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# Vertex-Disjoint Copies of $K_1 + (K_1 \cup K_2)$ in Graphs

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**Abstract.** Let $S$ denote the graph obtained from $K_4$ by removing two edges which have an endvertex in common. Let $k$ be an integer with $k \geq 2$. Let $G$ be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)|/2 + 2k - 1$, and suppose that $G$ contains $k$ vertex-disjoint triangles. In the case where $|V(G)| = 4k + 2$, suppose further that $G \ncong K_{4t+3} \cup K_{4k-4t-1}$ for any $t$ with $0 \leq t \leq k - 1$. Under these assumptions, we show that $G$ contains $k$ vertex-disjoint copies of $S$.

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## §1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph $G$, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of $G$, respectively. For a vertex $x$ of a graph $G$, the neighborhood of $x$ in $G$ is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. For a noncomplete graph $G$, let $\sigma_2(G) :=\min\{d_G(x) + d_G(y) |\ xy \notin E(G)\}$; if $G$ is a complete graph, let $\sigma_2(G) := \infty$. For a subset $L$ of $V(G)$, the subgraph induced by $L$ is denoted by $\langle L \rangle$. For a subset $M$ of $V(G)$, we let $G - M = \langle V(G) - M \rangle$ and, for a vertex $x$ of $G$, we let $G - x = \langle V(G) - \{x\} \rangle$. For subsets $L$ and $M$ of $V(G)$, we let $E(L, M)$ denote the set of edges of $G$ joining a vertex in $L$ and a vertex in $M$. When $L$ or $M$ consists of a single vertex, say $L = \{x\}$ or $M = \{y\}$, we write $E(x, M)$ or $E(L, y)$ for $E(L, M)$.

Let $K_n$ denote the complete graph of order $n$, and let $K_n^-$ be the graph obtained from $K_n$ by removing one edge. Also let $S$ be the graph obtained from $K_4$ by removing two edges which have an endvertex in common; thus $S = K_1 + (K_1 \cup K_2)$.

In this paper, we are concerned with conditions on $\sigma_2(G)$ for the existence of vertex-disjoint subgraphs. As examples of results concerning such conditions, we mention that it is proved in Justesen [2] that a graph $G$ of order at least $3k$ with $\sigma_2(G) \geq |V(G)| + k$ has $k$ vertex-disjoint triangles, and it is proved in Enomoto [1] that a graph $G$ of order at least $3k$ with $\sigma_2(G) \geq 4k - 1$ has $k$ vertex-disjoint cycles. This paper is concerned with the following theorem proved by Kawarabayashi in [3].

**Theorem 1.** *Let $k$ be an integer with $k \geq 2$, and let $G$ be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)| + k$. Then $G$ contains $k$ vertex-disjoint copies of $S$.*

In Theorem 1, the bound on $\sigma_2(G)$ is sharp. But this is simply because there exists a graph $G$ with $|V(G)| \geq 4k$ and $\sigma_2(G) = |V(G)|+k-1$ such that $G$ does not even contain $k$ vertex-disjoint triangles (see [2]). Based on this observation, Kawarabayashi and Ota [4] suggested the possibility of lowering the bound on $\sigma_2(G)$ by adding the assumption that $G$ contains $k$ vertex-disjoint triangles. Along this line, we prove the following theorem.

**Theorem 2.** *Let $k$ be an integer with $k \geq 2$. Let $G$ be a graph with $|V(G)| \geq 4k$ and $\sigma_2(G) \geq |V(G)|/2 + 2k - 1$, and suppose that $G$ contains $k$ vertex-disjoint triangles. In the case where $|V(G)| = 4k + 2$, suppose further that $G \not\cong K_{4t+3} \cup K_{4k-4t-1}$ for any $t$ with $0 \leq t \leq k - 1$. Then $G$ contains $k$ vertex-disjoint copies of $S$.*

It is easy to verify that if a graph $G$ with $|V(G)| \geq 4k$ and $\delta(G) \geq 4k - 1$ contains $k$ vertex-disjoint triangles, then it contains $k$ vertex-disjoint copies of $S$. Thus as an immediate corollary of Theorem 2, we obtain the following theorem.

**Theorem 3.** *Let $k$ be an integer with $k \geq 2$. Let $G$ be a graph with $|V(G)| \geq 4k$ and $\delta(G) \geq min\{|V(G)|/4 + k - 1/2, 4k - 1\}$, and suppose that $G$ contains $k$ vertex-disjoint triangles. In the case where $|V(G)| = 4k + 2$ and $k$ is odd, suppose further that $G \not\cong K_{2k+1} \cup K_{2k+1}$. Then $G$ contains $k$ vertex-disjoint copies of $S$.*

In the remainder of this section, we discuss the sharpness of conditions in Theorem 2 and 3. We first show that in Theorem 2, the bound on $\sigma_2(G)$ is sharp. For reference in the discussion of the sharpness of Theorem 3, we construct three families of examples.

**Example 1.** Let $k, n$ be integers with $k \geq 2$ and $n \geq 4k$, and let $s$ be an integer with $0 \leq s \leq k-1$. We construct a graph $F(n,k,s)$ of order $n$ as follows. Let $A, B, C, D$ be vertex-disjoint graphs with $|V(A)| = \lceil (n+1)/2 \rceil - 2k$, $|V(B)| = \lfloor (n+1)/2 \rfloor - 2k + s$, $|V(C)| = s$ and $|V(D)| = 4k - 2s - 1$ such that $A, B$ and $C$ have no edge and $D$ is a complete graph. Join $A$ completely to $B$, i.e., join each vertex of $A$ to all vertices of $B$. Further join $B$ completely to $C$, and $C$ completely to $D$. Let $F(n,k,s)$ denote the resulting graph. Then $F(n,k,s)$ satisfies $\sigma_2(F(n,k,s)) = \lfloor (n-1)/2 \rfloor + 2k - 1 (= \lceil n/2 \rceil + 2k - 2)$ and contains $k$ vertex-disjoint triangles, but does not contain $k$ vertex-disjoint copies of $S$.

**Example 2.** Let $k, n$ be integers with $k \geq 2$ and $n \geq 4k$, and let $r$ be an integer with $0 \leq r \leq k-1$. We construct a graph $G(n,k,r)$ of order $n$ as follows. Let $A, B, C, D, E$ be vertex-disjoint graphs with $|V(A)| = \lceil (n-2r-3)/2 \rceil - (2k-r-2)$, $|V(B)| = \lfloor (n-2r-3)/2 \rfloor$, $|V(C)| = 2(k-1-r)$, $|V(D)| = r$, $|V(E)| = 2r+3$ such that $A$ and $B$ have no edge and $C, D$ and $E$ are complete graphs. Join $A$ completely to $B$, $B$ completely to $C \cup D$, and $C \cup D$ completely to $E$. Let $G(n,k,r)$ denote the resulting graph. Then $G(n,k,r)$ satisfies $\sigma_2(G(n,k,r)) = \lceil n/2 \rceil + 2k - 2$ and contains $k$ vertex-disjoint triangles, but does not contain $k$ vertex-disjoint copies of $S$.

**Example 3.** Let $k, n$ be integers with $k \geq 2$ and $n \geq 4k$ such that $n$ is even, and let $q$ be an integer with $0 \leq q \leq k-2$. We construct a graph $H(n,k,q)$ of order $n$ as follows. Let $A, B, C, D, E, F$ be vertex-disjoint graphs with $|V(A)| = n/2 - 2k + 2$, $|V(B)| = n/2 - q - 2$, $|V(C)| = 2k - 2q - 4$, $|V(D)| = q$, $|V(E)| = 2q + 3$ and $|V(F)| = 1$ such that $A, B$ and $D$ have no edge and $C$ and $E$ are complete graphs. Join $A$ completely to $B$, $B$ completely to $C \cup D$, $C \cup D$ completely to $E$, and $A \cup B \cup C \cup D \cup E$ completely to $F$. Let $H(n,k,q)$ denote the resulting graph. Then $H(n,k,q)$ satisfies $\sigma_2(H(n,k,r)) = n/2 + 2k - 2$ and contains $k$ vertex-disjoint triangles, but does not contain $k$ vertex-disjoint copies of $S$.

We now show that in Theorem 3, the bound on $\delta(G)$ is sharp. First we consider the case where $n \geq 8k-1$. In this case, let $s = 0$ or $3k - \lfloor (n+1)/4 \rfloor - 1$ in Example 1, according as $n \geq 12k-5$ or $8k-1 \leq n \leq 12k-6$. Then $F(n,k,s)$ has minimum degree $4k-2$ or $\lfloor (n+1)/4 \rfloor + k - 1 (= \lceil (n-2)/4 \rceil + k - 1)$ according as $n \geq 12k-5$ or $8k-1 \leq n \leq 12k-6$, which means that the bound on $\delta(G)$ in Theorem 3 is sharp. Next we consider the case where $4k+1 \leq n \leq 8k-2$. In this case, let $r = \lfloor (n-1)/4 \rfloor - k$ in Example 2. Then $G(n,k,r)$ has minimum degree $\lfloor (n-3)/2 \rfloor - \lfloor (n-1)/4 \rfloor + k (= \lceil (n-2)/4 \rceil + k - 1)$. Finally we consider the case where $4k \leq n \leq 8k-6$ and $n$ is even (this includes the case where $n = 4k$, which is excluded from the preceding case). In this case, let $q = \lfloor n/4 \rfloor - k$ in Example 3. Then $H(n,k,q)$ has minimum degree $\lfloor n/4 \rfloor + k - 1$.

## §2. Preparation for the proof of Theorem 2

Let $k, G$ be as in Theorem 2. Write $|V(G)| = 4k + l$. By assumption, $G$ has $k$ vertex-disjoint triangles. Let $S_1, \dots, S_k$ be $k$ vertex-disjoint induced subgraphs of $G$ such that for each $i$, either $|V(S_i)| = 4$ and $S_i$ contains $S$ as a spanning subgraph, or $S_i \cong K_3$. We may assume that there exists $k'$ such that $S_i \supset S$ and $|V(S_i)| = 4$ for each $i$ with $1 \le i \le k'$ and $S_i \cong K_3$ for each $i$ with $k'+1 \le i \le k$. We choose $S_1, \dots, S_k$ so that $k'$ is maximum and, subject to the condition that $k'$ is maximum, $\sum_{i=1}^{k} |E(S_i)|$ is maximum. If $k' = k$, then the desired conclusion holds. Hence we may assume that $k' \le k-1$. Let $L := \cup_{i=1}^{k'} V(S_i)$ and $M := \cup_{i=k'+1}^{k-1} V(S_i)$. Let $v$ be a vertex in $G - L - M - V(S_k)$. For a subgraph $N$ of $G$, let $d_N = 3|E(v, V(N))| + \sum_{x \in V(S_k)} |E(x, V(N))|$. Note that $d_G = \sum_{x \in V(S_k)} (|E(v, V(G))| + |E(x, V(G))|) \ge 3\sigma_2(G)$ because $E(v, V(S_k)) = \emptyset$. Let $Z := G - L - M - V(S_k) - v$. For each $i$ with $1 \le i \le k'$, write $V(S_i) = \{a_i, b_i, c_i, d_i\}$ so that $d_{S_i}(b_i) \ge d_{S_i}(c_i) \ge d_{S_i}(d_i) \ge d_{S_i}(a_i)$; thus $d_{S_i}(a_i) = 1, d_{S_i}(b_i) = 3, d_{S_i}(c_i) = d_{S_i}(d_i) = 2$ if $S_i \cong S$, and $d_{S_i}(b_i) = d_{S_i}(c_i) = 3$ and $d_{S_i}(a_i) = d_{S_i}(d_i) = 2$ if $S_i \cong K_4^-$.

The main aim of this section is to prove that $d_{S_i} \le 13$ for each $1 \le i \le k'$ (see Lemma 2.4). We start with easy lemmas.

**Lemma 2.1.** *Let $i$ be an integer with $1 \le i \le k'$. Then the following statements hold:*

(i) *Suppose that there exists a subgraph $X$ of $S_i$ such that $X \cong K_3$ and $N_G(v) \supset V(X)$. Then $S_i \cong K_4$.*

(ii) *Suppose that there exists a subgraph $X$ of $S_i$ such that $X \cong K_3$ and $|N_G(v) \cap V(X)| \ge 2$. Then $S_i \not\cong S$.*

(iii) *If $S_i \cong K_4^-$, then $|E(v, V(S_i))| \le 3$.*

(iv) *If $S_i \cong K_4^-$ and $|E(v, V(S_i))| = 3$, then $|N_G(v) \cap \{b_i, c_i\}| = 1$.*

(v) *If $S_i \cong S$, then $|E(v, V(S_i))| \le 2$.*

(vi) *If $S_i \cong S$ and $|E(v, V(S_i))| = 2$, then $a_i v \in E(G)$.*

**Proof.** If $S_i \not\cong K_4$, and there exists a subgraph $X$ of $S_i$ such that $X \cong K_3$ and $N_G(v) \supset V(X)$, then by replacing $S_i$ by $\langle V(X) \cup \{v\} \rangle$, we get a contradiction to the maximality of $\sum_{i=1}^{k} |E(S_i)|$ because $\langle V(X) \cup \{v\}\rangle \cong K_4$. Thus (i) holds, and we can similarly prove (ii). Now (iii) and (iv) immediately follow from (i), and (v) and (vi) follow from (ii). $\square$

**Lemma 2.2.** *Let $x \in V(S_k)$, and let $i$ be an integer with $1 \leq i \leq k'$. Then the following statements hold:*

(i) *If $S_i \cong K_4$, then there exist no independent edges $xy, vz \in E(G)$ with $y, z \in V(S_i)$; in particular, $|E(\{x, v\}, V(S_i))| \leq 4$.*

(ii) *If $S_i \cong K_4^-$, then $|E(\{x, v\}, V(S_i))| \leq 4$.*

**Proof.** Suppose that $S_i \cong K_4$ and there exist two independent edges $xy, vz \in E(G)$ with $y, z \in V(S_i)$. Then each of $\langle \{y\} \cup V(S_k)\rangle$ and $\langle \{v\} \cup V(S_i - y)\rangle$ contains a copy of $S$, and these two copies of $S$ are vertex-disjoint, which contradicts the maximality of $k'$. Thus (i) follows. Next suppose that $S_i \cong K_4^-$ and $|E(\{v, x\}, V(S_i))| \geq 5$. Then there exist independent edges $xy, vz \in E(G)$ with $y, z \in V(S_i)$. If $y \in \{a_i, d_i\}$, then $\langle \{v\} \cup V(S_i - y)\rangle \supset S$ and $\langle \{y\} \cup V(S_k)\rangle \supset S$, which contradicts the maximality of $k'$. Thus there are no independent edges $xy, vz$ with $y, z \in V(S_i)$ such that $y \in \{a_i, d_i\}$. Since $|E(\{x, v\}, V(S_i))| \geq 5$, this implies $N_G(x) \cap V(S_i) \subseteq \{b_i, c_i\}$ and $|N_G(v) \cap V(S_i)| \geq 3$. In view of Lemma 2.1(iii), this forces $N_G(x) \cap V(S_i) = \{b_i, c_i\}$ and $|N_G(v) \cap V(S_i)| = 3$. By Lemma 2.1(iv), we may assume $N_G(v) \cap V(S_i) = \{a_i, b_i, d_i\}$. But then each of $\langle \{c_i\} \cup V(S_k)\rangle$ and $\langle \{v\} \cup V(S_i - c_i)\rangle$ contains $S$, which contradicts the maximality of $k'$. $\square$

**Lemma 2.3.** *Let $i$ be an integer with $1 \leq i \leq k'$. If $S_i \cong S$, then $|E(a_i, V(S_k)$ $)| \leq 1$, and equality holds only if $E(v, V(S_i - a_i)) = \emptyset$.*

**Proof.** Otherwise, we can easily get a contradiction to the maximality of $k'$ or $\sum_{i=1}^{k} |E(S_i)|$. $\square$

**Lemma 2.4.** *Let $1 \leq i \leq k'$. Then $d_{S_i} \leq 13$, and equality holds only if $S_i \cong S$, $c_i$ or $d_i$, say, $c_i$, is adjacent to $v$, $E(v, V(S_i)) = \{a_i v, c_i v\}$, $N_G(a_i) \cap V(S_k) = \emptyset$, $N_G(d_i) \supset V(S_k)$, $N_G(b_i) \cap V(S_k) = N_G(c_i) \cap V(S_k)$ and $|N_G(b_i) \cap V(S_k)| = 2$.*

**Proof.** If $S_i \cong K_4$ or $K_4^-$, then by Lemma 2.2, $|E(\{v, x\}, V(S_i))| \leq 4$ for any $x \in V(S_k)$, which implies $d_{S_i} \leq 12$. Thus we may assume $S_i \cong$

$S$. If $E(v, V(S_i)) = \emptyset$, then $d_{S_i} = \sum_{x \in V(S_k)} |E(x, V(S_i))| \leq 12$. Hence by Lemma 2.1(v), we may assume $1 \leq |E(v, V(S_i))| \leq 2$. Suppose that $|E(v, V(S_i))| = 1$. If $a_i v \notin E(G)$, then by Lemma 2.3, $E(a_i, V(S_k)) = \emptyset$, and hence $d_{S_i} = 3|E(v, V(S_i))| + \sum_{x \in V(S_k)} |E(x, V(S_i))| \leq 3 + 9 = 12$. Thus we may assume $a_i v \in E(G)$. If $|E(V(S_i), V(S_k))| \geq 10$, then it follows from Lemma 2.3 that there exists $x \in V(S_k)$ such that $N_G(x) \supset V(S_i)$, and we have $N_G(y) \supset V(S_i - a_i)$ for each $y \in V(S_k - x)$, and hence $\langle \{x, v, a_i, b_i\} \rangle \supset S$ and $\langle V(S_k - x) \cup \{c_i, d_i\} \rangle \supset S$, a contradiction. Thus $|E(V(S_i), V(S_k))| \leq 9$, and hence $d_{S_i} \leq 12$. Consequently we may assume $|E(v, V(S_i))| = 2$. If $|E(V(S_i), V(S_k))| \leq 6$, then $d_{S_i} \leq 12$. Thus we may assume $|E(V(S_i), V(S_k))| \geq 7$. Note that by Lemma 2.1(vi) and Lemma 2.3, $a_i v \in E(G)$ and $E(a_i, V(S_k)) = \emptyset$. Hence $|E(y, V(S_i))| \leq 3$ for each $y \in V(S_k)$, and there exists $x \in V(S_k)$ such that $|E(x, V(S_i))| = 3$ and $N_G(x) \cap V(S_i) = \{b_i, c_i, d_i\}$. If $vb_i \in E(G)$, then $\langle \{v, a_i, b_i, c_i\} \rangle \supset S$ and $\langle \{d_i\} \cup V(S_k) \rangle \supset S$, a contradiction. Thus we may assume $N_G(v) \cap V(S_i) = \{a_i, c_i\}$. If $|E(b_i, V(S_k))| = 3$, then $\langle \{a_i, b_i\} \cup V(S_k - x) \rangle \supset S$, $\langle V(S_i - \{a_i, b_i\}) \cup \{x, v\} \rangle \supset S$, a contradiction; similarly, if $|E(c_i, V(S_k))| = 3$, then $\langle \{v, c_i\} \cup V(S_k - x) \rangle \supset S$ and $\langle V(S_i - c_i) \cup \{x\} \rangle \supseteq S$, a contradiction. Thus $|E(b_i, V(S_k))| \leq 2$ and $|E(c_i, V(S_k))| \leq 2$. Since $|E(V(S_i), V(S_k))| \geq 7$, this forces $|E(b_i, V(S_k))| = 2$, $|E(c_i, V(S_k))| = 2$ and $|E(d_i, V(S_k))| = 3$, and hence $d_{S_i} = 13$. Now if $(N_G(b_i) \cap V(S_k)) \neq (N_G(c_i) \cap V(S_k))$, say, $N_G(b_i) \cap V(S_k) = \{x, y\}$ and $N_G(c_i) \cap V(S_k) = \{x, z\}$, then $\langle \{a_i, b_i, x, y\} \rangle \supset S$ and $\langle \{v, z, c_i, d_i\} \rangle \supset S$, a contradiction. Thus the lemma follows. □

**Lemma 2.5.** *$G - L - M - V(S_k) \not\supseteq K_3$.*

**Proof.** We see from the maximality of $k'$ that in $G - L - M - V(S_k)$, there is no subgraph isomorphic to $S$. Thus it suffices to show that there is no triangle component in $G - L - M - V(S_k)$. By way of contradiction, let $S_{k+1}$ be a triangle component in $G - L - M - V(S_k)$, and take $y \in V(S_{k+1})$ and $x \in V(S_k)$. Note that by the maximality of $k'$, $E(V(S_i), V(G - L - V(S_i))) = \emptyset$ for each $i$ with $k' + 1 \leq i \leq k + 1$. We separate the following point of the proof, and present it as a subclaim.

**Subclaim.** *Let $1 \leq i \leq k'$. Then there exist no independent edges $xu, yw \in E(G)$ such that $u, w \in V(S_i)$.*

**Proof.** If there exist two independent edges $xu, yw \in E(G)$ such that $u, w \in V(S_i)$, then by replacing $S_i$ by $\langle \{u\} \cup V(S_k) \rangle$ and $\langle \{w\} \cup V(S_{k+1}) \rangle$, we get a contradiction to the maximality of $k'$. □

Now by the subclaim, $|E(\{x, y\}, V(S_i))| \leq 4$ for each $i$ with $1 \leq i \leq k'$. Consequently $d_G(x) + d_G(y) \leq 4k' + 2 + 2 \leq 4(k-1) + 4 = 4k$. On the other hand, since $xy \notin E(G)$ by the maximality of $k'$, it follows from the assumption of Theorem 2 that $d_G(x)+d_G(y) \geq \sigma_2(G) \geq 4k+\frac{l}{2}-1$. Hence $k'=k-1, l=2$ and, for each $i$ with $1 \leq i \leq k-1$, $|E(\{x,y\},V(S_i))| = 4$. By the subclaim, this implies that for each $i$ with $1 \leq i \leq k-1$, either $|E(x,V(S_i))| = 4$ and $E(y,V(S_i)) = \emptyset$ or $|E(y,V(S_i))| = 4$ and $E(x,V(S_i))| = \emptyset$. We may assume there exists $t$ such that $E(x,V(S_i)) = \emptyset$ for each $1 \leq i \leq t$ and $|E(x,V(S_i))| = 4$ for each $t+1 \leq i \leq k-1$. Since $y \in V(S_{k+1})$ is arbitrary, for each $z \in V(S_{k+1})$, we have $|E(\{x,z\},V(S_i))| = 4$ for each $1 \leq i \leq k-1$, and hence $|E(z,V(S_i))| = 4$ for each $1 \leq i \leq t$ and $E(z,V(S_i)) = \emptyset$ for each $t+1 \leq i \leq k-1$. Thus $N_G(z) = V(S_{k+1}-\{z\}) \cup (\cup_{i=1}^{t} V(S_i))$ for each $z \in V(S_{k+1})$. Now let $1 \leq i \leq t$. Applying Lemma 2.1(i) to $y$, we see that $S_i \cong K_4$. Take $u \in V(S_i)$. Then arguing as above with $S_i$ and $S_{k+1}$ replaced by $\langle V(S_i-u)\cup\{y\}\rangle$ and $\langle V(S_{k+1}-\{y\})\cup\{u\}\rangle$, we obtain $N_G(u) = ((\cup_{i=1}^{t} V(S_i))-\{u\}) \cup V(S_{k+1})$. Consequently $\langle(\cup_{i=1}^{t} V(S_i)) \cup V(S_{k+1})\rangle$ is a component of $G$, and is isomorphic to $K_{4t+3}$. Arguing similarly with the roles of $S_k$ and $S_{k+1}$ replaced by each other, we also see that $\langle \cup_{i=t+1}^{k} V(S_i)\rangle$ is a component of $G$ and isomorphic to $K_{4k-4t-1}$. Therefore, $G \cong K_{4t+3} \cup K_{4k-4t-1}$, which contradicts the assumption of Theorem 2. □

## §3. Proof of Theorem 2

We continue with the notation of the preceding section. Note that Lemmas 2.1 through 2.4 hold for any choice of $v \in V(G-L-M-V(S_k))$. In this section, we assume that we have chosen $v$ so that $|E(v,V(Z))|$ is minimum.

**Lemma 3.1.** $|E(v,V(Z))| \leq \frac{|V(Z)|+1}{2}$.

**Proof.** If $N_{G-L-M-V(S_k)}(v) = \emptyset$, then the assertion of the lemma obviously holds. Hence we may assume there exists an edge $vw \in E(G-L-M-V(S_k))$. By Lemma 2.5, it follows that $N_{G-L-M-V(S_k)}(v) \cap N_{G-L-M-V(S_k)}(w) = \emptyset$. Consequently $|E(v,V(Z))| + |E(w,V(Z))| \leq |V(Z)|+1$. Hence by the choice of $v$, the assertion holds. □

**Lemma 3.2.** *The following statements hold:*

(i) *For each $i$ with $k'+1 \leq i \leq k-1$, $d_{S_i} \leq 9$.*

(ii) $d_Z \leq 3|E(v,V(Z))|$.

**Proof.** It follows from the maximality of $k'$ that $E(v, V(S_i)) = \emptyset$ for $i$ with $k'+1 \leq i \leq k-1$ and $E(V(Z), V(S_k)) = \emptyset$. Hence the desired results obviously hold. $\square$

**Lemma 3.3.** *There exists $i$ with $1 \leq i \leq k'$ such that $d_{S_i} = 13$.*

**Proof.** Suppose that $d_{S_i} \leq 12$ for all $i$ with $1 \leq i \leq k'$. Then by Lemma 3.1 and Lemma 3.2, $d_G = \sum_{i=1}^{k-1} d_{S_i} + d_Z + 2 + 2 + 2 \leq 12k' + 9(k-1-k') + \frac{3(|V(Z)|+1)}{2} + 6 = 3k' + 9k - 3 + \frac{3}{2}\{4k + l - 4k' - 3(k-1-k') - 4 + 1\} = 9k + \frac{3}{2}(k+k') + \frac{3}{2}l - 3 \leq 9k + \frac{3}{2}(k+k-1) + \frac{3}{2}l - 3 = 12k + \frac{3}{2}l - \frac{9}{2}$. On the other hand, by assumption, $d_G \geq 3\sigma_2(G) \geq 12k + \frac{3}{2}l - 3$. This is a contradiction. $\square$

By Lemma 2.4 and Lemma 3.3, we may assume that $d_{S_1} = 13, S_1 \cong S$, and $N_G(v) \cap V(S_1) = \{a_1, c_1\}$. Write $V(S_k) = \{a, b, c\}$. By Lemma 2.4, we may assume $N_{S_1}(a) = \{d_1\}$, and $N_{S_1}(b) = N_{S_1}(c) = \{b_1, c_1, d_1\}$. For a subgraph $N$ of $G$, let $d'_N = 2|E(v, V(N))| + |E(a_1, V(N))| + \sum_{x \in V(S_k)} |E(x, V(N))|$. Since $E(\{a_1, v\}, V(S_k)) = \emptyset$, it follows from the assumption of Theorem 2 that

$$d'_G \geq 3\sigma_2(G) \geq 12k + \frac{3}{2}l - 3. \tag{A}$$

Also, note that by the symmetry of the roles of $v$ and $a_1$ in $\langle V(S_1) \cup V(S_k) \cup \{v\}\rangle$, we can apply Lemmas 2.1 through 2.4 to $a_1$ as well; i.e., we can apply those lemmas with $S_1$ and $v$ replaced by $\langle\{v, b_1, c_1, d_1\}\rangle$ and $a_1$.

**Lemma 3.4.** *For each $i$ with $2 \leq i \leq k'$, $d'_{S_i} \leq 12$.*

**Proof.** Suppose that $d'_{S_i} \geq 13$. Let $p = 3|E(a_1, V(S_i))| + |E(V(S_k), V(S_i))|$. Applying Lemma 2.4 to $v$ and $a_1$, we get $d_{S_i} \leq 13$ and $p \leq 13$. Since $d'_{S_i} = \frac{2}{3}d_{S_i} + \frac{1}{3}p$, this implies $d_{S_i} = 13$ and $p = 13$. Hence, again applying Lemma 2.4 to $v$ or $a_1$, we see that $S_i \cong S$ and $a_i v, a_i a_1 \in E(G)$. Consequently, by replacing $S_1, S_i, S_k$ by $\langle\{v, a_1, a_i, b_1\}\rangle$, $\langle\{d_1, a, b, c\}\rangle$, $\langle\{b_i, c_i, d_i\}\rangle$, respectively, we get a contradiction to the maximality of $\sum_{i=1}^{k} |E(S_i)|$ because $\langle\{d_1, a, b, c\}\rangle \cong K_4$. $\square$

**Lemma 3.5.** *The following statements hold:*

(i) *For each $i$ with $k'+1 \leq i \leq k-1$, $d'_{S_i} \leq 9$.*

(ii) *For each $z \in V(Z)$, $|E(\{a_1, v\}, z)| \leq 1$.*

(iii) $d'_Z \leq \frac{3|V(Z)|+1}{2}$.

**Proof.** It follows from the maximality of $k'$ that $E(v, V(S_i)) = \emptyset$ for each $i$ with $k'+1 \le i \le k-1$. Also, by symmetry, we have $E(a_1, V(S_i)) = \emptyset$ for each $i$ with $k'+1 \le i \le k-1$. Hence (i) obviously holds. To show (ii), suppose that $|E(\{a_1, v\}, z)| \ge 2$. Then $\langle\{a_1, b_1, v, z\}\rangle \supset S$ and $\langle\{d_1, a, b, c\}\rangle \supset K_4$, which contradicts the maximality of $k'$. Thus (ii) holds. Now by (ii), $|E(a, V(Z))| \le |V(Z)| - |E(v, V(Z))|$. Since $E(V(S_k), V(Z)) = \emptyset$, this together with Lemma 3.1 implies $d'_Z = 2|E(v, V(Z))| + |E(a, V(Z))| \le |E(v, V(Z))| + |V(Z)| \le \frac{|V(Z)|+1}{2} + |V(Z)|$. This proves (iii). □

By Lemma 3.4 and (i) and (iii) of Lemma 3.5, we now obtain

$$\begin{aligned} d'_G &\le 12(k'-1) + 9(k-1-k') + \tfrac{3}{2}\{4k + l - 4k' - 3(k-k') - 1\} \\ &\quad + \tfrac{1}{2} + 4 + 2 + 5 + 5 + 3 = 9k + \tfrac{3}{2}(k+k') + \tfrac{3}{2}l - 3 \\ &\le 9k + \tfrac{3}{2}(k+k-1) + \tfrac{3}{2}l - 3 = 12k + \tfrac{3}{2}l - \tfrac{9}{2}, \end{aligned}$$

which contradicts (A). This completes the proof of Theorem 2. □

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