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Degree-Sum Conditions for Graphs to Have 2-Factors with Cycles Through Specified Vertices

Toshinori Sakai

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Abstract. Let $k \geq 2$ and $n \geq 1$ be integers, let G be a graph of order n with minimum degree at least $k+1$. Let v_1, v_2, $\cdots$, v_k be k distinct vertices of G, and suppose that there exist k vertex disjoint cycles $C_1, \cdots, C_k$ in G such that $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Suppose further that the minimum value of the sum of the degrees of two nonadjacent distinct vertices is greater than or equal to $n + \frac{k-4}{3}$. Under these assumptions, we show that there is a 2-factor of G with k cycles $D_1, D_2, \cdots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$.

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§1. Introduction

All graphs considered in this paper are finite simple undirected graphs with no loops and no multiple edges. For a graph G, we let $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively. For a vertex v of G, we let $d_G(v)$ denote the degree of v in G. We define $\delta(G)$ to be the minimum degree of G, and $\sigma_2(G)$ to be the minimum value of the sum of the degrees of two nonadjacent distinct vertices.

The following theorem appears in [1]:

Theorem A. *Let k and n be positive integers with $n \geq 3k$, and let G be a graph of order n. Suppose that one of the following four conditions is satisfied:*

a) $n = 3k$ *and* $\delta(G) \geq \frac{7k-2}{3}$*;*

b) $3k+1 \leq n \leq 4k$ *and* $\delta(G) \geq \frac{2n+k-3}{3}$*;*

c) $4k \leq n \leq 6k-3$ *and* $\delta(G) \geq 3k-1$*; or*

d) $n \geq 6k-3$ *and* $\delta(G) \geq \frac{n}{2}$.

Then for any k *distinct vertices* $v_1, \cdots, v_k$*, there is a 2-factor of* G *with* k *cycles* $C_i, 1 \leq i \leq k$ *such that* $v_i \in V(C_i)$ *for each* $1 \leq i \leq k$.

In [1], Theorem A is derived as an immediate corollary of the following two theorems:

Theorem B. *Let* k *and* n *be positive integers with* $n \geq 3k$*, and let* G *be a graph of order* n*. Suppose that one of the conditions a) through d) in Theorem A is satisfied. Then for any* k *distinct vertices* $v_1, \cdots, v_k$*, there exist* k *vertex disjoint cycles* C_i*,* $1 \leq i \leq k$*, such that* $|V(C_i)| \leq 5$ *and* $v_i \in V(C_i)$ *for each* $1 \leq i \leq k$.

Theorem C. *Let* k *and* n *be positive integers, and let* G *be a graph of order* n *such that* $\sigma_2(G) \geq n$*,* $\delta(G) \geq k+1$ *and* $\sigma_2(G) + \delta(G) \geq n + 3k - 2$*. Let* $v_1, \cdots, v_k$ *be distinct vertices of* G*, and suppose that there exist* k *vertex disjoint cycles* C_i*,* $1 \leq i \leq k$*, such that* $v_i \in V(C_i)$ *for each* $1 \leq i \leq k$*. Then there is a 2-factor of* G *with* k *cycles* D_i*,* $1 \leq i \leq k$*, such that* $v_i \in V(D_i)$ *for each* $1 \leq i \leq k$.

This paper is concerned with Theorem C. In Theorem C, the condition $\sigma_2(G) + \delta(G) \geq n + 3k - 2$ is of technical nature, and it has been conjectured that Theorem C holds even if we drop the condition $\sigma_2(G)+\delta(G) \geq n+3k-2$. This conjecture has partially been settled affirmatively by the following two theorems proved in [2] (Theorem D says that if we regard a path of length 0 or 1 as a cycle of length 1 or 2, respectively, then the conjecture is true; Theorem E says that if $n \geq 6k-4$, then the conjecture is true):

Theorem D. *Let* k *and* n *be positive integers, and let* G *be a graph of order* n*. Suppose that* $\sigma_2(G) \geq n$ *and* $\delta(G) \geq k+1$*. Then for any* k *distinct vertices* $v_1, \cdots, v_k$*, there is a spanning subgraph of* G *with* k *components* $C_i, 1 \leq i \leq k$*, such that for each* i*,* $v_i \in V(C_i)$ *and* C_i *is either a cycle or a path of length* 0 *or* 1.

Theorem E. *Let* k *and* n *be positive integers with* $n \geq 6k-4$*, and let* G *be a graph of order* n *such that* $\sigma_2(G) \geq n$ *and* $\delta(G) \geq k+1$*. Let* $v_1, \cdots, v_k$ *be distinct vertices of* G*, and suppose that there exist* k *vertex disjoint cycles* C_i*,* $1 \leq i \leq k$*, such that* $v_i \in V(C_i)$ *for each* $1 \leq i \leq k$*. Then there is a 2-factor of* G *with* k *cycles* D_i*,* $1 \leq i \leq k$*, such that* $v_i \in V(D_i)$ *for each* $1 \leq i \leq k$.

In this paper, we give the following (somewhat negative) solution to the conjecture (as we shall see in the second paragraph following the statement of Theorem 1, the condition $\sigma_2(G) \geq n + \frac{k-4}{3}$ in Theorem 1 is best possible, which means that if we drop the condition $\sigma_2(G) + \delta(G) \geq n + 3k - 2$ in Theorem C, then we need to replace the condition $\sigma_2(G) \geq n$ by the stronger condition $\sigma_2(G) \geq n + \frac{k-4}{3}$):

Theorem 1. *Let $k \geq 2$ and $n \geq 1$ be integers, and let G be a graph of order n such that $\sigma_2(G) \geq n+\frac{k-4}{3}$ and $\delta(G) \geq k+1$. Let $v_1, \cdots, v_k$ be distinct vertices of G, and suppose that there exist k vertex disjoint cycles C_i, $1 \leq i \leq k$, such that $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Then there is a 2-factor of G with k cycles D_i, $1 \leq i \leq k$, such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$.*

Theorem 1 does not hold for $k = 1$. To see this, let $l \geq 2$ be an integer, and consider a complete bipartite graph $G = K(l, l+1)$. Then $\sigma_2(G) = 2l = |V(G)| - 1 = |V(G)| + \frac{k-4}{3}$ and $\delta(G) = l \geq 2 = k+1$, but G does not have a 2-factor. Also the condition $\delta(G) \geq k+1$ is best. To verify this, let k, n be integers with $k \geq 2$ and $n \geq 3k+1$, and define a graph G of order n as follows: let L be a complete graph of order $n-1$ containing specified distinct vertices $v_1, v_2, \cdots, v_k$, let u be a vertex with $u \notin V(L)$, and let $V(G) = V(L) \cup \{u\}$ and $E(G) = E(L) \cup \{uv_i \,|\, 1 \leq i \leq k\}$. Then $\delta(G) = d_G(u) = k$, $\sigma_2(G) = (n-2)+k = n+k-2 \geq n+\frac{k-4}{3}$, and there exist k vertex disjoint cycles $C_1, \cdots, C_k$ in L (so in G) such that $v_i \in V(C_i)$ for each $1 \leq i \leq k$. But G does not contain a 2-factor with k cycles $D_1, D_2, \cdots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$ because any cycle containing u must passes through at least two vertices in $\{v_1, v_2, \cdots, v_k\}$.

Finally we verify that the condition $\sigma_2(G) \geq n + \frac{k-4}{3}$ is best for $3k+2 \leq n \leq \lceil \frac{13}{3}k \rceil$. Let k, n be integers with $k \geq 2$ and $3k+2 \leq n \leq \lceil \frac{13}{3}k \rceil$, and write $n = 3k+h$. We define a graph G of order n as follows. Let H be a complete graph of order h. Let $m = \lfloor \frac{2k}{3} \rfloor$ and let L be the graph of order $3k$ with vertex set

$$V(L) = \{v_i \,|\, 1 \leq i \leq k\} \cup \{u_i \,|\, 1 \leq i \leq k\} \cup \{w_i \,|\, 1 \leq i \leq k\}$$

such that the complement graph $\overline{L}$ of L satisfies

$$\begin{aligned} E(\overline{L}) \;\; = \;\; & \{w_i w_j \,|\, 1 \leq i < j \leq m\} \cup \{w_i w_k \,|\, 1 \leq i \leq m\} \\ & \cup \{u_i v_j \,|\, m+1 \leq i \leq k,\, j \neq i\} \\ & \cup \{v_i w_j \,|\, 1 \leq i \leq m,\; m+1 \leq j \leq k-1\} \end{aligned}$$

(Figure 1). Define G by $V(G) = V(H) \cup V(L)$ and $E(G) = E(H) \cup E(L) \cup \{hu_i \,|\, h \in V(H),\, 1 \leq i \leq k\}$.

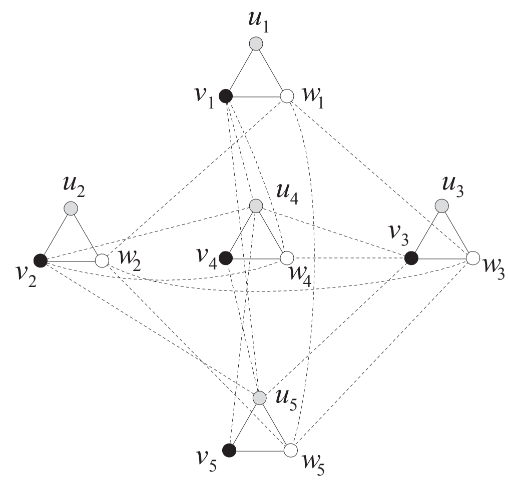

Figure 1: Dotted lines stand for the edges of $\overline{L}$ while solid lines stand for some of edges of L for the case where $k = 5$.

Then

$$\begin{aligned} d_G(h) &= h + k - 1 \quad (\text{for } h \in V(H)), \\ d_G(u_i) &= d_L(u_i) + |V(H)| = \begin{cases} n-1 & (\text{for } 1 \le i \le m), \\ n-k & (\text{for } m+1 \le i \le k), \end{cases} \\ d_G(v_i) &= d_L(v_i) = \begin{cases} k+2m & (\text{for } 1 \le i \le m), \\ 2k+m & (\text{for } m+1 \le i \le k), \end{cases} \\ d_G(w_i) &= d_L(w_i) = 3k - m - 1 \quad (\text{for } 1 \le i \le k), \end{aligned}$$

and hence $\delta(G) \ge k+1$. Next we verify that $\sigma_2(G) \ge n + \frac{k-4}{3} - 1$. Let $\varepsilon_k = 0, 2, 1$ and $c_k = 3, 4, 2$ according as $k \equiv 0, 1, 2 \pmod 3$. Then $m = \frac{2k-\varepsilon_k}{3}$, $\frac{k-c_k}{3}$ is an integer, the condition $\sigma_2(G) \ge n + \frac{k-4}{3} - 1$ is equivalent to $\sigma_2(G) \ge n + \frac{k-c_k}{3} - 1$, and

(i) $d_G(h) + d_G(v_i) - \left(n + \frac{k-c_k}{3} - 1\right) \ge \frac{c_k - 2\varepsilon_k}{3} \ge 0$ and $d_G(h) + d_G(w_i) - \left(n + \frac{k-c_k}{3} - 1\right) = \frac{\varepsilon_k + c_k}{3} - 1 \ge 0$ for any $h \in V(H)$ and any i with $1 \le i \le k$,

(ii) $d_G(w_i) + d_G(w_j) - \left(n + \frac{k-c_k}{3} - 1\right) = \frac{13k + 2\varepsilon_k + c_k - 3}{3} - n \ge \left\lceil \frac{13}{3}k \right\rceil - n \ge 0$ for any i and j with $1 \le i < j \le m$, or $1 \le i \le m$ and $j = k$,

(iii) $d_G(u_i) + d_G(v_j) - \left(n + \frac{k-c_k}{3} - 1\right) \geq k + \frac{c_k - 2\varepsilon_k}{3} + 1 \geq k + 1 > 0$ for any i with $m+1 \leq i \leq k$ and any j with $j \neq i$, and

(iv) $d_G(v_i) + d_G(w_j) - \left(n + \frac{k-c_k}{3} - 1\right) = \frac{13k + c_k - \varepsilon_k}{3} - n \geq \lceil \frac{13k+1}{3} \rceil - n > 0$ for any i with $1 \leq i \leq m$ and any j with $m+1 \leq j \leq k-1$.

Consequently, we have $\sigma_2(G) \geq n + \frac{k-c_k}{3} - 1$. Also G contains vertex disjoint cycles $C_i = u_i v_i w_i u_i$, $1 \leq i \leq k$, such that $v_i \in V(C_i)$ for each i. However, G does not contain a 2-factor with k cycles $D_1, D_2, \cdots, D_k$ such that $v_i \in V(D_i)$ for each i. To see this, by way of contradiction, suppose that G contains a 2-factor with k cycles $D_1, D_2, \cdots, D_k$ such that $v_i \in V(D_i)$ for each i. First note that each D_i must contain exactly three vertices in $V(L)$. Let D_{i_1} be a cycle containing a vertex in $V(H)$. Then D_{i_1} contains (exactly) two vertices in $\{u_1, \cdots, u_k\}$, and hence there is a cycle D_{i_2} that consists of v_{i_2} and two vertices w_{j_1} and w_{j_2}, where $j_1 \neq j_2$. For each i with $m+1 \leq i \leq k$, v_i is the only vertex in $\{v_1, \cdots, v_k\}$ that is adjacent to u_i, and hence u_i and v_i are in the same cycle. Therefore

$$1 \leq i_2 \leq m. \tag{1.1}$$

On the other hand, $w_i w_j \in E(G)$ only when at least one of i and j is in $\{m+1, \cdots, k-1\}$. Thus we may assume $m+1 \leq j_1 \leq k-1$. But then $w_{j_1} v_{i_2} \notin E(G)$ by (1.1) and the construction of G, a contradiction.

Our notation is standard with the possible exception of the following: Let G be a graph. For a subset U of $V(G)$, we let $\langle U \rangle_G$ denote the subgraph of G induced by U. For a vertex v of G, we denote by $N_G(v)$ the set of neighbours of v. For a vertex v of G and for a subgraph H of G with $v \notin V(H)$, we let $N_H(v) = N_G(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. Furthermore, for subgraphs H and K of G with $V(H) \cap V(K) = \emptyset$, we let $N_H(K) = \cup_{v \in V(K)} N_H(v)$. Let $C = u_0 u_1 \cdots u_{m-1} u_0$ be a cycle (indices are to be read modulo m). For two vertices u_i, u_j on C with $i \leq j \leq i+m-1$, we let $C[u_i, u_j]$ and $C(u_i, u_j)$ denote the paths $u_i u_{i+1} \cdots u_j$ and $u_{i+1} u_{i+2} \cdots u_{j-1}$, respectively (if $j = i$ or $j = i+1$, $C(u_i, u_j)$ denotes an empty path).

§2. Proof of Theorem 1

Throughout this section, let k, n, G, v_i and C_i be as in Theorem 1. We may assume we have chosen k cycles C_i so that $\sum_{i=1}^{k} |V(C_i)|$ is maximum. Let $L = \langle \cup_{i=1}^{k} V(C_i) \rangle_G$, $l = |V(L)|$ and $H = G - L$. If $V(H) = \emptyset$, then there is nothing to be proved. Thus suppose that

$$V(H) \neq \emptyset. \tag{2.1}$$

Let H_0 be a connected component of H. We shall obtain a contradiction in a series of claims. The first five claims, Claims 1 through 5, are essentially proved in [1], but we include their proofs for the convenience of the reader.

Claim 1. *Let D_i, $1 \le i \le k$, be vertex disjoint subgraphs of L such that D_1 is a path (we allow D_1 to consist of a single vertex), and D_i is a cycle for each $2 \le i \le k$. Suppose that D_i contains exactly one vertex in $\{v_1, \cdots, v_k\}$ for each $1 \le i \le k$, and write $V(D_i) \cap \{v_1, \cdots, v_k\} = \{v_{j_i}\}$. Suppose further that $d_L(u) \ge l - \frac{1}{2}|\cup_{i=1}^k V(D_i)|$ for all $u \in V(L) - \cup_{i=1}^k V(D_i)$. Then there are vertex disjoint subgraphs D_i^*, $1 \le i \le k$, of L such that D_1^* is a path, D_i^* is a cycle for each $2 \le i \le k$, $v_{j_i} \in V(D_i^*)$ for each $1 \le i \le k$, and $\cup_{i=1}^k V(D_i^*) = V(L)$, and such that D_1^* has the same initial vertex and the same terminal vertex as D_1 (thus if D_1 consists of a single vertex, then $D_1^* = D_1$).*

Proof. Choose vertex disjoint subgraphs $D_i^*, 1 \le i \le k$, of L satisfying the same conditions as the D_i so that $\sum_{i=1}^k |V(D_i^*)|$ is as large as possible, subject to the condition that $V(D_i^*) \supseteq V(D_i)$ for each $1 \le i \le k$ and D_1^* has the same initial vertex and the same terminal vertex as D_1. Let $A = \cup_{i=1}^k V(D_i^*)$. Suppose that $A \ne V(L)$, and take $u \in V(L) - A$. Then $d_L(u) \ge l - \frac{1}{2}|\cup_{i=1}^k V(D_i)| \ge l - \frac{1}{2}|A|$ by assumption, and hence $d_{\langle A \rangle_G}(u) = d_L(u) - d_{L-A}(u) \ge (l - \frac{1}{2}|A|) - (l - |A| - 1) = \frac{1}{2}|A| + 1$. Consequently there exist v, $v' \in N_G(u)$ such that v and v' are consecutive on some $D_i^*, 1 \le i \le k$. Inserting u into D_i^*, we get a contradiction to the maximality of $|A|$. Thus $A = V(L)$, as desired. □

Claim 2.

(i) $|N_{C_i}(H_0)| \le 1$ *for each* $1 \le i \le k$.

(ii) $H = H_0$.

Proof. (i) First note that for each $1 \le i \le k$,

(2.2) no two vertices in $N_{C_i}(H_0)$ are consecutive on C_i

because of the maximality of $\sum_{i=1}^k |V(C_i)|$. By way of contradiction, suppose $|N_{C_i}(H_0)| \ge 2$ for some i, say $i = 1$. Take $u_1, u_2 \in V(C_1)$ and $h_1, h_2 \in V(H_0)$ (we allow the possibly that $h_1 = h_2$) such that $u_i h_i \in E(G)$ for $i = 1, 2$, $V(C_1(u_1, u_2)) \not\ni v_1$ and such that $N_{H_0}(C_1(u_1, u_2)) = \emptyset$. Let $P = C_1[u_2, u_1]$ and let Q be a path in H_0 having h_1 and h_2 as its initial vertex and its terminal vertex, respectively. Then $C = PQu_2$ is a cycle containing v_1. Let $R = C_1(u_1, u_2)$ and $r = |V(R)|$, and take $w \in V(R)$ (note that $V(R) \ne \emptyset$ by

(2.2)). Then since $d_G(h_1) \le |V(H_0)| - 1 + \frac{l-r+1}{2}$ by (2.2), $d_G(w) + |V(H_0)| - 1 + \frac{l-r+1}{2} \ge d_G(w) + d_G(h_1) \ge n + \lceil \frac{k-4}{3} \rceil \ge n$. Consequently

$$\begin{aligned} d_L(w) &\ge d_G(w) - |V(H) - V(H_0)| \ge l - \frac{1}{2}(l-r) + \frac{1}{2} \\ &= l - \frac{1}{2}|V(P) \cup (\cup_{i=2}^{k} V(C_i))| + \frac{1}{2}, \end{aligned}$$

and hence it follows from Claim 1 that there exist vertex disjoint subgraphs $P', C_2', \cdots, C_k'$ of L such that P' is a path with $v_1 \in V(P')$ joining u_2 and u_1, C_i' is a cycle with $v_i \in V(C_i')$ for each $2 \le i \le k$, and $V(P') \cup (\cup_{i=2}^{k} V(C_i')) = V(L)$. Set $C_1' = P'Qu_2$. Then $C_1', C_2', \cdots, C_k'$ are vertex disjoint cycles such that $v_i \in V(C_i')$ for each $1 \le i \le k$ and such that $\sum_{i=1}^{k} |V(C_i')| = \sum_{i=1}^{k} |V(C_i)| + |V(Q)|$. This contradicts the maximality of $\sum_{i=1}^{k} |V(C_i)|$.
(ii) Suppose $H \ne H_0$. Take $h \in V(H_0)$ and $h' \in V(H) - V(H_0)$. Then $n \le d_G(h) + d_G(h') \le (|V(H_0)| - 1 + d_L(h)) + (|V(H)| - |V(H_0)| - 1 + d_L(h')) = d_L(h) + d_L(h') + |V(H)| - 2$, and hence $d_L(h) + d_L(h') \ge n - |V(H)| + 2 = l + 2 \ge 3k + 2$. On the other hand, it follows from (i) of this claim that $d_L(h) + d_L(h') \le k + k = 2k$. Consequently $3k + 2 \le 2k$, a contradiction. □

Set $S = \{u \in V(L) \,|\, d_H(u) \ge 2\}$ and let $s = |S|$. By Claim 2 (i), $|S \cap V(C_i)| \le 1$ for each $i \in \{1, \cdots, k\}$. We may assume $S \cap V(C_i) = \{u_i\}$ for each $i \in \{1, \cdots, s\}$ and $S \cap V(C_i) = \emptyset$ for each $i \in \{s+1, \cdots, k\}$.

Claim 3.

(i) $u_i \ne v_i$ *for each* $i \in \{1, \cdots, s\}$.

(ii) $|V(H)| > k - s$.

Proof. (i) Suppose $v_i = u_i$ for some $i \in \{1, \cdots, s\}$, say $i = 1$. Take $h_1, h_2 \in N_H(v_1)$ with $h_1 \ne h_2$. Since H is connected by Claim 2 (ii), there is a path Q in H connecting h_1 and h_2. Take $u \in V(C_1) - \{v_1\}$. By Claim 2 (i), u is adjacent to no vertex in $V(H)$. Hence $n \le d_G(u) + d_G(h_1) = d_L(u) + d_G(h_1)$, which implies $d_L(u) \ge n - (|V(H)| - 1 + k) = l - (k-1) > l - \frac{1}{2}\sum_{i=2}^{k} |V(C_i)|$. Thus applying Claim 1 with $D_1 = v_1$, we obtain $k-1$ vertex disjoint cycles $C_i' \,(2 \le i \le k)$ in L such that $v_i \in V(C_i')$ for each $2 \le i \le k$ and $\{v_1\} \cup (\cup_{i=2}^{k} V(C_i')) = V(L)$. Set $C_1' = v_1 Q v_1$. Then the $C_i' \,(1 \le i \le k)$ are vertex disjoint cycles such that $v_i \in V(C_i')$ for each $1 \le i \le k$, which contradicts the maximality of $\sum_{i=1}^{k} |V(C_i)|$.
(ii) Suppose $|V(H)| \le k - s$. Let $E(H, L)$ denote the set of edges which have one endvertex in $V(H)$ and the other in $V(L)$. Since $\delta(G) \ge k+1$,

$|V(H)|(k+1-(|V(H)|-1)) \leq |E(H,L)| \leq s|V(H)|+(k-s) = s(|V(H)|-1)+k$, and hence $|V(H)|(k+2-|V(H)|) \leq (k-|V(H)|)(|V(H)|-1)+k$ by the assumption that $|V(H)| \leq k-s$. This implies $2|V(H)| \leq |V(H)|$, which contradicts (2.1). □

Recall $\sigma_2(G) \geq n + \frac{k-4}{3}$. Let $c_k = 3, 4, 2$ according to whether $k \equiv 0, 1, 2 \pmod 3$. Then we have

$$\sigma_2(G) \geq n + \frac{k-c_k}{3} \tag{2.3}$$

because $\sigma_2(G)$ is an integer.

Claim 4. *$d_L(u) \geq l - s + \frac{k-c_k}{3} + 1$ for all $u \in V(L) - N_L(H)$.*

Proof. Since $\sum_{h \in V(H)} d_G(h) \leq |V(H)|(|V(H)|-1) + s|V(H)| + k - s$, there exists $h \in V(H)$ such that $d_G(h) \leq |V(H)| - 1 + s + \frac{k-s}{|V(H)|}$. In view of Claim 3 (ii), we have $d_G(h) \leq |V(H)| - 1 + s$. Thus for every $u \in V(L) - N_L(H)$,

$$d_L(u) = d_G(u) \geq n + \frac{k-c_k}{3} - d_G(h) \geq l - s + \frac{k-c_k}{3} + 1$$

by (2.3). □

For each $u \in V(L) - N_L(H)$, $d_L(u) \geq l - s + 1 = |(V(L) - \{u\}) - S| + 2$ by Claim 4, and hence $|N_G(u) \cap S| \geq 2$. In view of Claims 2 (i) and 3 (i), this in particular implies that $s \geq 2$, and

$$N_G(v_i) \cap (S - \{u_i\}) \neq \emptyset \text{ for each } i \in \{1, \cdots, s\}. \tag{2.4}$$

Claim 5. *There exist no vertex disjoint subgraphs P, C'_i, $2 \leq i \leq k$, in L such that*

$$\begin{cases} P \text{ is a path joining two distinct vertices in } \{u_1, \cdots, u_s\},\ C'_i \text{ is a cycle} \\ \text{for each } 2 \leq i \leq k, \text{ each of } P \text{ and the } C'_i,\ 2 \leq i \leq k, \text{ contains exactly} \\ \text{one vertex in } \{v_1, \cdots, v_k\}, \text{ and } V(P) \cup \left(\cup_{i=2}^k V(C'_i)\right) \supseteq N_L(H). \end{cases} \tag{2.5}$$

Proof. Suppose that there exist vertex disjoint subgraphs $P, C'_i, 2 \leq i \leq k$, in L which satisfy (2.5). Write $V(P) \cap \{v_1, \cdots, v_k\} = \{v_{j_1}\}$ and $V(C'_i) \cap \{v_1, \cdots, v_k\} = \{v_{j_i}\}$ for each $2 \leq i \leq k$. Let u_r and u_t be the initial vertex and the terminal vertex of P, respectively. Since $d_L(u) > l - s \geq l - k > l - \frac{1}{2}(|V(P)| + \sum_{i=2}^k |V(C'_i)|)$ for all $u \in V(L) - N_L(H)$ by Claim 4, it follows from Claim 1 that there are vertex disjoint subgraphs P', C''_i, $2 \leq i \leq k$, in L

such that P' is a path with $v_{j_1} \in V(P')$ connecting u_r and u_t, C''_i is a cycle with $v_{j_i} \in V(C''_i)$ for each $2 \le i \le k$, and $V(P') \cup (\cup_{i=2}^{k} V(C''_i)) = V(L)$. Take $h \in N_H(u_r)$ and $h' \in N_H(u_t) - \{h\}$ (note that $d_H(u_t) \ge 2$ because $u_t \in S$). Combining P' and a path in H connecting h and h', we obtain a cycle C'_1. Then C'_1 and the C''_i, $2 \le i \le k$, are vertex disjoint cycles in G such that $v_{j_1} \in V(C'_1)$, $v_{j_i} \in V(C''_i)$ for each $2 \le i \le k$, and $|V(C'_1)| + \sum_{i=2}^{k} |V(C''_i)| > \sum_{i=1}^{k} |V(C_i)|$. This contradicts the maximality of $\sum_{i=1}^{k} |V(C_i)|$. □

For each $i \in \{1, \cdots, s\}$, take $w_i \in V(C_i) - \{u_i, v_i\}$ and let $W_i = \langle \{u_i, v_i, w_i\} \rangle_G$. We redefine the orientation of each C_i, $1 \le i \le s$, so that $w_i \in V(C_i(v_i, u_i))$.

Claim 6. *Let $1 \le h, j \le s$ with $h \ne j$, and suppose that $v_h u_j \in E(G)$. Then $w_h v_j \notin E(G)$ or $w_h w_j \notin E(G)$.*

Proof. At the cost of relabeling, we may assume $h = 1$ and $j = 2$. Suppose $v_1 u_2 \in E(G)$, $w_1 v_2 \in E(G)$ and $w_1 w_2 \in E(G)$. Let $P' = C_1[u_1, v_1]u_2$ and $C'_2 = w_1 C_2[v_2, w_2]w_1$, and let $C'_i = C_i$ for $3 \le i \le k$. Then P' and the C'_i $(2 \le i \le k)$ satisfy (2.5), a contradiction. □

Having (2.4) in mind, take i_1 and i_2 with $1 \le i_1, i_2 \le s$ and $i_1 \ne i_2$ satisfying $u_{i_2} \in N_G(v_{i_1})$ so that

$$\sum_{i=1}^{s} d_{W_i}(w_{i_1}) \text{ is minimum.} \tag{2.6}$$

We may assume $i_1 = 1$ and $i_2 = 2$. We remark in advance that we make use of (2.6) only in the last stage of the proof (see the paragraph following Claim 12).

Claim 7. *We have $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) \le 7$. Further if $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) = 7$, then*

$$\begin{cases} w_1 w_2 \notin E(G),\ w_1 u_1 \in E(G),\ w_1 v_1 \in E(G); \\ v_2 x \in E(G) \text{ for each } x \in \{u_1, v_1, w_1\}; \text{ and} \\ w_2 x \in E(G) \text{ for each } x \in \{u_1, v_1\}. \end{cases} \tag{2.7}$$

Proof. Applying Claim 6 with $h = 1$ and $j = 2$, we get $v_2 w_1 \notin E(G)$ or $w_2 w_1 \notin E(G)$. This implies $d_{W_1}(v_2) + d_{W_1}(w_2) \le 5$, and hence $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) \le 7$. Now assume that $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) = 7$, i.e., $d_{W_1}(v_2) + d_{W_1}(w_2) = 5$ and $d_{W_1}(w_1) = 2$. Assume further that $w_1 w_2 \in E(G)$. Then $v_2 w_1 \notin E(G)$ by Claim 6. Hence by the assumption that $d_{W_1}(v_2) + d_{W_1}(w_2) = 5$, we in particular have $v_2 u_1 \in E(G)$ and $w_2 v_1 \in E(G)$. But then applying Claim 6 with $h = 2$ and $j = 1$, we get a contradiction. Thus the equality $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) = 7$ implies (2.7). □

Claim 8. *$d_{W_2}(w_1) + d_{W_2}(v_2) + d_{W_2}(w_2) \le 6$.*

Proof. Since $d_{W_2}(w_1) \le 2$ by Claim 6, the desired inequality follows. □

Claim 9. *$d_{W_i}(w_1) + d_{W_i}(v_2) + d_{W_i}(w_2) \le 7$ for each $i \in \{3, \cdots, s\}$.*

Proof. Let $3 \le i \le s$. If $v_2u_i \notin E(G)$, then $d_{W_i}(v_2) \le 2$; if $v_2u_i \in E(G)$, then $d_{W_i}(w_2) \le 2$ by Claim 6. In either case, we have $d_{W_i}(v_2) + d_{W_i}(w_2) \le 5$. Now assume that $d_{W_i}(v_2) + d_{W_i}(w_2) = 5$ and $d_{W_i}(w_1) = 3$. Then $u_i \in N_G(v_2) \cap N_G(w_2)$ or $w_i \in N_G(v_2) \cap N_G(w_2)$. If $u_i \in N_G(v_2) \cap N_G(w_2)$, then $P' = C_1[u_1, v_1]u_2$, $C'_2 = u_iC_2[v_2, w_2]u_i$, $C'_i = w_1C_i[v_i, w_i]w_1$ and $C'_j = C_j$, $j \ne 1, 2, i$, satisfy (2.5), a contradiction; if $w_i \in N_G(v_2) \cap N_G(w_2)$, then $P' = C_1[u_1, v_1]u_2$, $C'_2 = w_iC_2[v_2, w_2]w_i$, $C'_i = w_1C_i[u_i, v_i]w_1$ and $C'_j = C_j$, $j \ne 1, 2, i$, satisfy (2.5), a contradiction. □

Let $W = \langle \cup_{i=1}^{s} V(W_i)\rangle_G$. Clearly

$$\text{(2.8)} \qquad \begin{cases} d_L(x) \le |V(L) - V(W)| + d_W(x) = l - 3s + d_W(x) \\ \text{for each } x \in \{w_1, v_2, w_2\}. \end{cases}$$

From this and Claims 7, 8 and 9, it follows that

$$\text{(2.9)} \quad d_L(w_1) + d_L(v_2) + d_L(w_2) \le 3l - 9s + 7 + 6 + 7(s-2) = 3l - 2s - 1.$$

Now if $d_L(w_1) + d_L(v_2) + d_L(w_2) \le 3l - 2s - 2$, then $3l - 2s - 2 \ge 3\left(l - s + \frac{k - c_k}{3} + 1\right)$ by Claim 4, and hence $s \ge k - c_k + 5 \ge k + 1$, a contradiction. Thus $d_L(w_1) + d_L(v_2) + d_L(w_2) = 3l - 2s - 1$. Then again by Claim 4, $3l - 2s - 1 \ge 3\left(l - s + \frac{k - c_k}{3} + 1\right)$, i.e., $s \ge k + 4 - c_k$. This inequality holds only when $c_k = 4$ and $s = k$. Thus $s = k \equiv 1 \pmod 3$. Consequently, $s \ge 4$, and all of the following equalities, (2.10) through (2.12), must hold:

$$\text{(2.10)} \qquad d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) = 7;$$

$$\text{(2.11)} \qquad d_{W_2}(w_1) + d_{W_2}(v_2) + d_{W_2}(w_2) = 6; \text{ and}$$

$$\text{(2.12)} \qquad d_{W_i}(w_1) + d_{W_i}(v_2) + d_{W_i}(w_2) = 7 \text{ for each } i \in \{3, \cdots, s\}.$$

Note that we have not made use of (2.6) so far. Note also that we have

$$\text{(2.13)} \qquad u_1 \in N_G(v_2)$$

by (2.7) and (2.10). Thus $d_{W_2}(w_2) + d_{W_2}(v_1) + d_{W_2}(w_1) = 7$, and hence (2.7) holds with the roles of W_1 and W_2 replaced by each other. Consequently

$$\text{(2.14)} \qquad \begin{cases} w_1w_2 \notin E(G),\ w_1u_1 \in E(G),\ w_1v_1 \in E(G), \text{ and every vertex} \\ \text{in } \{u_1, v_1, w_1\} \text{ and every vertex in } \{u_2, v_2, w_2\}, \text{ except } w_1 \text{ and} \\ w_2 \text{ and except possibly } u_1 \text{ and } u_2, \text{ are adjacent to each other.} \end{cases}$$

Again note that we have not yet used (2.6). Thus

(2.15) $\begin{cases} \text{for any } i, j \in \{1, \cdots, s\} \text{ with } i \neq j \text{ such that } u_j \in N_G(v_i), \text{ the} \\ \text{statements corresponding to (2.10) through (2.12) and (2.14) hold.} \end{cases}$

Let $I = \{\, i \,|\, d_{W_i}(w_1) = 3,\ 3 \leq i \leq s \,\}$ and let $J = \{3, \cdots, s\} - I$. Since $d_W(w_1) \geq 2s + 1$ by Claim 4 and (2.8), and since $d_{W_1}(w_1) = d_{W_2}(w_1) = 2$ by (2.14), we have

(2.16) $$I \neq \emptyset.$$

Claim 10. *Let $i \in I$. Then w_iv_1, $w_iv_2 \notin E(G)$ and $d_{W_i}(w_2) = 3$.*

Proof. We first show that $u_iv_2 \notin E(G)$. Suppose that $u_iv_2 \in E(G)$. Since $d_{W_i}(w_1) = 3$ by assumption, and since $d_{W_i}(v_2) + d_{W_i}(w_2) = 5$ by (2.14) and (2.15), we have $d_{W_i}(w_1) + d_{W_i}(v_2) + d_{W_i}(w_2) = 8$, which contradicts (2.12). Thus $u_iv_2 \notin E(G)$. Next suppose that $w_iv_2 \in E(G)$. If $u_iw_2, v_iw_2 \in E(G)$, then $P' = C_1[u_1, v_1]u_2$, $C_2' = w_1v_2w_iw_1$, $C_i' = w_2C_i[u_i, v_i]w_2$ and $C_j' = C_j$, $j \neq 1, 2, i$, satisfy (2.5), a contradiction; otherwise, in view of (2.12), $w_iw_2 \in E(G)$, and hence $P' = C_1[u_1, v_1]u_2$, $C_2' = w_iC_2[v_2, w_2]w_i$, $C_i' = w_1C_i[u_i, v_i]w_1$ and $C_j' = C_j$, $j \neq 1, 2, i$, satisfy (2.5), a contradiction. Thus $w_iv_2 \notin E(G)$. Consequently $d_{W_i}(v_2) = 1$ and

(2.17) $$d_{W_i}(w_2) = 3$$

by (2.12). Now by (2.13) and (2.17), we can argue as above with the roles of W_1 and W_2 replaced by each other, to obtain ($u_iv_1 \notin E(G)$ and) $w_iv_1 \notin E(G)$. □

Claim 11. *Let $j \in J$. Then $w_jw_1 \notin E(G)$, and $u_jv_1, u_jw_1, v_jw_1 \in E(G)$.*

Proof. We have

(2.18) $$d_{W_j}(w_1) + d_{W_j}(v_2) + d_{W_j}(w_2) = 7$$

by (2.12), and $d_{W_j}(w_1) \leq 2$ by the assumption that $j \in J$. Furthermore, it follows from (2.12), (2.13) and (2.15) that

(2.19) $$d_{W_j}(w_2) + d_{W_j}(v_1) + d_{W_j}(w_1) = 7.$$

If $d_{W_j}(w_2) = 3$, then applying Claim 10 with the roles of W_1 and W_2 replaced by each other, we obtain $d_{W_j}(w_1) = 3$, which contradicts the assumption that $j \in J$. Thus $d_{W_j}(w_2) \leq 2$. Consequently by (2.18) and (2.19), $d_{W_j}(v_1) = d_{W_j}(v_2) = 3$. Therefore $u_jv_1 \in E(G)$, and hence $w_jw_1 \notin E(G)$ and $u_jw_1, v_jw_1 \in E(G)$ by (2.14) and (2.15). □

Since $v_1u_j \in E(G)$ for each $j \in J$ by Claim 11, we can apply Claim 10 to W_1 and W_j in place of W_1 and W_2, to obtain the following claim:

Claim 12. *Let $i \in I$ and let $j \in J$. Then $w_iv_j \notin E(G)$.* □

Take $i' \in I$ (recall (2.16)). Since $\sum_{i=1}^{s} d_{W_i}(w_{i'}) \leq 2|J \cup \{1, 2, i'\}| + 3|I - \{i'\}| = 3|I| + 2|J| + 3$ by Claims 10 and 12, it follows from the choice of $i_1 = 1$ and $i_2 = 2$ (recall (2.6)) that

$$\sum_{i=1}^{s} d_{W_i}(w_1) \leq 3|I| + 2|J| + 3. \tag{2.20}$$

On the other hand, it follows from (2.14) and Claim 11 that

$$\sum_{i=1}^{s} d_{W_i}(w_1) = |\{u_1, v_1, u_2, v_2\}| + 3|I| + 2|J| = 3|I| + 2|J| + 4,$$

which contradicts (2.20).

This completes the proof of Theorem 1.

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Toshinori Sakai
Research Institute of Education, Tokai University
2-28-4 Tomigaya, Shibuya-ku, Tokyo 151-8677, Japan
E-mail: tsakai@yoyogi.ycc.u-tokai.ac.jp