SUT Journal of Mathematics
Vol. 38, No. 2 (2002), 185–199

On cohomology rings of a cyclic group and a ring of integers

Takao Hayami and Katsunori Sanada†

(Received November 6, 2002)

Abstract. We determine the ring homomorphism $HH^*(\Gamma) \to H^*(G, \Gamma)$ explicitly, where G denotes the cyclic group of order p^ν and Γ denotes the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta)$ for a primitive p^ν-th root of unity ζ.

AMS 1991 *Mathematics Subject Classification.* 16E40, 20J06.

Key words and phrases. Hochschild cohomology, group cohomology, cup product, cohomology ring.

Introduction

We have investigated some kinds of cohomology rings of generalized quaternion groups in [H], [HS] and [S2]. These results depends on the fact that generalized quaternion groups have a periodic resolution of period 4 and so it is easy to compute the group cohomology. We also know that cyclic groups have a periodic resolution of period 2. So, it may be natural to ask a cyclic group analogy of [S2] and [HS]. Our objective in this paper is to determine a ring homomorphism between a group cohomology ring of a cyclic group with coefficients in an order and the Hochschild cohomology ring of the order.

Let $G = \langle x \rangle$ denote the cyclic group of order p^ν for any prime number p and any positive integer $\nu \geqslant 1$. The rational group ring $\mathbb{Q}G$ is isomorphic to the direct sum of the cyclotomic fields $\mathbb{Q}(\zeta_d)$, where ζ_d denotes a primitive d-th root of 1 for d dividing p^ν, and there exist primitive idempotents e_i for $0 \leqslant i \leqslant \nu$ such that $\mathbb{Q}Ge_i \simeq \mathbb{Q}(\zeta_{p^i})$. Then we have a ring homomorphism $\phi : \mathbb{Z}G \to \mathbb{Z}Ge_\nu; x \mapsto xe_\nu$. Since xe_ν is a primitive p^ν-th root of e_ν, we identify xe_ν with ζ_{p^ν} under the isomorphism stated above. We set $\Gamma = \mathbb{Z}Ge_\nu(=$

†This research was partially supported by Grant-in-Aid for Joint Research (No. 108), Tokyo University of Science, Japan.

$\mathbb{Z}[\zeta_{p^\nu}]$). In this paper, we explicitly determine the ring homomorphism F^* : $HH^*(\Gamma) := \bigoplus_{n \geqslant 0} HH^n(\Gamma) \to H^*(G, \Gamma) := \bigoplus_{n \geqslant 0} H^n(G, \Gamma)$ induced by the ring homomorphism ϕ. In the above, Γ in the right hand side is regarded as a G-module by conjugation, so it is a trivial G-module.

In Section 1, as preliminaries, we describe the detail of defining ring homomorphism F^* stated above.

In Section 2.1, we give a chain transformation lifting the identity map on $\mathbb{Z}$ between the well known periodic resolution of period 2 and the standard resolution for G (Proposition 1). In Section 2.2, we give a pair of dual bases of Γ as a Frobenius $\mathbb{Z}$-algebra (Lemma 2). Furthermore, we give initial parts of a chain transformation lifting the identity map on Γ between a periodic resolution of period 2 (see [BF], [LL]) and the standard complex of Γ (Proposition 3).

In Section 3, as a main result of this paper, we will determine the ring homomorphism $F^* : HH^*(\Gamma) \to H^*(G, \Gamma)$ by investigating the image of a generator of $HH^*(\Gamma)$ under F^2 (Theorem).

§1. Preliminaries

Let R be a commutative ring and Λ an R-algebra which is a finitely generated projective R-module. If M is a left $\Lambda^{\mathrm{e}}(= \Lambda \otimes_R \Lambda^{\mathrm{op}})$-module, then the n-th Hochschild cohomology of Λ with coefficients in M is defined by

$$H^n(\Lambda, M) := \mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}(\Lambda, M).$$

Suppose M' is another Λ^{e}-module. Then for every pair of integers $p, q \geqslant 0$ there is a (Hochschild) cup product

$$H^p(\Lambda, M) \otimes_R H^q(\Lambda, M') \overset{\smile}{\longrightarrow} H^{p+q}(\Lambda, M \otimes_\Lambda M').$$

If we put $M = M' = \Lambda$, then the cup product gives $HH^*(\Lambda) := \bigoplus_{n \geqslant 0} HH^n(\Lambda)$ the structure of a graded ring with identity $1 \in Z(\Lambda) \simeq HH^0(\Lambda)$, where $HH^n(\Lambda)$ denotes $H^n(\Lambda, \Lambda)$ and $Z(\Lambda)$ denotes the center of Λ. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of Λ.

Let G be a finite group and e a central idempotent of the rational group ring $\mathbb{Q}G$. In the following, we set $\Lambda = \mathbb{Z}G$ and $\Lambda' = \mathbb{Z}Ge$, and we regard Λ' as a $\mathbb{Z}$-algebra. Then there is a ring homomorphism $\psi : \Lambda \to \Lambda'^{\mathrm{e}}; x \mapsto xe \otimes (x^{-1}e)^\circ$ for $x \in G$. Let M be a left Λ'^{e}-module, which is regarded as a left Λ-module using ψ above, hence we will denote it by ${}_\psi M$. Then we have a homomorphism of $\mathbb{Z}$-modules (see [S2, Section 1] for example)

$$F^n : H^n(\Lambda', M) \longrightarrow H^n(G, {}_\psi M) := \mathrm{Ext}^n_\Lambda(\mathbb{Z}, {}_\psi M).$$

In the above, $H^n(G, {}_\psi M)$ denotes the ordinary n-th group cohomology. Let (X_G, d_G) be the standard resolution of G, that is,

$$(X_G)_n = \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{n+1 \text{ times}} \quad \text{for } n \geqslant 0,$$

and the boundaries are given by

$$\begin{aligned}(d_G)_1([\sigma]) &= \sigma[\cdot] - [\cdot],\\ (d_G)_n([\sigma_1|\dots|\sigma_n]) &= \sigma_1[\sigma_2|\dots|\sigma_n]\\ &\quad + \sum_{i=1}^{n-1}(-1)^i[\sigma_1|\dots|\sigma_{i-1}|\sigma_i\sigma_{i+1}|\sigma_{i+2}|\dots|\sigma_n]\\ &\quad + (-1)^n[\sigma_1|\dots|\sigma_{n-1}] \qquad \text{for } n \geqslant 2,\end{aligned}$$

where $\sigma[\cdot]$ denotes $\sigma \in (X_G)_0$ and $\sigma_0[\sigma_1|\dots|\sigma_n]$ denotes $\sigma_0 \otimes \sigma_1 \otimes \cdots \otimes \sigma_n \in (X_G)_n$ for $\sigma, \sigma_0, \sigma_1, \dots, \sigma_n \in G$. Furthermore, let $(X_{\Lambda'}, d_{\Lambda'})$ be the standard complex of Λ', that is,

$$(X_{\Lambda'})_n = \underbrace{\Lambda' \otimes \cdots \otimes \Lambda'}_{n+2 \text{ times}} \quad \text{for } n \geqslant 0,$$

and the boundaries are given by

$$\begin{aligned}(d_{\Lambda'})_1\left([\lambda']\right) &= \lambda'[\cdot] - [\cdot]\lambda',\\ (d_{\Lambda'})_n\left([\lambda'_1, \dots, \lambda'_n]\right) &= \lambda'_1[\lambda'_2, \dots, \lambda'_n]\\ &\quad + \sum_{i=1}^{n-1}(-1)^i[\lambda'_1, \dots, \lambda'_{i-1}, \lambda'_i\lambda'_{i+1}, \lambda'_{i+2}, \dots, \lambda'_n]\\ &\quad + (-1)^n[\lambda'_1, \dots, \lambda'_{n-1}]\lambda'_n \qquad \text{for } n \geqslant 2,\end{aligned}$$

where $\lambda'_0[\cdot]\lambda'_1$ denotes $\lambda'_0 \otimes \lambda'_1 \in (X_{\Lambda'})_0$ and $\lambda'_0[\lambda'_1, \dots, \lambda'_n]\lambda'_{n+1}$ denotes $\lambda'_0 \otimes \lambda'_1 \otimes \cdots \otimes \lambda'_{n+1} \in (X_{\Lambda'})_n$ for $\lambda', \lambda'_0, \lambda'_1, \dots, \lambda'_{n+1} \in \Lambda'$. The homomorphism F^n is induced by

$$\begin{aligned}\tilde{F}^n &: \mathrm{Hom}_{\Lambda'^{\mathrm{e}}}((X_{\Lambda'})_n, M) \longrightarrow \mathrm{Hom}_\Lambda((X_G)_n, {}_\psi M),\\ &\tilde{F}^n(f)\,(x_0[x_1|\dots|x_n]) = f\left(x_0 e[x_1 e, \dots, x_n e](x_0 \cdots x_n)^{-1} e\right),\end{aligned}$$

for $x_0, x_1, \dots, x_n \in G$.

Suppose A and B are G-modules. Then for every pair of integers $p, q \geqslant 0$ there exists a homomorphism called (ordinary) cup product

$$H^p(G, A) \otimes H^q(G, B) \xrightarrow{\smile} H^{p+q}(G, A \otimes B).$$

Note that F^n preserves cup products, that is, the following diagram is commutative:

$$\begin{array}{ccc} H^p(\Lambda', M) \otimes H^q(\Lambda', M') & \xrightarrow{\smile} & H^{p+q}(\Lambda', M \otimes_{\Lambda'} M') \\ {\scriptstyle F^p \otimes F^q}\big\downarrow & & \big\downarrow{\scriptstyle F^{p+q}} \\ H^p(G, {}_\psi M) \otimes H^q(G, {}_\psi M') & \xrightarrow[\smile_\mu]{} & H^{p+q}\left(G, {}_\psi(M \otimes_{\Lambda'} M')\right), \end{array}$$

where M' is another Λ'^{e}-module. In the above, $\smile_\mu$ denotes the map induced by the (ordinary) cup product and a left Λ-homomorphism $\mu : {}_\psi M \otimes {}_\psi M' \to {}_\psi(M \otimes_{\Lambda'} M')$; $m \otimes m' \mapsto m \otimes_{\Lambda'} m'$. If we put $M = M' = \Lambda'$ and identify Λ' with $\Lambda' \otimes_{\Lambda'} \Lambda'$ as a Λ'^{e}-module, then we have the following ring homomorphism:

$$F^* : HH^*(\Lambda') \longrightarrow H^*(G, {}_\psi\Lambda') := \bigoplus_{n \geqslant 0} H^n(G, {}_\psi\Lambda').$$

We treat the case $M = M' = \Lambda'$ only in the following. We make $\mathrm{Hom}_{\Lambda'^{\mathrm{e}}}((X_{\Lambda'})_n, \Lambda')$ and $\mathrm{Hom}_\Lambda((X_G)_n, {}_\psi\Lambda')$ into left $Z(\Lambda')$-modules by putting $(z \cdot f)(x) = z \cdot f(x)$, $(z \cdot g)(y) = z \cdot g(y)$ for $f \in \mathrm{Hom}_{\Lambda'^{\mathrm{e}}}((X_{\Lambda'})_n, \Lambda')$, $x \in (X_{\Lambda'})_n$, $g \in \mathrm{Hom}_\Lambda((X_G)_n, {}_\psi\Lambda')$, $y \in (X_G)_n$ and $z \in Z(\Lambda')$. Note that $(d_{\Lambda'})_{n+1}^{\#} : \mathrm{Hom}_{\Lambda'^{\mathrm{e}}}((X_{\Lambda'})_n, \Lambda') \to \mathrm{Hom}_{\Lambda'^{\mathrm{e}}}((X_{\Lambda'})_{n+1}, \Lambda')$ is a $Z(\Lambda')$-homomorphism, where $(d_{\Lambda'})_{n+1}^{\#}$ is induced by the differential $(d_{\Lambda'})_{n+1} : (X_{\Lambda'})_{n+1} \to (X_{\Lambda'})_n$. Similarly, $(d_G)_{n+1}^{\#} : \mathrm{Hom}_\Lambda((X_G)_n, {}_\psi\Lambda') \to \mathrm{Hom}_\Lambda((X_G)_{n+1}, {}_\psi\Lambda')$ is a $Z(\Lambda')$-homomorphism, where $(d_G)_{n+1}^{\#}$ is induced by the differential $(d_G)_{n+1} : (X_G)_{n+1} \to (X_G)_n$. Then $HH^n(\Lambda')$ and $H^n(G, {}_\psi\Lambda')$ are also left $Z(\Lambda')$-modules. Note that $\tilde{F}^n$ is a $Z(\Lambda')$-homomorphism.

On the other hand, let α be the image of $z \in Z(\Lambda')$ under the isomorphism $Z(\Lambda') \xrightarrow{\sim} HH^0(\Lambda')$. We make $HH^n(\Lambda')$ into a left $Z(\Lambda')$-module by putting $z \cdot \beta = \alpha \smile \beta$ for $\beta \in HH^n(\Lambda')$. Similarly, let α' be the image of the above z under the isomorphism $({}_\psi\Lambda')^G = Z(\Lambda') \xrightarrow{\sim} H^0(G, {}_\psi\Lambda')$. We make $H^n(G, {}_\psi\Lambda')$ into a left $Z(\Lambda')$-module by putting $z \cdot \beta' = \alpha' \smile_\mu \beta'$ for $\beta' \in H^n(G, {}_\psi\Lambda')$. Note that $F^0(\alpha) = \alpha'$ holds. Then it is easy to see that the $Z(\Lambda')$-module structure of $HH^n(\Lambda')$ and $H^n(G, {}_\psi\Lambda')$ by the cochain level operations corresponds to the one by the cup products, respectively. Since F^* is a ring homomorphism, we have $F^n(z \cdot \beta) = F^n(\alpha \smile \beta) = F^0(\alpha) \smile_\mu F^n(\beta) = \alpha' \smile_\mu F^n(\beta) = z \cdot F^n(\beta)$. Thus F^* is a homomorphism of graded $Z(\Lambda')$-algebras.

§2. Resolutions and chain transformations

2.1. The cyclic group of order m

Let $G = \langle x \rangle$ denote the cyclic group of order m for any positive integer $m \geqslant 2$. We set $\Lambda = \mathbb{Z}G$. Then the following periodic Λ-free resolution for $\mathbb{Z}$ of period

2 is well known (see [CE, Chapter XII, Section 7] for example):

$$(Y_G, \delta_G): \quad \cdots \longrightarrow \Lambda \xrightarrow{(\delta_G)_1} \Lambda \xrightarrow{(\delta_G)_2} \Lambda \xrightarrow{(\delta_G)_1} \Lambda \xrightarrow{(\delta_G)_2} \Lambda \xrightarrow{(\delta_G)_1} \Lambda \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

$$(\delta_G)_1(c) = c(x-1),$$

$$(\delta_G)_2(c) = c\sum_{i=0}^{m-1} x^i.$$

In the following, we set $(\delta_G)_{2k+i} = (\delta_G)_i$ for any integer $k \geqslant 0$ and $i = 1, 2$ because (Y_G, δ_G) is a periodic resolution.

(X_G, d_G) denotes the standard resolution of G stated in Section 1. We introduce the notation $*$ for basis elements in $(X_G)_i$ $(i \geqslant 0)$ as follows:

$$\sigma_0[\sigma_1] * \sigma_2[\cdot] := \sigma_0[\sigma_1\sigma_2] \ (\in (X_G)_1),$$
$$\sigma_0[\sigma_1] * \sigma_2[\sigma_3|\dots|\sigma_i] := \sigma_0[\sigma_1\sigma_2|\sigma_3|\dots|\sigma_i] \ (\in (X_G)_{i-1})$$

for $\sigma_0, \sigma_1, \dots, \sigma_i \in G$. It is easy to see that the following equations hold:

$$[\sigma_1] * \sigma_2[\cdot] = [\sigma_1\sigma_2] * [\cdot],$$
$$[\sigma_1] * \sigma_2[\sigma_3|\dots|\sigma_i] = [\sigma_1\sigma_2] * [\sigma_3|\dots|\sigma_i];$$
$$(d_G)_1([\sigma_1] * \sigma_2[\cdot]) = \sigma_1\sigma_2[\cdot] - [\cdot],$$
$$(d_G)_{i-1}([\sigma_1] * \sigma_2[\sigma_3|\dots|\sigma_i]) = \sigma_1\sigma_2[\sigma_3|\dots|\sigma_i] - [\sigma_1] * (d_G)_{i-2}(\sigma_2[\sigma_3|\dots|\sigma_i]) \qquad \text{for} \quad i \geqslant 3.$$

Proposition 1. *A chain transformation $u_n : (Y_G)_n \to (X_G)_n$ $(n \geqslant 0)$ lifting the identity map on $\mathbb{Z}$ is given inductively as follows:*

$$u_0(1) = [\cdot];$$
$$u_{2k+1}(1) = [x] * u_{2k}(1) \quad \textit{for } k \geqslant 0;$$
$$u_{2k+2}(1) = \sum_{i=0}^{m-1} [x^i] * u_{2k+1}(1) \quad \textit{for } k \geqslant 0,$$

where each u_n is a left Λ-homomorphism.

Proof. It suffices to show that the equation $(d_G)_n \cdot u_n = u_{n-1} \cdot (\delta_G)_n$ holds for $n \geqslant 1$. By induction on k. First we verify the case $k = 0$, that is, $n = 1, 2$. In the case $n = 1$, noting that $u_1(1) = [x]$, we have the following:

$$((d_G)_1 \cdot u_1)(1) = (d_G)_1([x]) = x[\cdot] - [\cdot] = u_0(x-1) = (u_0 \cdot (\delta_G)_1)(1).$$

In the case $n = 2$, we have the following:

$$\begin{aligned}
\left((d_G)_2 \cdot u_2\right)(1) &= (d_G)_2 \left(\sum_{i=0}^{m-1} [x^i] * u_1(1)\right) \\
&= \sum_{i=0}^{m-1} x^i u_1(1) - \sum_{i=0}^{m-1} [x^i] * (d_G)_1 \left(u_1(1)\right) \\
&= u_1 \left(\sum_{i=0}^{m-1} x^i\right) - \sum_{i=0}^{m-1} [x^i] * (x-1) u_0(1) \\
&= \left(u_1 \cdot (\delta_G)_2\right)(1).
\end{aligned}$$

Suppose that the result holds for $k-1$. In the case $n = 2k+1$, using the assumption of induction, we have the following:

$$\begin{aligned}
\left((d_G)_{2k+1} \cdot u_{2k+1}\right)(1) &= (d_G)_{2k+1}([x] * u_{2k}(1)) \\
&= x u_{2k}(1) - [x] * (d_G)_{2k} \left(u_{2k}(1)\right) \\
&= x u_{2k}(1) - [x] * \left(u_{2k-1} \cdot (\delta_G)_{2k}\right)(1) \\
&= x u_{2k}(1) - [x] * \left(\sum_{i=0}^{m-1} x^i u_{2k-1}(1)\right) \\
&= x u_{2k}(1) - \sum_{i=0}^{m-1} [x^{i+1}] * u_{2k-1}(1) \\
&= x u_{2k}(1) - u_{2k}(1) \\
&= \left(u_{2k} \cdot (\delta_G)_{2k+1}\right)(1).
\end{aligned}$$

In the case $n = 2k+2$, using the above calculation, we have the following:

$$\begin{aligned}
\left((d_G)_{2k+2} \cdot u_{2k+2}\right)(1) &= (d_G)_{2k+2} \left(\sum_{i=0}^{m-1} [x^i] * u_{2k+1}(1)\right) \\
&= \sum_{i=0}^{m-1} x^i u_{2k+1}(1) - \sum_{i=0}^{m-1} [x^i] * (d_G)_{2k+1} \left(u_{2k+1}(1)\right) \\
&= u_{2k+1} \left(\sum_{i=0}^{m-1} x^i\right) - \sum_{i=0}^{m-1} [x^i] * (x-1) u_{2k}(1) \\
&= \left(u_{2k+1} \cdot (\delta_G)_{2k+2}\right)(1).
\end{aligned}$$

This completes the proof. □

The chain transformation u_2 will be used in Section 3, in the case $m = p^\nu$ for a prime number p and a positive integer ν.

2.2. The ring of integers $\mathbb{Z}[\zeta]$

Let ζ be a primitive p^ν-th root of 1. We consider the ring of integers $\Gamma = \mathbb{Z}[\zeta]$ of the cyclotomic field $\mathbb{Q}(\zeta)$. It is well-known that $\{\zeta^i\}_{i=0}^{\varphi(p^\nu)-1}$ is a $\mathbb{Z}$-basis of Γ, where φ denotes the Euler function, so $\varphi(p^\nu) = p^{\nu-1}(p-1)$ (see [W, Lemma 7-5-3]).

We take a matrix $P \in M_{\varphi(p^\nu)}(\mathbb{Z})$ as follows:

$$P = \underbrace{\begin{pmatrix} P' & \cdots & \cdots & P' \\ \vdots & & \iddots & O \\ \vdots & \iddots & \iddots & \vdots \\ P' & O & \cdots & O \end{pmatrix}}_{p-1} \quad \text{where } P' = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \in M_{p^{\nu-1}}(\mathbb{Z}).$$

Then it is easy to see that P is an invertible matrix in $M_{\varphi(p^\nu)}(\mathbb{Z})$. We define a set of elements $\{\zeta^{[i]}\}_{i=0}^{\varphi(p^\nu)-1}$ of Γ by

$$\left(\zeta^{[0]},\quad \zeta^{[1]},\quad \ldots,\quad \zeta^{[\varphi(p^\nu)-1]}\right) = \left(\zeta^0,\quad \zeta^1,\quad \ldots,\quad \zeta^{\varphi(p^\nu)-1}\right) P.$$

Lemma 2. *Γ is a Frobenius $\mathbb{Z}$-algebra with a pair of $\mathbb{Z}$-bases $\{\zeta^i\}_{i=0}^{\varphi(p^\nu)-1}$, $\{\zeta^{[i]}\}_{i=0}^{\varphi(p^\nu)-1}$ which satisfy the following equations:*

$$\gamma\zeta^i = \sum_{j=0}^{\varphi(p^\nu)-1} \zeta^j \alpha_{ji}(\gamma), \quad \zeta^{[j]}\gamma = \sum_{i=0}^{\varphi(p^\nu)-1} \alpha_{ji}(\gamma)\zeta^{[i]}$$

for any $\gamma \in \Gamma$ and for some $\alpha_{ji}(\gamma) \in \mathbb{Z}$.

Proof. It is clear that $\{\zeta^{[i]}\}_{i=0}^{\varphi(p^\nu)-1}$ is a $\mathbb{Z}$-basis of Γ. The equations are verified for $\gamma = \zeta$ by direct computation, so they hold for any $\gamma \in \Gamma$. Hence, it follows that the homomorphism $\chi : \Gamma \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ induced by $\chi(\zeta^i)(\zeta^{[j]}) = \delta_{ij}$ is an isomorphism of left Γ-modules. Therefore Γ is a Frobenius $\mathbb{Z}$-algebra. □

Remark. The *norm* $N_\Gamma(\gamma)$ of $\gamma \in \Gamma$ is defined by

$$N_\Gamma(\gamma) = \sum_{i=0}^{\varphi(p^\nu)-1} \zeta^i \gamma \zeta^{[i]} = \left(\sum_{i=0}^{\varphi(p^\nu)-1} \zeta^i \zeta^{[i]} \right) \gamma$$

(cf. [S1, Section 1.1]). It is easy to see that $\sum_{i=0}^{\varphi(p^\nu)-1} \zeta^i \zeta^{[i]} = \Phi'(\zeta)$, where $\Phi'(x)$ denotes the derivative of the p^ν-th cyclotomic polynomial $\Phi(x) = x^{p^{\nu-1}(p-1)} + x^{p^{\nu-1}(p-2)} + \cdots + x^{p^{\nu-1}} + 1$. The ideal of Γ generated by $\Phi'(\zeta)$ coincides with the *different* $\pi^{\nu p^{\nu-1}(p-1)-p^{\nu-1}}\Gamma$ of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$, where π denotes $\zeta-1$, which generates the prime ideal of $\mathbb{Q}(\zeta)$ lying above p (see [W, Propositions 4-8-18 and 7-4-1]). Hence we have

$$N_\Gamma(\Gamma) = \pi^{\nu p^{\nu-1}(p-1)-p^{\nu-1}}\Gamma. \qquad \square$$

Then there exists a Γ^{e}-projective resolution $(Y_\Gamma, \delta_\Gamma)$ for Γ of period 2 (see [BF], [LL]):

$$(Y_\Gamma, \delta_\Gamma): \quad \cdots \longrightarrow \Gamma\otimes\Gamma \xrightarrow{(\delta_\Gamma)_1} \Gamma\otimes\Gamma \xrightarrow{(\delta_\Gamma)_2} \Gamma\otimes\Gamma \xrightarrow{(\delta_\Gamma)_1} \Gamma\otimes\Gamma \xrightarrow{\varepsilon} \Gamma \to 0,$$
$$(\delta_\Gamma)_1([\cdot]) = \zeta[\cdot] - [\cdot]\zeta,$$
$$(\delta_\Gamma)_2([\cdot]) = \sum_{i=0}^{\varphi(p^\nu)-1} \zeta^{[i]}[\cdot]\zeta^i.$$

In the above, $[\cdot]$ denotes $1\otimes 1 \in \Gamma\otimes\Gamma$.

Proposition 3. *An initial part of a chain transformation $v_n : (X_\Gamma)_n \to (Y_\Gamma)_n$ lifting the identitiy map on Γ is given as follows:*

$$v_0([\cdot]) = [\cdot];$$
$$v_1\left([\zeta^i]\right) = \begin{cases} 0 & \text{if } i = 0, \\ [\cdot]\zeta^{i-1} + \zeta[\cdot]\zeta^{i-2} + \cdots + \zeta^{i-1}[\cdot] & \text{if } i \geqslant 1; \end{cases}$$
$$v_2\left([\zeta^i, \zeta^j]\right) = \begin{cases} 0 & \text{if } 0 \leqslant i+j < \varphi(p^\nu), \\ \zeta^{i+j-\varphi(p^\nu)}[\cdot] & \text{if } \varphi(p^\nu) \leqslant i+j < p^\nu, \\ \zeta^{i+j-p^\nu}(\zeta^{p^{\nu-1}} - 1)[\cdot] & \text{if } p^\nu \leqslant i+j, \end{cases}$$

for $0 \leqslant i, j < \varphi(p^\nu)$, where each v_n is a left Γ^{e}-homomorphism.

Proof. It suffices to show that the equation $v_{n-1}\cdot(d_\Gamma)_n = (\delta_\Gamma)_n \cdot v_n$ holds for $n = 1, 2$. In the case $n = 1$, the left hand side is as follows:

$$\begin{aligned}(v_0\cdot(d_\Gamma)_1)\left([\zeta^i]\right) &= v_0\left(\zeta^i[\cdot] - [\cdot]\zeta^i\right) \\ &= \zeta^i[\cdot] - [\cdot]\zeta^i \qquad \text{for } i \geqslant 0.\end{aligned}$$

The right hand side is divided into two cases:
Case $i = 0$:

$$((\delta_\Gamma)_1 \cdot v_1)([1]) = 0.$$

Case $i \geqslant 1$:

$$\begin{aligned}
&((\delta_\Gamma)_1 \cdot v_1)\left([\zeta^i]\right) \\
&\qquad = (\delta_\Gamma)_1\left([\cdot]\zeta^{i-1} + \zeta[\cdot]\zeta^{i-2} + \cdots + \zeta^{i-1}[\cdot]\right) \\
&\qquad = (\zeta[\cdot] - [\cdot]\zeta)\,\zeta^{i-1} + \zeta\,(\zeta[\cdot] - [\cdot]\zeta)\,\zeta^{i-2} + \cdots + \zeta^{i-1}\,(\zeta[\cdot] - [\cdot]\zeta) \\
&\qquad = \zeta^i[\cdot] - [\cdot]\zeta^i.
\end{aligned}$$

In the case $n = 2$, the left hand side is divided into six cases:
Case $ij = 0$:

$$(v_1 \cdot (d_\Gamma)_2)\left([\zeta^i, \zeta^j]\right) = 0.$$

Case $0 < i + j < \varphi(p^\nu),\ ij \neq 0$:

$$\begin{aligned}
(v_1 \cdot (d_\Gamma)_2)\left([\zeta^i, \zeta^j]\right) &= v_1\left(\zeta^i[\zeta^j] - [\zeta^{i+j}] + [\zeta^i]\zeta^j\right) \\
&= \zeta^i\left([\cdot]\zeta^{j-1} + \zeta[\cdot]\zeta^{j-2} + \cdots + \zeta^{j-1}[\cdot]\right) \\
&\quad - \left([\cdot]\zeta^{i+j-1} + \zeta[\cdot]\zeta^{i+j-2} + \cdots + \zeta^{i+j-1}[\cdot]\right) \\
&\quad + \left([\cdot]\zeta^{i-1} + \zeta[\cdot]\zeta^{i-2} + \cdots + \zeta^{i-1}[\cdot]\right) \\
&= 0.
\end{aligned}$$

Case $i + j = \varphi(p^\nu)$:

$$\begin{aligned}
&(v_1 \cdot (d_\Gamma)_2)\left([\zeta^i, \zeta^j]\right) \\
&\qquad = v_1\left(\zeta^i[\zeta^j] - [\zeta^{i+j}] + [\zeta^i]\zeta^j\right) \\
&\qquad = v_1\left(\zeta^i[\zeta^j] + \sum_{k=0}^{p-2}[\zeta^{kp^{\nu-1}}] + [\zeta^i]\zeta^j\right) \\
&\qquad = \zeta^i\left([\cdot]\zeta^{j-1} + \zeta[\cdot]\zeta^{j-2} + \cdots + \zeta^{j-1}[\cdot]\right) \\
&\qquad\quad + \sum_{k=1}^{p-2}\left([\cdot]\zeta^{kp^{\nu-1}-1} + \zeta[\cdot]\zeta^{kp^{\nu-1}-2} + \cdots + \zeta^{kp^{\nu-1}-1}[\cdot]\right) \\
&\qquad\quad + \left([\cdot]\zeta^{i-1} + \zeta[\cdot]\zeta^{i-2} + \cdots + \zeta^{i-1}[\cdot]\right)\zeta^j \\
&\qquad = \sum_{k=1}^{p-1}\left([\cdot]\zeta^{kp^{\nu-1}-1} + \zeta[\cdot]\zeta^{kp^{\nu-1}-2} + \cdots + \zeta^{kp^{\nu-1}-1}[\cdot]\right) \\
&\qquad = \sum_{k=0}^{\varphi(p^\nu)-1} \zeta^{[k]}[\cdot]\zeta^k.
\end{aligned}$$

Case $\varphi(p^\nu) < i + j < p^\nu$:

$$(v_1 \cdot (d_\Gamma)_2)\left([\zeta^i, \zeta^j]\right)$$

$$
\begin{aligned}
&= v_1\left(\zeta^i[\zeta^j] - [\zeta^{i+j}] + [\zeta^i]\zeta^j\right)\\
&= v_1\left(\zeta^i[\zeta^j] + \sum_{k=0}^{p-2}[\zeta^{kp^{\nu-1}+i+j-\varphi(p^\nu)}] + [\zeta^i]\zeta^j\right)\\
&= \zeta^i\left([\cdot]\zeta^{j-1} + \zeta[\cdot]\zeta^{j-2} + \cdots + \zeta^{j-1}[\cdot]\right)\\
&\quad + \sum_{k=0}^{p-2}\left([\cdot]\zeta^{i+j-\varphi(p^\nu)-1} + \zeta[\cdot]\zeta^{i+j-\varphi(p^\nu)-2} + \cdots + \zeta^{i+j-\varphi(p^\nu)-1}[\cdot]\right)\zeta^{kp^{\nu-1}}\\
&\quad + \sum_{k=1}^{p-2}\zeta^{i+j-\varphi(p^\nu)}\left([\cdot]\zeta^{kp^{\nu-1}-1} + \zeta[\cdot]\zeta^{kp^{\nu-1}-2} + \cdots + \zeta^{kp^{\nu-1}-1}[\cdot]\right)\\
&\quad + \left([\cdot]\zeta^{i-1} + \zeta[\cdot]\zeta^{i-2} + \cdots + \zeta^{i-1}[\cdot]\right)\zeta^j\\
&= \zeta^{i+j-\varphi(p^\nu)}\left(\sum_{k=1}^{p-1}\left([\cdot]\zeta^{kp^{\nu-1}-1} + \zeta[\cdot]\zeta^{kp^{\nu-1}-2} + \cdots + \zeta^{kp^{\nu-1}-1}[\cdot]\right)\right)\\
&= \zeta^{i+j-\varphi(p^\nu)}\left(\sum_{k=0}^{\varphi(p^\nu)-1}\zeta^{[k]}[\cdot]\zeta^k\right).
\end{aligned}
$$

Case $i + j = p^\nu$:

$$
\begin{aligned}
&(v_1\cdot(d_\Gamma)_2)\left([\zeta^i,\zeta^j]\right)\\
&\quad= v_1\left(\zeta^i[\zeta^j] - [1] + [\zeta^i]\zeta^j\right)\\
&\quad= [\cdot]\zeta^{p^\nu-1} + \zeta[\cdot]\zeta^{p^\nu-2} + \cdots + \zeta^{p^{\nu-1}-1}[\cdot]\zeta^{p^{\nu-1}(p-1)}\\
&\qquad + \zeta^{p^{\nu-1}}[\cdot]\zeta^{p^{\nu-1}(p-1)-1} + \cdots + \zeta^{p^\nu-1}[\cdot]\\
&\quad= -\sum_{k=1}^{p^{\nu-1}}\zeta^{k-1}[\cdot]\zeta^{p^{\nu-1}-k}\left(\zeta^{p^{\nu-1}(p-2)} + \zeta^{p^{\nu-1}(p-3)} + \cdots + 1\right)\\
&\qquad + \zeta^{p^{\nu-1}}[\cdot]\zeta^{p^{\nu-1}(p-1)-1} + \zeta^{p^{\nu-1}+1}[\cdot]\zeta^{p^{\nu-1}(p-1)-2} + \cdots + \zeta^{p^\nu-1}[\cdot]\\
&\quad= \sum_{m=1}^{p-1}(\zeta^{mp^{\nu-1}} - 1)\left(\sum_{k=1}^{p^{\nu-1}}\zeta^{k-1}[\cdot]\zeta^{p^{\nu-1}(p-m)-k}\right)\\
&\quad= (\zeta^{p^{\nu-1}} - 1)\left(\sum_{m=1}^{p-1}\left(\zeta^{p^{\nu-1}(m-1)} + \zeta^{p^{\nu-1}(m-2)} + \cdots + 1\right)\right.\\
&\qquad\qquad\qquad\qquad \left.\times\left(\sum_{k=1}^{p^{\nu-1}}\zeta^{k-1}[\cdot]\zeta^{p^{\nu-1}(p-m)-k}\right)\right)\\
&\quad= (\zeta^{p^{\nu-1}} - 1)\left(\sum_{k=0}^{\varphi(p^\nu)-1}\zeta^{[k]}[\cdot]\zeta^k\right).
\end{aligned}
$$

Case $i + j > p^\nu$:

$$\begin{aligned}
&(v_1 \cdot (d_\Gamma)_2)\left([\zeta^i, \zeta^j]\right) \\
&\quad = v_1\left(\zeta^i[\zeta^j] - [\zeta^{i+j-p^\nu}] + [\zeta^i]\zeta^j\right) \\
&\quad = [\cdot]\zeta^{i+j-1} + \zeta[\cdot]\zeta^{i+j-2} + \cdots + \zeta^{i+j-1}[\cdot] \\
&\qquad - \left([\cdot]\zeta^{i+j-p^\nu-1} + \zeta[\cdot]\zeta^{i+j-p^\nu-2} + \cdots + \zeta^{i+j-p^\nu-1}[\cdot]\right) \\
&\quad = \zeta^{i+j-p^\nu}\left([\cdot]\zeta^{p^\nu-1} + \zeta[\cdot]\zeta^{p^\nu-2} + \cdots + \zeta^{p^\nu-1}[\cdot]\right) \\
&\quad = \zeta^{i+j-p^\nu}(\zeta^{p^{\nu-1}} - 1)\left(\sum_{k=0}^{\varphi(p^\nu)-1} \zeta^{[k]}[\cdot]\zeta^k\right).
\end{aligned}$$

The above last equality follows from the calculation in the case $i + j = p^\nu$. The right hand side is divided into three cases:
Case $0 \leqslant i + j < \varphi(p^\nu)$:

$$((\delta_\Gamma)_2 \cdot v_2)\left([\zeta^i, \zeta^j]\right) = 0.$$

Case $\varphi(p^\nu) \leqslant i + j < p^\nu$:

$$\begin{aligned}
((\delta_\Gamma)_2 \cdot v_2)\left([\zeta^i, \zeta^j]\right) &= (\delta_\Gamma)_2\left(\zeta^{i+j-\varphi(p^\nu)}[\cdot]\right) \\
&= \zeta^{i+j-\varphi(p^\nu)}\left(\sum_{k=0}^{\varphi(p^\nu)-1} \zeta^{[k]}[\cdot]\zeta^k\right).
\end{aligned}$$

Case $i + j \geqslant p^\nu$:

$$\begin{aligned}
((\delta_\Gamma)_2 \cdot v_2)\left([\zeta^i, \zeta^j]\right) &= (\delta_\Gamma)_2\left(\zeta^{i+j-p^\nu}(\zeta^{p^{\nu-1}} - 1)[\cdot]\right) \\
&= \zeta^{i+j-p^\nu}(\zeta^{p^{\nu-1}} - 1)\left(\sum_{k=0}^{\varphi(p^\nu)-1} \zeta^{[k]}[\cdot]\zeta^k\right).
\end{aligned}$$

This completes the proof of Proposition 3. □

§3. The ring homomorphism $HH^*(\Gamma) \to H^*(G, \Gamma)$

Let $G = \langle x \rangle$ denote the cyclic group of order p^ν for any prime number p and any positive integer $\nu \geqslant 1$ (we do not consider the case $p^\nu = 2$). Then the rational group ring $\mathbb{Q}G$ is isomorphic to the direct sum of the cyclotomic fields $\mathbb{Q}(\zeta_d)$, where ζ_d denotes a primitive d-th root of 1 for d dividing p^ν:

$$\mathbb{Q}G \simeq \bigoplus_{d \mid p^\nu} \mathbb{Q}(\zeta_d).$$

There exist primitive idempotents e_i for $0 \leqslant i \leqslant \nu$ (${e_i}^2 = e_i,\ e_ie_j = 0$ for $i \neq j,\ 1 = \sum_i e_i$) such that $\mathbb{Q}Ge_i \simeq \mathbb{Q}(\zeta_{p^i})$. Then we have a ring homomorphism $\phi : \mathbb{Z}G \to \mathbb{Z}Ge_\nu; x \mapsto xe_\nu$. Note that xe_ν is a primitive p^ν-th root of e_ν. Under the isomorphism stated above, we identify xe_ν with ζ_{p^ν}. In the following, we set $\Lambda = \mathbb{Z}G$ and $\Gamma = \mathbb{Z}Ge_\nu (= \mathbb{Z}[\zeta_{p^\nu}])$, and we regard Γ as a $\mathbb{Z}$-algebra. In the rest of this section, we write ζ in place of ζ_{p^ν} for brevity. By Section 1, the ring homomorphism ϕ induces the following Γ-algebra homomorphism between the cohomology rings:

$$F^* : HH^*(\Gamma) \longrightarrow H^*(G, \Gamma).$$

In the above, Γ in the right hand side is regarded as a G-module using a ring homomorphism $\psi : \Lambda \to \Gamma^{\mathrm{e}}; x \mapsto xe_\nu \otimes (x^{-1}e_\nu)^\circ = \zeta \otimes (\zeta^{-1})^\circ$, so it is a trivial G-module. In this section, we will determine the ring homomorphism $F^* : HH^*(\Gamma) \to H^*(G, \Gamma)$ by investigating the image of a generator of $HH^*(\Gamma)$ in degree 2 under F^2.

First, we state the cohomologies $H^n(G, \Gamma)$ and $HH^n(\Gamma)$.

Lemma 4. *The cohomology $H^n(G, \Gamma)$ is as follows:*

$$H^n(G,\Gamma) \simeq \begin{cases} \Gamma & \text{for } n = 0, \\ 0 & \text{for } n \equiv 1 \mod 2, \\ \Gamma/\pi^{\nu p^{\nu-1}(p-1)}\Gamma & \text{for } n \equiv 0 \mod 2,\ n \neq 0. \end{cases}$$

Moreover, the cohomology ring $H^*(G, \Gamma)$ is isomorphic to

$$\Gamma[X]/(\pi^{\nu p^{\nu-1}(p-1)}X),$$

where $\pi = \zeta - 1$ and $\deg X = 2$.

Proof. Applying the functor $\mathrm{Hom}_\Lambda(-, \Gamma)$ to the periodic resolution (Y_G, δ_G) in Section 2.1, we have the following complex which gives $H^n(G, \Gamma)$ where we identify $\mathrm{Hom}_\Lambda(\Lambda, \Gamma)$ with Γ as Γ-modules:

$$\begin{aligned} &\Big(\mathrm{Hom}_\Lambda(Y_G, \Gamma), (\delta_G)^\#\Big) : 0 \longrightarrow \Gamma \xrightarrow{(\delta_G)_1^\#} \Gamma \xrightarrow{(\delta_G)_2^\#} \Gamma \xrightarrow{(\delta_G)_1^\#} \Gamma \longrightarrow \cdots, \\ &\quad (\delta_G)_1^\#(\gamma) = (x-1)\gamma = 0, \\ &\quad (\delta_G)_2^\#(\gamma) = \sum_{i=0}^{p^\nu - 1} x^i \gamma = p^\nu \gamma. \end{aligned}$$

Since $p^\nu \Gamma = (\zeta - 1)^{\nu p^{\nu-1}(p-1)}\Gamma$ holds (see [W, Proposition 7-4-1]), we have the module structure of $H^n(G, \Gamma)$. Now we put $X = e_\nu$ which is a generator of $H^2(G, \Gamma)$. Note that $H^{2n}(G, \Gamma)$ is generated by $X^n = e_\nu$ (see [CE, Chapter XII, Section 7]). This completes the proof. □

Lemma 5. *The Hochschild cohomology of* Γ *is as follows:*

$$HH^n(\Gamma) \simeq \begin{cases} \Gamma & \text{for } n = 0, \\ 0 & \text{for } n \equiv 1 \mod 2, \\ \Gamma/\pi^{\nu p^{\nu-1}(p-1)-p^{\nu-1}}\Gamma & \text{for } n \equiv 0 \mod 2, \; n \neq 0. \end{cases}$$

Moreover, the Hochschild cohomology ring $HH^*(\Gamma)$ *is isomorphic to*

$$\Gamma[Y]/(\pi^{\nu p^{\nu-1}(p-1)-p^{\nu-1}}Y),$$

where $\pi = \zeta - 1$ *and* $\deg Y = 2$*.*

Proof. Applying the functor $\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(-,\Gamma)$ to the periodic resolution $(Y_\Gamma, \delta_\Gamma)$ in Section 2.2, we have the following complex which gives $HH^n(\Gamma)$, where we identify $\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma \otimes \Gamma, \Gamma)$ with Γ as Γ-modules:

$$\Big(\mathrm{Hom}_{\Gamma^{\mathrm{e}}}(Y_\Gamma, \Gamma), (\delta_\Gamma)^{\#}\Big) : 0 \longrightarrow \Gamma \xrightarrow{(\delta_\Gamma)_1^{\#}} \Gamma \xrightarrow{(\delta_\Gamma)_2^{\#}} \Gamma \xrightarrow{(\delta_\Gamma)_1^{\#}} \Gamma \longrightarrow \cdots,$$

$$(\delta_\Gamma)_1^{\#}(\gamma) = \zeta\gamma - \gamma\zeta = 0,$$

$$(\delta_\Gamma)_2^{\#}(\gamma) = \sum_{i=0}^{\phi(p^\nu)-1} \zeta^{[i]}\gamma\zeta^i = \Phi'(\zeta)\gamma.$$

Therefore we have the above Γ-module structure of $HH^n(\Gamma)$ by Remark in Section 2.2. Since Γ is a Frobenius algebra, we can consider the *complete* cohomology $\hat{H}^*(\Gamma,\Gamma) = \bigoplus_{i\in\mathbb{Z}} \hat{H}^i(\Gamma,\Gamma)$. This cohomology is periodic of period 2. So, $\hat{H}^*(\Gamma,\Gamma)$ has an invertible element $Y \in \hat{H}^2(\Gamma,\Gamma)$ $\big(= HH^2(\Gamma)\big)$ (cf. [S1, Section 3]). $\square$

Next, we determine the ring homomorphism $F^* : HH^*(\Gamma) \to H^*(G,\Gamma)$ by calculating the image $F^2(Y)$ for the generator Y of $HH^*(\Gamma)$.

Theorem. *The ring homomorphism* $F^* : HH^*(\Gamma) \to H^*(G,\Gamma)$ *is induced by* $F^2(Y) = (\zeta^{p^{\nu-1}} - 1)X$*.*

Proof. It is easy to see that F^n is an isomorphism for $n = 0$ and the zero map for n odd. Thus we calculate $F^2(Y)$. This is obtained by the composition of the following maps on the cochain level:

$$\Gamma \xrightarrow{\beta} \mathrm{Hom}_{\Gamma^{\mathrm{e}}}((Y_\Gamma)_2, \Gamma) \xrightarrow{v_2^{\#}} \mathrm{Hom}_{\Gamma^{\mathrm{e}}}((X_\Gamma)_2, \Gamma)$$

$$\xrightarrow{\tilde{F}^2} \mathrm{Hom}_\Lambda((X_G)_2, \Gamma) \xrightarrow{u_2^{\#}} \mathrm{Hom}_\Lambda((Y_G)_2, \Gamma) \xrightarrow{\alpha} \Gamma.$$

In the above, α denotes the isomorphism $\mathrm{Hom}_{\Lambda}((Y_G)_2, \Gamma) \to \Gamma$ and β denotes the isomorphism $\Gamma \to \mathrm{Hom}_{\Gamma^e}((Y_\Gamma)_2, \Gamma)$. For $\gamma \in \Gamma$, we have

$$
\begin{aligned}
&\left(\alpha \cdot u_2^{\#} \cdot \tilde{F}^2 \cdot v_2^{\#} \cdot \beta\right)(\gamma) \\
&= \left(\tilde{F}^2(\beta(\gamma) \cdot v_2)\right)(u_2(1)) \\
&= \left(\tilde{F}^2(\beta(\gamma) \cdot v_2)\right)\left(\sum_{k=0}^{p^\nu-1}[x^k|x]\right) \\
&= (\beta(\gamma) \cdot v_2)\left(\sum_{k=0}^{p^\nu-1}[\zeta^k, \zeta]\zeta^{-k-1}\right) \\
&= (\beta(\gamma) \cdot v_2)\left(\sum_{k=0}^{\varphi(p^\nu)-1}[\zeta^k, \zeta]\zeta^{-k-1} + \sum_{l=0}^{p^{\nu-1}-1}[\zeta^{\varphi(p^\nu)+l}, \zeta]\zeta^{-\varphi(p^\nu)-l-1}\right) \\
&= (\beta(\gamma) \cdot v_2)\left(\sum_{k=0}^{\varphi(p^\nu)-1}[\zeta^k, \zeta]\zeta^{-k-1} - \sum_{l=0}^{p^{\nu-1}-1}\sum_{k=0}^{p-2}[\zeta^{p^{\nu-1}k+l}, \zeta]\zeta^{-\varphi(p^\nu)-l-1}\right) \\
&= \beta(\gamma)\left([\cdot]\zeta^{-p^{\nu-1}(p-1)} - [\cdot]\right) \\
&= \left(\zeta^{p^{\nu-1}} - 1\right)\gamma.
\end{aligned}
$$

This completes the proof. □

Corollary. F^{2n} $(n \geqslant 1)$ *is a monomorphism if and only if* $n = 1$*. Moreover,* F^{2n} *is the zero map if and only if* $n \geqslant \nu(p-1)$*.*

Proof. Noting that $(\zeta^{p^{\nu-1}} - 1)\Gamma = (\zeta - 1)^{p^{\nu-1}}\Gamma = \pi^{p^{\nu-1}}\Gamma$, we have

$$
\begin{aligned}
\pi^k Y^n \in \mathrm{Ker}\ F^{2n} &\iff F^{2n}(\pi^k Y^n) = 0 \text{ in } H^{2n}(G, \Gamma) \\
&\iff (\pi^k(\zeta^{p^{\nu-1}} - 1)^n)X^n \subset (\pi^{\nu p^{\nu-1}(p-1)})X^n \\
&\iff (\pi^k(\zeta^{p^{\nu-1}} - 1)^n) \subset (\pi^{\nu p^{\nu-1}(p-1)}) \\
&\iff (\pi^{k+np^{\nu-1}}) \subset (\pi^{\nu p^{\nu-1}(p-1)}) \\
&\iff k + np^{\nu-1} \geqslant \nu p^{\nu-1}(p-1) \\
&\iff k \geqslant \nu p^{\nu-1}(p-1) - np^{\nu-1}.
\end{aligned}
$$

Hence, considering the case $k = 0$, it follows that F^{2n} is the zero map if and only if $n \geqslant \nu(p-1)$. By Lemma 5, it is easy to see that F^{2n} is a monomorphism if and only if $n = 1$. □

Acknowledgement

The authors would like to express their gratitude to Professor T. Nozawa and the referee for valuable comments and many helpful suggestions.

References

[BF] F. R. Bobovich and D. K. Faddeev, *Hochschild cohomologies for $\mathbb{Z}$-rings with a power basis*, Mat. Zametki **4-2** (1968), 141–150 = Math. Notes **4** (1968), 575–581.

[CE] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton. NJ., 1956.

[H] T. Hayami, *Hochschild cohomology ring of the integral group ring of the generalized quaternion group*, SUT J. of Math. **38** (2002), 83–126.

[HS] T. Hayami and K. Sanada, *Cohomology ring of the generalized quaternion group with coefficients in an order*, Comm. Algebra **30** (2002), 3611–3628.

[LL] M. Larsen and A. Lindenstrauss, *Cyclic Homology of Dedekind Domains*, K-Theory **6** (1992), 301–334.

[S1] K. Sanada, *On the cohomology of Frobenius algebras*, J. Pure Appl. Algebra **80** (1992), 65–88.

[S2] K. Sanada, *Remarks on cohomology rings of the quaternion group and the quaternion algebra*, SUT J. of Math. **31** (1995), 85–92.

[W] E. Weiss, *Algebraic Number Theory*, Dover Publications, Inc., N. Y. , 1998.

Takao Hayami
Department of Mathematics, Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku, Tokyo 162-0827, Japan
E-mail: `hayami@minserver.ma.kagu.sut.ac.jp`

Katsunori Sanada
Department of Mathematics, Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku, Tokyo 162-0827, Japan
E-mail: `sanada@rs.kagu.tus.ac.jp`