ 4n-2-2i+j & \text{if } j>0,\, m+3 \le i \le n-2,\, i \equiv 0 \pmod 2, \\ 3m-3+i+j & \text{if } j>0,\, 3 \le i \le m,\, i \equiv 1 \pmod 2, \\ m-3+i+j & \text{if } j>0,\, m+2 \le i \le n-1,\, i \equiv 1 \pmod 2, \\ 3m-2+j & \text{if } j>0,\, i=n. \end{cases}$$

One can check that ζ_n is a bijection which satisfies (P) for $\mu = 2+3m$. Therefore, nP_3 is super edge-magic.

F. $n > 2$, $k = 1$. In this case n is odd and $m := (n+1)/2$ is an integer. Clearly, the value $\zeta_{n+1}(w_{2,m+1}) = 3(n+1)$ and the sum $\zeta_{n+1}(w_{0,m+1}) + \zeta_{n+1}(w_{2,m+1}) = 3(n+1)+m+1$ are maximal. So, a mapping ζ'_{n+1} from $V((n+1)P_3 - w_{2,m+1})$ into integers, given by $\zeta'_{n+1}(x) = \zeta_{n+1}(x)$, satisfies (P). Therefore, $nP_3 \cup P_2$ (isomorphic to $(n+1)P_3 - w_{2,m+1}$) is super edge-magic. □

References

1. Z. Chen, *On super edge-magic graphs*, J. Comb. Math. Comb. Comput. **38** (2001), 55–64.
2. H. Enomoto, A. S. Llado, T. Nakamigawa and G. Ringel, *Super edge-magic graphs*, SUT J. Math. **34** (1998), 105–109.
3. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, *The place of super edge-magic labeling among other classes of labelings*, Discrete Math. **231** (2001), 153–168.
4. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner, *On super edge-magic graphs*, preprint.
5. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, *On edge-magic labelings of certain disjoint unions of graphs*, preprint.
6. Y. Fukuchi, *A recursive theorem for super-edge-magic labelings of trees*, SUT J. Math. **36** (2000), 279–285.
7. J. A. Gallian, *A dynamic survey of graph labeling*, Electronic J. Combinatorics **#DS6** (2001).
8. A. Kotzig and A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull. **13** (1970), 451–461.
9. G. Ringel and A. S. Llado, *Another tree conjecture*, Bull. ICA **18** (1996), 83–85.
10. K. Wijaya and E. T. Baskoro, *Edge-magic total labelling on disconnected graphs*, Proceedings of the eleventh AWOCA, 2000, pp. 139–144.
11. V. Yegnanarayanan, *On magic graphs*, Util. Math. **59** (2001), 181–204.

Jaroslav Ivančo
Institute of Mathematics, P. J. Šafárik University
041 54 Košice, Jesenná 5, Slovakia

E-mail: ivanco@duro.science.upjs.sk