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BLOW-UP SOLUTIONS FOR SOME NONLINEAR ELLIPTIC EQUATIONS INVOLVING A FINSLER-LAPLACIAN

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Abstract: In this paper we prove existence results and asymptotic behavior for strong solutions $u \in W^{2,2}_{\rm loc}(\Omega)$ of the nonlinear elliptic problem

$$\text{(P)} \qquad \begin{cases} -\Delta_H u + H(\nabla u)^q + \lambda u = f & \text{in } \Omega, \\ u \to +\infty & \text{on } \partial\Omega, \end{cases}$$

where H is a suitable norm of $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, Δ_H is the Finsler Laplacian, $1 < q \leq 2$, $\lambda > 0$, and f is a suitable function in $L^\infty_{\rm loc}$. Furthermore, we are interested in the behavior of the solutions when $\lambda \to 0^+$, studying the so-called ergodic problem associated to (P). A key role in order to study the ergodic problem will be played by local gradient estimates for (P).

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1. Introduction

Let Ω be a C^2 bounded domain of $\mathbb{R}^n$, $n \geq 2$, and let us consider the following Finsler-Laplacian of u, namely the operator $\Delta_H u$ defined as

$$\Delta_H u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(H(\nabla u)H_{\xi_i}(\nabla u)),$$

where H is a suitable smooth norm of $\mathbb{R}^n$ (see Subsection 2.1 for the precise assumptions). The aim of the paper is to study the existence of solutions of the equation

$$(1.1) \qquad -\Delta_H u + H(\nabla u)^q + \lambda u = f(x) \quad \text{in } \Omega,$$

where $1 < q \leq 2$, $\lambda > 0$, and f is a suitable function in $L^\infty_{\rm loc}$, bounded from below, with the boundary condition

$$(1.2) \qquad \lim_{x \to \partial\Omega} u(x) = +\infty.$$

We will refer to the solutions of (1.1) which satisfy (1.2) as blow-up solutions. We are also interested in the asymptotic behavior of the solutions. Moreover, we study the behavior of the blow-up solutions of (1.1) when $\lambda \to 0^+$.

Problems which deal with Finsler-Laplacian type operators have been studied in several contexts (see, for example, [**AFTL, BFK, FK, CS, WX, CFV, DG1, DG2, DG3, Ja**]).

When H is the Euclidean norm, namely $H(\xi) = |\xi| = \sqrt{\sum \xi_i^2}$, blow-up problems for equations depending on the gradient have been studied by many authors. We refer the reader, for example, to [**LL, BG, GNR, PV, Le, Po, BPT, FGMP**]. In the Euclidean setting, problem (1.1)–(1.2) reduces to

$$\begin{cases} -\Delta u + |\nabla u|^q + \lambda u = f(x) & \text{in } \Omega, \\ \lim\limits_{x \to \partial\Omega} u(x) = +\infty. \end{cases} \tag{1.3}$$

The interest in problems modeled by (1.3) has been grown since the seminal paper by Lasry and Lions [**LL**]. The equation in (1.3) is a particular case of Hamilton–Jacobi–Bellman equations, which are related to stochastic differential problems. Indeed, in [**LL**] the authors enlightened the relation between problem (1.3) and a model of a stochastic control problem involving constraints on the state of the system by means of unbounded drifts. We briefly recall a few facts about this link.

Let us consider the stochastic differential equation

$$dX_t = a(X_t)\,dt + dB_t, \quad X_0 = x \in \Omega,$$

where B_t is a standard Brownian motion. We assume that $a(\cdot) \in \mathcal{A}$, where $\mathcal{A}$ is the class of feedback controls such that the state process X_t, solution to the above SDE, remains in Ω with probability 1, for all $t \geq 0$ and for any $x \in \Omega$. Thanks to the dynamic programming principle due to Bellman, the function $u_\lambda \in W^{2,r}_{\mathrm{loc}}(\Omega)$, $r < \infty$, which solves (1.3) can be represented as the value function

$$u_\lambda = \inf_{a \in \mathcal{A}} E \int_0^\infty \left[f(X_t) + c_q |a(X_t)|^{q'} \right] e^{-\lambda t}\,dt,$$

where E is the expected value, $1 < q \leq 2$, $q' = \frac{q}{q-1}$, $c_q = (q-1)q^{-q'}$, and $e^{-\lambda t}$ is a discount factor.

In [**LL**] there are several results regarding the existence, uniqueness, and asymptotic behavior of the solutions of (1.3).

When λ tends to zero, the limit of λu_λ is known as ergodic limit. This kind of problems have been largely studied (see, for example, [**BF1, LL,**

BF2, Po, FGMP]). A typical result states that λu_λ tends to a value $u_0 \in \mathbb{R}$ and $u_\lambda(x) - u_\lambda(x_0)$, for fixed $x_0 \in \Omega$, tends to a function v which solves

$$\begin{cases} -\Delta v + |\nabla v|^q + u_0 = f(x) & \text{in } \Omega, \\ \lim\limits_{x \to \partial\Omega} v(x) = +\infty. \end{cases} \tag{1.4}$$

Problem (1.4) is seen as the ergodic limit, as $\lambda \to 0^+$, of the stochastic control problem just described.

The scope of the present paper is to obtain existence, uniqueness, and asymptotic behavior of the solutions to problem (1.1)–(1.2), in the spirit of the work by Lasry and Lions [**LL**], when H is a general norm of $\mathbb{R}^n$.

The interest in this kind of problems is twofold. First, in analogy with the relation between the quoted SDE and the elliptic problem (1.3), we stress that the Finsler Laplacian Δ_H can be interpreted as the generator of a "h-Finslerian diffusion", which generalizes the standard Brownian motion in $\mathbb{R}^n$. Stochastic processes of this type arise in some Biology problems, as in the theory of evolution by endo-symbiosis in which modern cells of plants and animals arise from separately living bacterial species. We refer the reader to [**AZ1, AZ2**] (and to the bibliography cited therein) for the stochastic interpretation of Δ_H and for the quoted applications. Second, apart from the stochastic motivation, the nonlinear elliptic problem we study is of interest in its own right. In our case, the operator in (1.1) is, in general, anisotropic and quasilinear, with a strong nonlinearity in the gradient, and generalizes to this setting some extensively studied problems in the isotropic case. Actually, this brings several difficulties and differencies with respect to the Euclidean case. Moreover, in [**LL**] the asymptotic behavior of the solutions of (1.3) near to the boundary of Ω is strongly related to a precise behavior of f with respect to the distance to $\partial\Omega$. In our case, the anisotropy of the operator leads to use an appropriate distance function to the boundary related to H. On the other hand, the function $\nabla^2 H^2(\xi)$ is always discontinuous at $\xi = 0$, unless it is constant. Hence, also giving smoothness assumptions on H and on the data, it is not possible to apply classical Calderón–Zygmund type regularity results to get strong solutions in $W^{2,r}_{\text{loc}}(\Omega)$, $r < \infty$. We deal, in fact, only with solutions in $W^{2,2}_{\text{loc}}(\Omega)$. Furthermore, this lack of regularity does not permit to obtain, in general, the same gradient estimates for the solutions of (1.1)–(1.2) proved in the Euclidean case, which play a central role in the study of the ergodic problem. Actually, we are able to treat also the case $\lambda \to 0^+$, obtaining existence results for the limit problem

$$\begin{cases} -\Delta_H v + H(\nabla v)^q + u_0 = f(x) & \text{in } \Omega, \\ \lim_{x\to\partial\Omega} v(x) = +\infty, \end{cases}$$

and some properties of the ergodic constant u_0. We refer the reader to Subsection 2.3 for the complete scheme of the obtained results.

The paper is organized as follows.

In Section 2 we give the precise assumptions on H and recall some basic facts of convex analysis. Moreover, we state our results. In Section 3 we prove some a priori estimates for the gradient. Finally, in Section 4 we give the proof of the main results.

2. Assumptions, main results, and comments

2.1. Notation and preliminaries. Throughout the paper we will consider a convex even 1-homogeneous function

$$\xi \in \mathbb{R}^n \mapsto H(\xi) \in [0, +\infty[,$$

that is, a convex function such that

$$H(t\xi) = |t|H(\xi), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n, \tag{2.1}$$

and such that

$$a|\xi| \le H(\xi), \quad \xi \in \mathbb{R}^n, \tag{2.2}$$

for some constant $0 < a$. Under this hypothesis it is easy to see that there exists $b \ge a$ such that

$$H(\xi) \le b|\xi|, \quad \xi \in \mathbb{R}^n.$$

Moreover, we will assume that

$$H^2 \in C^3(\mathbb{R}^n \setminus \{0\}), \text{ and } \nabla^2_\xi H^2 \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}. \tag{2.3}$$

In all the paper we will denote by Ω a set of $\mathbb{R}^n$, $n \ge 2$ such that

$$\Omega \text{ is a bounded connected open set with } C^2 \text{ boundary.} \tag{2.4}$$

The hypothesis (2.3) on H assures that the operator Δ_H is elliptic, hence there exists a positive constant γ such that

$$\gamma|\xi|^2 \le \sum_{i,j=1}^n \frac{\partial}{\partial \xi_j}(H(\eta)H_{\xi_i}(\eta))\xi_i\xi_j, \tag{2.5}$$

for any $\eta \in \mathbb{R}^n \setminus \{0\}$ and for any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

We will consider as solutions of equation (1.1) the strong solutions, namely functions $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ such that the equality in (1.1) holds almost everywhere in Ω.

In this context, an important role is played by the polar function of H, namely the function H^o defined as

$$x \in \mathbb{R}^n \mapsto H^o(x) = \sup_{\xi \neq 0} \frac{\xi \cdot x}{H(\xi)}.$$

It is not difficult to verify that H^o is a convex, 1-homogeneous function that satisfies (2.2) (with different constants). Moreover,

$$H(\xi) = (H^o)^o(\xi) = \sup_{x \neq 0} \frac{\xi \cdot x}{H^o(x)}.$$

The assumption (2.3) on H^2 implies that $\{\xi \in \mathbb{R}^n : H(\xi) < 1\}$ is strongly convex, in the sense that it is a C^2 set and all the principal curvatures are strictly positive functions on $\{\xi : H(\xi) = 1\}$. This ensures that $H^o \in C^2(\mathbb{R}^n \setminus \{0\})$ (see [**Sch**] for the details).

The following well-known properties hold true:

$$\nabla_\xi H(\xi) \cdot \xi = H(\xi), \qquad \xi \neq 0, \tag{2.6}$$

$$\nabla_\xi H(t\xi) = \operatorname{sign} t \cdot \nabla_\xi H(\xi), \qquad \xi \neq 0,\ t \neq 0, \tag{2.7}$$

$$\nabla^2_\xi H(t\xi) = \frac{1}{|t|} \nabla^2_\xi H(\xi), \qquad \xi \neq 0,\ t \neq 0, \tag{2.8}$$

$$H(\nabla_x H^o(x)) = 1, \qquad \forall\, x \neq 0, \tag{2.9}$$

$$H^o(x)\, \nabla_\xi H(\nabla_x H^o(x)) = x, \qquad \forall\, x \neq 0. \tag{2.10}$$

Analogous properties hold interchanging the roles of H and H^o.

The open set

$$\mathcal{W} = \{x \in \mathbb{R}^n : H^o(x) < 1\}$$

is the so-called Wulff shape centered at the origin. More generally, we set

$$\mathcal{W}_r(x_0) = r\mathcal{W} + x_0 = \{x \in \mathbb{R}^2 : H^o(x - x_0) < r\},$$

and $\mathcal{W}_r(0) = \mathcal{W}_r$.

2.2. Anisotropic distance function. Due to the nature of the problem, it seems to be natural to consider a suitable notion of distance to the boundary. The anisotropic distance of $x \in \overline{\Omega}$ to the boundary of $\partial\Omega$ is the function

$$d_H(x) = \inf_{y \in \partial\Omega} H^o(x - y), \qquad x \in \overline{\Omega}. \tag{2.11}$$

It is not difficult to prove that $d_H \in W^{1,\infty}(\Omega)$. Moreover, property (2.9) gives that $d_H(x)$ satisfies

$$H(\nabla d_H(x)) = 1 \quad \text{a.e. in } \Omega. \tag{2.12}$$

Furthermore, if $\partial\Omega$ is C^2, then d_H is C^2 in a suitable neighborhood of $\partial\Omega$ in $\overline{\Omega}$ (see [**CM**]).

Since $\partial\Omega$ is C^2, it is possible to extend d_H outside $\overline{\Omega}$ to a function which is still C^2 in a suitable neighborhood of $\partial\Omega$ in $\mathbb{R}^n$. Indeed, let

$$\tilde{d}_H(x) = \inf_{y\in\partial\Omega} H^o(x-y), \quad x \in \mathbb{R}^n \setminus \Omega,$$

and define the signed anisotropic distance function d_H^s as

$$d_H^s(x) = \begin{cases} d_H(x) & \text{if } x \in \overline{\Omega}, \\ -\tilde{d}_H(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{2.13}$$

The following result is proved in [**CM**].

Theorem 2.1. *Let Ω be as in* (2.4)*. Then there exists $\mu > 0$ such that d_H^s is $C^2(A_\mu)$, with $A_\mu = \{x \in \mathbb{R}^n : -\mu < d_H^s(x) < \mu\}$.*

2.3. Main results. The first result concerns the case when f blows up at the boundary at most as $d_H(x)^{-q'}$, with $q' = q/(q-1)$.

Theorem 2.2. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ bounded from below and such that*

$$\lim_{d_H(x)\to 0} f(x)\, d_H(x)^{q'} = C_1, \quad \textit{for some } 0 \le C_1 < +\infty. \tag{2.14}$$

Then there exists a unique solution $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ of (1.1) *such that u blows up at $\partial\Omega$. Moreover, any subsolution $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$ of* (1.1) *is such that $u \ge v$ in Ω. Finally, if C_0 is the unique positive solution of $\left(\frac{2-q}{q-1}\right)^q C_0^q - \frac{2-q}{(q-1)^2} C_0 - C_1 = 0$ if $q < 2$, $C_0^2 - C_0 - C_1 = 0$ if $q = 2$, then*

$$u(x) \sim \begin{cases} \dfrac{C_0}{d_H(x)^{\frac{2-q}{q-1}}} & \textit{if } q < 2, \\ C_0 \log \dfrac{1}{d_H(x)} & \textit{if } q = 2, \end{cases} \tag{2.15}$$

as $d_H(x) \to 0$.

The second main result we are able to prove is the case in which f blows up very fast on $\partial\Omega$.

Theorem 2.3. *Let us suppose that $f \in L^\infty_{\rm loc}(\Omega)$ is bounded from below and satisfies*

$$\liminf_{d_H \to 0} f(x)\, d_H^\beta(x) > 0, \quad \text{for some } \beta \geq q'. \tag{2.16}$$

Then, any solution $u \in W^{2,2}_{\rm loc}(\Omega)$ of (1.1) bounded from below blows up at $\partial\Omega$. Moreover there exists a maximum solution of (1.1) in $W^{2,2}_{\rm loc}(\Omega)$ and, among all the solutions bounded from below in Ω, there exists a minimum one which is the increasing limit of sequences of subsolutions of (1.1).

If in addition there exists $C_1 > 0$ such that

$$f(x) \sim \frac{C_1}{d_H^\beta(x)}, \quad \text{for some } \beta > q', \tag{2.17}$$

then the blow up solution u is unique and, as $d_H(x) \to 0$,

$$u(x) \sim \frac{C_0}{d_H(x)^{\frac{\beta}{q}-1}},$$

with $C_0 = (\alpha^{-1} C_1)^{1/q}$.

Finally, we prove what happens when $\lambda \to 0^+$. We will denote by u_λ a blow up solution of (1.1), and $v_\lambda = u_\lambda - u_\lambda(x_0)$, where x_0 is any fixed point chosen in Ω.

Theorem 2.4. *Let $1 < q \leq 2$, and suppose that $f \in W^{1,\infty}_{\rm loc}(\Omega)$ is bounded from below and such that, as $d_H(x) \to 0$,*

$$f(x) = o\left(\frac{1}{d_H(x)^{q'}}\right). \tag{2.18}$$

Denote by u_λ the unique solution of (1.1) in $W^{2,2}_{\rm loc}(\Omega)$ such that u_λ blows up at $\partial\Omega$. Then, ∇u_λ and λu_λ are bounded in $L^\infty_{\rm loc}(\Omega)$ and $\lambda u_\lambda \to u_0 \in \mathbb{R}$, $v_\lambda \to v \in W^{2,2}_{\rm loc}(\Omega)$ as $\lambda \to 0^+$, where the convergence is uniform on compact sets of Ω. Moreover, v verifies (2.15) and it is a solution of the ergodic equation

$$-\Delta_H v + H(\nabla v)^q + u_0 = f \quad \text{in } \Omega. \tag{2.19}$$

In addition, if $\tilde u_0$ is such that the equation $-\Delta_H w + H(\nabla w)^q + \tilde u_0 = f$ admits a blow-up solution in $W^{2,2}_{\rm loc}(\Omega)$, then necessarily $\tilde u_0 = u_0$.

We will refer to the unique constant u_0 such that (2.19) admits a blow-up solution as the ergodic constant relative to (2.19).

Remark 2.1. We observe that the ergodic constant u_0, in the case $q = 2$, is related to an eigenvalue problem. Indeed, if v is a solution of the ergodic problem, performing the change of variable $w = e^{-v}$ and using the properties of H we have that w satisfies

$$\begin{cases} -\Delta_H w + f(x)\, w = u_0\, w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ w > 0 & \text{in } \Omega. \end{cases} \tag{2.20}$$

This observation will be useful in the proof of the uniqueness, up to an additive constant, of the blow-up solutions of (2.19) (Theorem 2.5 below). As a matter of fact, u_0 is the smallest eigenvalue of (2.20). We refer to the proof of Theorem 2.5 for the details.

When $q \in]1, 2[$, due to the nonlinearity of the principal part of the operator, and the fact that problem (2.19) is non-variational, the uniqueness up to an additive constant of the solution of (2.19) does not seem to be easy to prove.

Theorem 2.5. *If $q = 2$, under the hypotheses of Theorem 2.4, and assuming also that $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ satisfies*

$$|\nabla f(x)| \le \frac{C_1}{d_H^3(x)} \tag{2.21}$$

for some $C_1 \ge 0$, if v and $\tilde{v}$ are blow-up solutions in $W^{2,2}_{\mathrm{loc}}(\Omega)$ of (2.19), then $\tilde{v} = v + C$, for some constant $C \in \mathbb{R}$.

3. Gradient bounds

In this section we prove a local gradient bound for the solutions of

$$-\Delta_H u + H(\nabla u)^q + \lambda u = f, \quad u \in W^{2,2}_{\mathrm{loc}}(\Omega). \tag{3.1}$$

Such estimate is crucial in order to prove Theorem 2.4 on the ergodic problem. The method we will use relies in a local version, contained in [**LL**] (see also [**Lio1, Lio2**]), of the classical Bernstein technique (see[**GT, LU**]).

Theorem 3.1. *Let Ω be a bounded open set, and suppose that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ solves (3.1). For any $\delta > 0$, let us consider the set $\Omega_\delta = \{x \in \Omega : d_H(x) > \delta\}$. If $f \in C^{1,\vartheta}_{\mathrm{loc}}(\Omega)$, for some $\vartheta \in]0, 1[$, then*

$$|\nabla u| \le C_\delta \quad \text{for any } x \in \Omega_\delta, \tag{3.2}$$

where the constant C_δ depends on $\|\nabla f\|_\infty$, $\sup(f - \lambda u)$, δ, and q.

Actually, we will prove in Section 4 that the estimate (3.2) holds also under different assumptions on f (see Remark 4.1).

In order to prove Theorem 3.1, we will need the following matrix trace inequality.

Lemma 3.1. *Let A, B symmetric $n \times n$ matrices, with $A \geq 0$. The following inequality holds:*

$$[\mathrm{Tr}(AB)]^2 \leq \mathrm{Tr}(AB^2)\,\mathrm{Tr}(A), \tag{3.3}$$

where $\mathrm{Tr}(A)$ *is the trace of A.*

Proof: By diagonalizing, we can write $A = PDP^T$, with P orthogonal matrix, $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, and $0 \leq \lambda_i$ i-th eigenvalue of A. Moreover, let $D^{\frac{1}{2}} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$.

For two given $n \times n$ matrices X, Y, the Cauchy–Schwarz inequality can be written as

$$[\mathrm{Tr}(XY)]^2 \leq \mathrm{Tr}(X^TX)\,\mathrm{Tr}(Y^TY);$$

hence

$$\begin{aligned}
[\mathrm{Tr}(AB)]^2 &= \left[\mathrm{Tr}\left(D^{\frac{1}{2}}D^{\frac{1}{2}}P^TBP\right)\right]^2 \\
&\leq \mathrm{Tr}(D)\,\mathrm{Tr}\left[\left(D^{\frac{1}{2}}P^TBP\right)^T D^{\frac{1}{2}}P^TBP\right] \\
&= \mathrm{Tr}(A)\,\mathrm{Tr}(AB^2).
\end{aligned}$$

□

Proof of Theorem 3.1: If u is constant, there is nothing to prove. Then, let us assume that u is not constant in Ω.

The regularity assumptions on H imply that $u \in C^3(\{\nabla u \neq 0\}) \cap C^{1,\gamma}(\Omega)$ (see [**To, CFV, CS, LU**]).

For the sake of simplicity, we put

$$a^{ij}(\xi) = \frac{1}{2}\{[H(\xi)]^2\}_{\xi_i\xi_j}.$$

Hence the equation (3.1) can be written as (here and in the following the Einstein summation convention is understood)

$$-a^{ij}(\nabla u)\,u_{x_ix_j} + [H(\nabla u)]^q + \lambda u = f.$$

If $\nabla u \neq 0$, we can derive the equation with respect to x_k, obtaining that

$$-a^{ij}u_{x_ix_jx_k} - a^{ij}_{\xi_m}\,u_{x_mx_k}\,u_{x_ix_j} + q\,H^{q-1}H_{\xi_m}u_{x_mx_k} + \lambda u_{x_k} = f_{x_k}.$$

Let us consider $\varphi \in \mathcal{D}(\Omega)$ such that $0 \leq \varphi \leq 1$ in Ω, $\varphi \equiv 1$ on Ω_δ, and

$$|\Delta\varphi| \leq C\,\varphi^\theta, \quad |\nabla\varphi|^2 \leq C\,\varphi^{1+\theta} \quad \text{in } \Omega, \tag{3.4}$$

for some $\theta \in]0,1[$ that will be determined later, and for some constant $C = C(\delta,\theta)$. For example, a function φ with the above properties can be determined by taking a cut-off function $\psi \in \mathcal{D}(\Omega)$, with $\psi \equiv 1$ in Ω_δ, $0 \le \psi \le 1$ in Ω, and then choosing $\varphi = \psi^{\frac{2}{1-\theta}}$.

In the following, C will denote a constant whose value may change from line to line.

Multiplying by $\varphi\, u_{x_k}$ and summing we get

$$
\begin{aligned}
(3.5)\quad -a^{ij} u_{x_i x_j x_k} \varphi\, u_{x_k} - a^{ij}_{\xi_m} u_{x_m x_k} u_{x_i x_j} \varphi\, u_{x_k} + q\, H^{q-1} H_{\xi_m} u_{x_m x_k} \varphi u_{x_k} \\
+ \lambda \varphi\, u_{x_k} u_{x_k} = \varphi\, f_{x_k} u_{x_k}.
\end{aligned}
$$

Denoting $v = |\nabla u|^2$, equation (3.5) can be rewritten as

$$
\begin{aligned}
&- a^{ij}\, v_{x_i x_j}\, \varphi + 2\varphi a^{ij} u_{x_i x_k} u_{x_j x_k} - a^{ij}_{\xi_m} u_{x_i x_j} \varphi\, v_{x_m} \\
&\qquad + q\, H^{q-1} \nabla_\xi H \cdot \nabla v\, \varphi + 2\lambda \varphi\, v = 2\varphi\, \nabla f \cdot \nabla u,
\end{aligned}
$$

or

$$
\begin{aligned}
&-a^{ij} (\varphi\, v)_{x_i x_j} + 2\varphi a^{ij} u_{x_i x_k} u_{x_j x_k} - a^{ij}_{\xi_m} u_{x_i x_j} (\varphi\, v)_{x_m} \\
&\qquad + q\, H^{q-1} \nabla_\xi H \cdot \nabla(v\, \varphi) + 2\lambda\varphi\, v + \frac{2}{\varphi}(a^{ij} \varphi_{x_i} (\varphi v)_{x_j}) \\
&\qquad = 2\varphi\, \nabla f \cdot \nabla u + \left[-a^{ij}_{\xi_m} u_{x_i x_j}\, \varphi_{x_m} + q H^{q-1} \nabla_\xi H \cdot \nabla \varphi \right] v \\
&\qquad\quad - a^{ij} \varphi_{x_i x_j} v + 2\frac{v}{\varphi}(a^{ij} \varphi_{x_i} \varphi_{x_j}).
\end{aligned}
$$

Let x_0 be a maximum point for φv in Ω. Obviously, $\nabla u(x_0) \neq 0$, otherwise $\varphi v \equiv 0$ in Ω, which contradicts the fact that u is not constant. For the same reason, we can assume that $x_0 \in \operatorname{Supp} \varphi$. Then by the maximum principle we get the following inequality in x_0:

$$
\begin{aligned}
2\varphi a^{ij} u_{x_i x_k} u_{x_j x_k} + 2\lambda \varphi\, v \le\ & 2\varphi\, \nabla f \cdot \nabla u \\
& + \left[-a^{ij}_{\xi_m} u_{x_i x_j}\, \varphi_{x_m} + q H^{q-1} \nabla_\xi H \cdot \nabla \varphi \right] v \qquad (3.6) \\
& - a^{ij} \varphi_{x_i x_j} v + 2\frac{v}{\varphi}(a^{ij} \varphi_{x_i} \varphi_{x_j}).
\end{aligned}
$$

Now, being $H(\xi)$ 1-homogeneous, and recalling that $a^{ij} = H\, H_{\xi_i \xi_j} + H_{\xi_i} H_{\xi_j}$, it follows that $a^{ij}_{\xi_m}$ are homogeneous of degree -1, and then

$$
|\nabla u| \left| a^{ij}_{\xi_m}(\nabla u) \right| = \left| a^{ij}_{\xi_m} \left(\frac{\nabla u}{|\nabla u|} \right) \right| \le C.
$$

Hence, using the above inequality, the boundedness of a^{ij}, Young inequality, and the equation, we get

$$\begin{aligned}\left|a^{ij}_{\xi_m}(\nabla u)u_{x_ix_j}\varphi_{x_m}v\right| &= |\nabla u|\left|a^{ij}_{\xi_m}\left(\frac{\nabla u}{|\nabla u|}\right)u_{x_ix_j}\varphi_{x_m}\right| \\ &\le C|\nabla u||\nabla\varphi|(a^{ij}(\nabla u)\,u_{x_ix_j}) \\ &\le \varepsilon\varphi(a^{ij}(\nabla u)\,u_{x_ix_j})^2 + C(\varepsilon)\frac{|\nabla\varphi|^2}{\varphi}|\nabla u|^2 \\ &= \varepsilon\,\varphi(H(\nabla u)^q + \lambda u - f)^2 + C(\varepsilon)\frac{|\nabla\varphi|^2}{\varphi}|\nabla u|^2.\end{aligned}$$

On the other hand, from (3.3) and the equation it follows that

$$a^{ij}(\nabla u)\,u_{x_ix_k}u_{x_jx_k} \ge \frac{(a^{ij}(\nabla u)\,u_{x_ix_j})^2}{\operatorname{Tr}[a^{ij}]} \ge C(H(\nabla u)^q + \lambda u - f)^2.$$

Hence, for ε sufficiently small, recalling (3.6) and that $\lambda u - f$ is bounded from below, we have

$$\begin{aligned}[(H(\nabla u)^q - C_1)^+]^2\varphi \le C\Bigg\{&\frac{|\nabla\varphi|^2}{\varphi}|\nabla u|^2 + 2\varphi|\nabla f||\nabla u| \\ &+ q|H(\nabla u)|^{q-1}|\nabla_\xi H(\nabla u)||\nabla\varphi|v \\ &- a^{ij}\ \varphi_{x_ix_j}v + 2\frac{v}{\varphi}(a^{ij}\varphi_{x_i}\varphi_{x_j})\Bigg\}.\end{aligned}$$

Now using conditions (3.4), (2.2), the boundedness of a^{ij}, and the 0-homogeneity of $\nabla_\xi H$, we get

$$[(H(\nabla u)^q - C_1)^+]^2\varphi \le C\left(\varphi v^{\frac{1}{2}} + \varphi^\theta v^{\frac{q+1}{2}} + \varphi^\theta v\right)$$

that means

$$\varphi v^q \le C\left(1 + \varphi v^{\frac{1}{2}} + \varphi^\theta v^{\frac{q+1}{2}} + \varphi^\theta v\right).$$

Choosing $\theta \ge \frac{3-q}{2}$ and recalling that $1 < q \le 2$ and $0 \le \varphi \le 1$, the above inequality implies that

$$\begin{aligned}X^q &\le C(\varphi^{q-1} + \varphi^{q-\frac{1}{2}}X^{\frac{1}{2}} + \varphi^{\theta-\frac{3-q}{2}}X^{\frac{q+1}{2}} + \varphi^{\theta-2+q}X) \\ &\le C(1 + X^{\frac{1}{2}} + X^{\frac{q+1}{2}} + X),\end{aligned}$$

where $X = \varphi v$. Then necessarily

$$\max_\Omega \varphi v = \varphi v(x_0) \le C.$$

Being $\varphi \equiv 1$ in Ω_δ, we get that

$$|\nabla u| = v^{1/2} \le C_\delta \quad \text{in } \Omega_\delta,$$

and the proof is complete. □

Actually, we can prove a more precise estimate of the gradient of the solutions when we specify the behavior of the datum f near the boundary.

Theorem 3.2. *Let Ω be a bounded open set, and suppose that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ solves (3.1). Supposing that $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ satisfies*

$$|f(x)| \le \frac{C_1}{d_H^\beta(x)}, \quad |\nabla f(x)| \le \frac{C_1}{d_H^{\beta+1}(x)} \tag{3.7}$$

for some $\beta \le q'$, $C_1 \ge 0$, and

$$\lambda u \ge -C_2$$

for some $C_2 \ge 0$. Then

$$|\nabla u| \le \frac{C_3}{d_H^{\frac{1}{q-1}}(x)} \quad \text{in } \Omega,$$

where C_3 only depends on C_1, C_2, β, and the diameter of Ω.

Proof: Let $x_0 \in \Omega$, define $r = \frac{1}{2}\, d_H(x_0)$ and consider $v(x) = r^\alpha u(x_0 + rx)$, $\alpha = \frac{2-q}{q-1}$, for $x \in \mathcal{W}_1(0) = \mathcal{W}$. The function $v \in W^{2,2}_{\mathrm{loc}}(\mathcal{W})$ solves

$$-\Delta_H v + H(\nabla v)^q + \lambda r^2 v = r^{q'} f(x_0 + rx) \quad \text{in } \mathcal{W}.$$

The hypothesis (3.7) on f gives that

$$|r^{q'} f(x_0 + rx)| \le C_1 2^\beta r^{q'-\beta} \le C_1 2^\beta [\mathrm{diam}_H(\Omega)]^{q'-\beta} = C_4,$$

where $\mathrm{diam}_H(\Omega) = \sup\limits_{x,y\in\Omega} H^o(x-y)$, and, similarly,

$$|r^{q'} \nabla_x f(x_0 + rx)| \le C_1 2^\beta r^{q'-\beta} \le C_4.$$

Now, using the estimate (3.2), we have

$$|\nabla v(0)| = |\nabla u(x_0)| r^{\frac{1}{q-1}} \le C_3,$$

where C_3 depends on C_4. □

4. Proof of the main results

Proof of Theorem 2.2: We split the proof considering first the case of f bounded, then we study the general case, with $f \in L^\infty_{\mathrm{loc}}(\Omega)$ such that (2.14) holds.

Case 1: $f \in L^\infty(\Omega)$. We look for solutions which blow up approaching the boundary. To this aim, we consider functions of the type $u(x) = C_0\, d_H(x)^{-\alpha}$, with $C_0 > 0$ and $\alpha > 0$. Recall that the anisotropic distance function is $C^2(\Gamma)$, where $\Gamma = \{x \in \overline{\Omega} : d_H(x) \le \delta_0\}$, with $\delta_0 > 0$ sufficiently small, is a tubular neighborhood of $\partial\Omega$. If we substitute such functions in (1.1), by (2.1) and property (2.12) we get that

$$H(\nabla d_H^{-\alpha}) = \alpha C_0\, d_H^{-\alpha-1}.$$

Moreover, if $\overline{y}_x$ is the unique minimum point of (2.11), that is $d_H(x) = H^o(x - \overline{y}_x)$, then

$$\nabla d_H(x) = \nabla_x H^o(x - \overline{y}_x)$$

(see [**CM**, Proposition 3.3]), and then by (2.10) we have

$$\nabla_\xi H(\nabla d_H(x)) = \nabla_\xi H(\nabla_x H^o(x - \overline{y}_x)) = \frac{x - \overline{y}_x}{H^o(x - \overline{y}_x)}.$$

Moreover, using (2.7), we finally have

$$\nabla_\xi H(\nabla d_H^{-\alpha}) = -\nabla_\xi H(\nabla d_H) = -\frac{x - \overline{y}_x}{H^o(x - \overline{y}_x)}. \tag{4.1}$$

Hence, computing the anisotropic Laplacian and using (4.1) and (2.6) it follows that

$$\begin{aligned}
\Delta_H(C_0\, d_H^{-\alpha}) &= -C_0\,\alpha\, \mathrm{div}\left[d_H(x)^{-\alpha-1}\nabla_\xi H(\nabla d_H(x))\right] \\
&= C_0\alpha(\alpha+1)\, d_H(x)^{-\alpha-2}\, \frac{\nabla_x H^o(x - \overline{y}_x)\cdot(x - \overline{y}_x)}{H^o(x - \overline{y}_x)} \\
&\quad - C_0\,\alpha\, d_H(x)^{-\alpha-1} \sum_{i,j=1}^n H_{\xi_i\xi_j}(\nabla d_H(x))\partial_{x_i x_j} d_H(x) \\
&= C_0\alpha(\alpha+1)\, d_H(x)^{-\alpha-2} - K(x)\, d_H(x)^{-\alpha-1},
\end{aligned}$$

where

$$K(x) = C_0\,\alpha \sum_{i,j=1}^{n} H_{\xi_i\xi_j}(\nabla d_H(x))\partial_{x_ix_j}\, d_H(x)$$

is bounded in Γ, being $\nabla^2_\xi H$ bounded on $\{\xi : H(\xi) = 1\}$, and $d_H \in C^2(\Gamma)$. Hence

$$\begin{aligned}(4.2)\qquad -\Delta_H u + H(\nabla u)^q + \lambda u - f &= -C_0\alpha(\alpha+1)\, d_H^{-\alpha-2} \\ &\quad + K(x)\, d_H(x)^{-\alpha-1} + C_0^q\alpha^q\, d_H^{-(\alpha+1)q} + \lambda C_0\, d_H^{-\alpha} - f.\end{aligned}$$

If f is in L^∞, the leading term in (4.2) is

$$-C_0\alpha(\alpha+1)\, d_H^{-\alpha-2} + C_0^q\alpha^q\, d_H^{-(\alpha+1)q}.$$

If $q < 2$ this leads to the choice of

$$(4.3)\qquad \alpha = \frac{2-q}{q-1}, \quad C_0 = \frac{1}{\alpha}(\alpha+1)^{\frac{1}{q-1}}.$$

For $q = 2$ we choose $u(x) = -C_0 \log d_H$ and $C_0 = 1$.

We construct, by means of the signed distance function $d_H^s(x)$, defined in (2.13), a suitable family of subsolutions and supersolutions of (1.1). To this aim, recalling that $d_H^s(x) \in C^2(A_\mu)$, where A_μ is given in Theorem 2.1, it is possible to construct a function $d(x)$ in $C^2(\mathbb{R}^n)$ such that

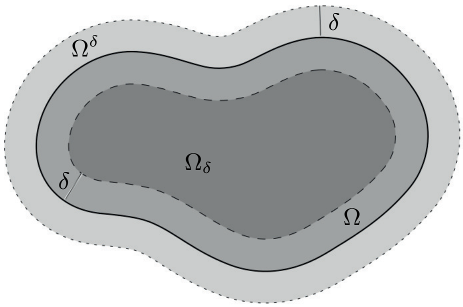
$$(4.4)\qquad \begin{cases} d(x) = d_H(x) & \text{if } x \in \overline{\Omega}, \text{ and } d_H(x) \le \delta_0, \\ d(x) \ge \delta_0 & \text{if } x \in \Omega, \text{ and } d_H(x) > \delta_0, \\ d(x) = -\tilde d_H(x) & \text{if } x \notin \overline{\Omega}, \text{ and } \tilde d_H(x) \le \delta_0, \\ d(x) \le -\delta_0 & \text{if } x \notin \Omega, \text{ and } \tilde d_H(x) > \delta_0, \end{cases}$$

where δ_0 is a positive constant smaller than μ. Hence, if $q < 2$, for $\varepsilon \ge 0$ and δ such that $0 \le \delta \le \delta_0$, we define

$$(4.5)\qquad \begin{aligned} \underline{w}_{\varepsilon,\delta}(x) &= (C_0 - \varepsilon)(d(x) + \delta)^{-\alpha} - C_\varepsilon \qquad x \in \Omega^\delta, \\ \overline{w}_{\varepsilon,\delta}(x) &= (C_0 + \varepsilon)(d(x) - \delta)^{-\alpha} + C_\varepsilon \qquad x \in \Omega_\delta, \end{aligned}$$

where C_ε is a constant which will be chosed later, and

$$\begin{aligned} \Omega^\delta &:= \{x \in \mathbb{R}^n : d(x) \ge -\delta\} \supset \Omega, \\ \Omega_\delta &:= \{x \in \Omega : d(x) > \delta\} \subset \Omega. \end{aligned}$$



If $q = 2$, the functions $(d \pm \delta)^{-\alpha}$ in (4.5) have to be substituted with $-\log(d \pm \delta)$.

For suitable choices of C_ε, the functions in (4.5) are a supersolution and a subsolution of (1.1) in Ω^δ and Ω_δ, respectively (we may assume $f \equiv 0$ in $\Omega^\delta \setminus \Omega$). Indeed, for α and C_0 as in (4.3), we get

$$\begin{aligned}
&-\Delta_H \overline{w}_{\varepsilon,\delta} + H(\nabla \overline{w}_{\varepsilon,\delta})^q + \lambda \overline{w}_{\varepsilon,\delta} - f \\
&\quad = -\alpha(\alpha+1)(C_0+\varepsilon)(d-\delta)^{-\alpha-2} H(\nabla d)^2 + \alpha(C_0+\varepsilon)(d-\delta)^{-\alpha-1}\Delta_H d \\
&\qquad + \alpha^q (C_0+\varepsilon)^q (d-\delta)^{-q(\alpha+1)} H(\nabla d)^q + \lambda(C_0+\varepsilon)(d-\delta)^{-\alpha} + \lambda C_\varepsilon - f \\
&\quad \geq \alpha(\alpha+1)(C_0+\varepsilon)(d-\delta)^{-\alpha-2}\left[\left(1+\frac{\varepsilon}{C_0}\right)^{q-1} H(\nabla d)^q - H(\nabla d)^2\right] \\
&\qquad + \lambda C_\varepsilon - \overline{C}(1 + (d-\delta)^{-\alpha-1}) \\
&\quad \geq \nu\varepsilon(d-\delta)^{-\alpha-2} + \lambda C_\varepsilon - C(1 + (d-\delta)^{-\alpha-1}),
\end{aligned}$$

for some $\nu > 0$ and $C > 0$. We stress that $\Delta_H d = \frac{1}{2}\operatorname{div}(\nabla_\xi(H^2)(\nabla d))$ is bounded in Ω_δ being $d \in C^2(\mathbb{R}^n)$ and $\nabla_\xi^2 H^2 \in L^\infty(\mathbb{R}^n)$.

By choosing C_ε sufficiently large, the last term in the above inequalities is nonnegative, and $\overline{w}_{\varepsilon,\delta}$ is a supersolution of (1.1) in Ω_δ. The same argument shows that $\underline{w}_{\varepsilon,\delta}$ is a subsolution of (1.1) in Ω^δ. Now, fixed $M > 0$, let us consider the approximating problem

$$\begin{cases} -\Delta_H u_M + H(\nabla u_M)^q + \lambda u_M = f & \text{in } \Omega, \\ u_M = \underline{w}_{\varepsilon,1/M} & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Observe that $w_{\varepsilon,\frac{1}{M}} = (C_0-\varepsilon)M^\alpha - C_\varepsilon =: C_{\varepsilon,M}$ on $\partial\Omega$. Then, by setting $v_M = u_M - (C_0-\varepsilon)M^\alpha - C_\varepsilon$, problem (4.6) can be rewritten as

$$\begin{cases} -\Delta_H v_M + H(\nabla v_M)^q + \lambda v_M = f - \lambda C_{\varepsilon,M} & \text{in } \Omega, \\ v_M = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

Problem (4.7) admits a subsolution and a supersolution in $L^\infty(\Omega)$ (it is sufficient to take two suitable constants). Under hypotheses (2.2) and (2.5), by [**BMP**, Theorem 2.1], we get that problem (4.7) admits a weak solution $v_M \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, namely v_M satifies

$$\int_\Omega [H(\nabla v_M)\nabla_\xi H(\nabla v_M)\cdot\nabla\varphi + H(\nabla v_M)^q\varphi + \lambda v_M\varphi]\,dx \\ = \int_\Omega (f-\lambda C_{\varepsilon,M})\,\varphi\,dx, \quad \forall\,\varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega).$$

Then also (4.6) admits a weak solution $u_M \in W^{1,2}(\Omega)\cap L^\infty(\Omega)$. Moreover, such solutions are in $W^{2,2}(\Omega)$ (see [**To**], and the remarks contained in [**CS, CFV**]), and in $C^{1,\vartheta}(\overline{\Omega})$ (see [**LU, Lie**]). Now we apply the comparison principle contained in [**BBGK**, Theorem 3.1] (see also [**BM**, Theorem 3.1]). We stress that the hypothesis (22) in [**BBGK**] holds, because the function $H(\xi)\nabla_\xi H(\xi)$ is 1-homogeneous, and then

$$\nabla_\xi[H(\xi)\nabla_\xi H(\xi)]\xi - H(\xi)\nabla_\xi H(\xi) = 0.$$

Hence we have that, for $0 < M < N$ and for any $\varepsilon' > 0$,

$$\underline{w}_{\varepsilon,1/M} \le u_M \le u_N \le \overline{w}_{\varepsilon',0} \quad \text{in } \Omega. \tag{4.8}$$

Last inequality in (4.8) follows observing that $u_N < \overline{w}_{\varepsilon',0}$ near the boundary of Ω, being u_N finite on $\partial\Omega$, while $\overline{w}_{\varepsilon',0}|_{\partial\Omega} = +\infty$, and using the comparison principle. Hence (4.8) gives that the functions u_M, $M > 0$ are uniformly bounded in $L^\infty_{\mathrm{loc}}(\Omega)$. This estimate, since $f \in L^\infty_{\mathrm{loc}}(\Omega)$, allows to apply [**To**, Theorem 1] in order to obtain a $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ estimate. Actually, in any compact set $\Omega' \Subset \Omega$, by [**To**, Theorem 1] we have

$$|\nabla u_M(x) - \nabla u_M(x')| \le C|x-x'|^\vartheta, \quad \forall\, x, x' \in \Omega',$$

where C is a constant which depends only on n, γ, Γ, Ω', ϑ, and on the L^∞ bound of u_M in Ω'. Then by Ascoli–Arzelà Theorem u_M, as $M \to +\infty$, converges locally uniformly to a function $\underline{u} \in C^1(\Omega)$. Moreover, $\underline{u}$ is a weak solution of (1.1) and, recalling (4.8),

$$\underline{w}_{\varepsilon,0} \le \underline{u} \le \overline{w}_{\varepsilon',0}, \quad \forall\,\varepsilon' > 0. \tag{4.9}$$

Using again [**To**], $\underline{u} \in W^{2,2}_{\mathrm{loc}}(\Omega) \cap C^{1,\vartheta}_{\mathrm{loc}}(\Omega)$. Then, by the chain rule for vector-valued functions contained in [**MM**], we have that $\underline{u}$ is a strong solution of (1.1).

As a matter of fact, $\underline{u} \geq \underline{w}_{\varepsilon',0}$, for any $\varepsilon' > 0$. By comparison principle, if $v \in W^{2,2}_{1\,oc}(\Omega)$ is another solution of (1.1) which blows up on the boundary, then $u_M \leq v$. Hence, $\underline{u}$ is the minimal blow up solution.

The next step consists in constructing a maximum blow-up solution of (1.1). To this end, we may argue as before to get the minimal solution $\underline{u}_\delta$ of (1.1) in Ω_δ which diverges on $\partial\Omega_\delta$. We have that

$$\underline{w}_{\varepsilon,\delta} \leq \underline{u}_\delta \leq \overline{w}_{\varepsilon,\delta}, \quad \forall\, \varepsilon > 0. \tag{4.10}$$

Moreover, if $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$ is any blow up solution of (1.1), being v bounded on $\partial\Omega_\delta$, we have that

$$v \leq \underline{u}_\delta. \tag{4.11}$$

Passing to the limit as $\delta \to 0$ in (4.10), using (4.9) and (4.11), reasoning as before we get a maximal blow-up solution $\overline{u} = \lim_{\delta\to 0} \underline{u}_\delta$ of (1.1) such that

$$\underline{w}_{\varepsilon,0} \leq \underline{u} \leq v \leq \overline{u} \leq \overline{w}_{\varepsilon,0} \tag{4.12}$$

for any $\varepsilon > 0$. As a matter of fact, we claim that

$$\underline{u} = \overline{u}.$$

Indeed, by (4.12) it follows that

$$\lim_{d(x)\to 0} \frac{\overline{u}(x)}{\underline{u}(x)} = 1.$$

Hence, being $\overline{u}(x)$ and $\underline{u}(x)$ divergent near the boundary, we get that for any $\theta \in]0,1[$ there exists a neighborhood of $\partial\Omega$, dependent on θ, in which

$$\underline{u}(x) > \theta\overline{u}(x) + (1-\theta)\frac{m}{\lambda} =: w_\theta(x),$$

with $m = \inf_\Omega f$. The function w_θ is a subsolution of (1.1), and by maximum principle $w_\theta \leq \underline{u}$ in all Ω. As $\theta \to 1$, we have that $\overline{u} \leq \underline{u}$ in Ω, and we get the claim.

We further emphasize that inequality (4.11) clearly holds also if $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$ is any subsolution of problem (1.1). Passing to the limit, we obtain $v \leq \underline{u}$.

Case 2: f unbounded. The proof runs analogously as in the previous case, except what concerns the existence of the minimum explosive solution. Indeed, if $f \sim C_1\, d_H(x)^{-q'}$ or $f = o(d_H^{-q'})$ near $\partial\Omega$, substituting

$u(x) = C_0\, d_H(x)^{-\alpha}$ in (1.1), with $\alpha = (2-q)/(q-1)$, we have that the leading term in (4.2), when x approaches the boundary, is

$$\left[\left(\frac{2-q}{q-1}\right)^q C_0^q - \frac{2-q}{(q-1)^2} C_0 - C_1\right] d_H^{-q'}(x).$$

Hence, as before we can construct a maximum explosive solution $\overline{u}$ of (1.1) such that

$$(C_0 - \varepsilon)\, d^{-\alpha} - C_\varepsilon \le \overline{u} \le (C_0 + \varepsilon)\, d^{-\alpha} + C_\varepsilon, \tag{4.13}$$

where d is the function defined in (4.4). As regards the existence of the minimum solution, differently from the bounded case we have that $\underline{w}_{\varepsilon,\delta}$ defined in (4.5) is a subsolution of (1.1) in Ω^δ, with f replaced by

$$f_\delta = \begin{cases} \min\left\{f, C_2 + C_3(d+\delta)^{-q'}\right\} & \text{in } \Omega, \\ C_2 + C_3(d+\delta)^{-q'} & \text{in } \Omega^\delta \setminus \Omega, \end{cases}$$

with C_2, C_3 positive constants such that $C_3 > C_1$, and $C_2 + C_3 d^{-q'} > f$ in Ω. Now, f_δ is bounded in Ω, and from the first case we get that there exists a unique explosive solution u_δ of (1.1) with f replaced by f_δ, and $u_\delta \ge \underline{w}_{\varepsilon,\delta}$. Hence, being $f \ge f_\delta$, the comparison principle gives that $\overline{u} \ge u_\delta$. Passing to the limit, we obtain a minimal solution $\underline{u}(x) = \lim_{\delta\to 0} u_\delta(x)$ of (1.1), with $\underline{u} \le \overline{u}$, that satisfies (4.13). Again, the uniqueness and the comparison with subsolutions follows as before. □

Remark 4.1. We observe that by taking a closer look to the proof of Theorem 2.2, we are able to conclude that the thesis of the Theorem 3.1 holds also if $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and (2.14) is satisfied. Indeed, by using the approximating problems

$$\begin{cases} -\Delta_H \tilde{u}_M + H(\nabla \tilde{u}_M)^q + \lambda \tilde{u}_M = f_M & \text{in } \Omega, \\ \tilde{u}_M = \underline{w}_{\varepsilon,1/M} & \text{on } \partial\Omega, \end{cases}$$

with f_M sequence of smooth functions such that $f_M \to f$ in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, the solutions $\tilde{u}_M$ are uniformly bounded in $L^\infty_{\mathrm{loc}}(\Omega)$ and converge, up to a subsequence, to the unique blow-up solution u of problem (3.1). Then applying the bound (3.2) in Ω_δ to $\tilde{u}_M$ and passing to the limit we get the same bound also for u.

Proof of Theorem 2.3: The main part of the proof relies in the following statement.

Claim. *If* (2.16) *holds, then any solution of* (1.1) *in* $W^{2,2}_{\mathrm{loc}}(\Omega)$*, which is bounded from below, blows up when* $d_H \to 0$.

Once we prove the claim, the thesis of the theorem follows by adapting the proof contained in [**LL**, Theorems III.2 and III.3] and the arguments used in Theorem 2.2 in order to construct a minimum and a maximum solution and, under the additional hypothesis (2.17), to prove that such solutions coincide.

In order to prove the claim, we may suppose, without loss of generality, that $u \geq 0$ in Ω and $f \geq \tilde{K}\, d_H^{-q'}$ for some positive constant $\tilde{K}$. Let x_0 be a point in Ω such that $d_H(x_0) = 2r$. Hence $\mathcal{W}_r(x_0) \Subset \Omega$, and from the equation we get that

$$\begin{cases} -\Delta_H u + H(\nabla u)^q + \lambda u \geq K r^{-q'} & \text{in } \mathcal{W}_r(x_0), \\ u \geq 0 & \text{on } \partial\mathcal{W}_r(x_0), \end{cases}$$

where $K = 3^{-q'}\tilde{K}$. This means that u is a supersolution of

$$\begin{cases} -\Delta_H \tilde{u}_r + H(\nabla \tilde{u}_r)^q + \lambda \tilde{u}_r = K r^{-q'} & \text{in } \mathcal{W}_r(x_0), \\ \tilde{u}_r = 0 & \text{on } \partial\mathcal{W}_r(x_0), \end{cases} \tag{4.14}$$

and, obviously, $w = 0$ is a subsolution of (4.14). Applying again [**BMP**, Theorem 2.1] and [**To**], there exists a strong solution $\tilde{u}_r \in W^{2,2}(\mathcal{W}_r(x_0)) \cap C^{1,\vartheta}(\overline{\mathcal{W}_r(x_0)})$ of (4.14). Hence, $u(x) \geq \tilde{u}_r(x) \geq 0$ in $\mathcal{W}_r(x_0)$. Defining $u_r(x) = r^\alpha \tilde{u}_r(rx + x_0)$, for $x \in \mathcal{W}_1(0) = \mathcal{W}$, with $\alpha = (2-q)/(q-1)$, it follows that u_r solves

$$\begin{cases} -\Delta_H u_r + H(\nabla u_r)^q + \lambda r^2 u_r = K & \text{in } \mathcal{W}, \\ u_r = 0 & \text{on } \partial\mathcal{W}. \end{cases} \tag{4.15}$$

For $k > 0$, multiplying the above equation by $(u_r - k)^+$ and integrating, by (2.6), (2.2), since $u_r \geq 0$, we easily get

$$a \int_{u_r > k} |\nabla u_r|^2\, dx \leq \int_{\mathcal{W}} H(\nabla u_r) \nabla_\xi H(\nabla u_r) \cdot \nabla (u_r - k)^+\, dx \leq K |\{u_r > k\}|,$$

and, for $h > k$,

$$|\{u_r > h\}| \leq C(h-k)^{-2^*} |\{u_r > k\}|^{2^*/2},$$

where C is a constant independent of r. Hence, the classical Stampacchia Lemma (see [**St**]) assures that u_r is uniformly bounded in $L^\infty(\mathcal{W})$. Moreover, by [**BBGK**] u_r is the unique bounded solution of (4.15), which is also radial with respect to H^o, due to the symmetry of the data. That is, $u_r(x) = U_r(H^o(x))$, $x \in \mathcal{W}$.

Reasoning as in Theorem 2.2 we get that $u_r \to u_0 \in W^{2,2}_{\mathrm{loc}}(\mathcal{W})$, where $u_0(x) = U_0(H^o(x))$, $x \in \mathcal{W}$ solves

$$\begin{cases} -\Delta_H u_0 + H(\nabla u_0)^q = K & \text{in } \mathcal{W}, \\ u_0 = 0 & \text{on } \partial\mathcal{W}. \end{cases}$$

As a matter of fact, U_0 solves the problem

$$\begin{cases} -U_0'' - \dfrac{n-1}{r} U_0' + |U_0'(r)|^q = K & \text{in } [0,1], \\ U_0(1) = 0,\ U_0'(0) = 0. \end{cases}$$

Hence, by the maximum principle $U_0(0) = u_0(0) > 0$. This implies that, for $q < 2$, $u(x)$ diverges as $d_H \to 0$. As regards the case $q = 2$, this method allows only to say that

$$\liminf_{d_H \to 0} u \geq u_0(0) = K_0.$$

As a matter of fact, arguing as in [**LL**], we have that for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that for $x \in \Omega$ with $d_H(x) < s_\varepsilon$, then $u(x) \geq K_0 - \varepsilon$. Now, putting $v = u - (K_0 - \varepsilon)$, and repeating exactly the above argument for v (at least for $2r < s_\varepsilon$), we get that

$$\liminf_{d_H \to 0} u \geq K_0 + K_0 - \varepsilon = 2K_0 - \varepsilon.$$

Letting ε go to zero, and iterating the argument, we get that u diverges as $d_H \to 0$ also if $q = 2$. □

Proof of Theorem 2.4: The argument of the proof of Theorem 2.2 allows to obtain that the solution u_λ of problem (1.1) satisfies, if $1 < q < 2$,

$$\frac{C_0 - \varepsilon}{d^\alpha} - \frac{C_\varepsilon}{\lambda} \leq u_\lambda \leq \frac{C_0 + \varepsilon}{d^\alpha} + \frac{C_\varepsilon}{\lambda}, \tag{4.16}$$

for all $\varepsilon > 0$, $\lambda \in]0,1]$, and for some $C_\varepsilon > 0$. In the case $q = 2$, the function $d^{-\alpha}$ has to be replaced with $|\log d|$. By (4.16), λu_λ is uniformly bounded from below and in $L^\infty_{\mathrm{loc}}(\Omega)$. Moreover using Theorem 3.1 and Remark 4.1 we get that also ∇u_λ is uniformly bounded in $L^\infty_{\mathrm{loc}}(\Omega)$. Then, $v_\lambda = u_\lambda(x) - u_\lambda(x_0)$, for some fixed $x_0 \in \Omega$, is uniformly bounded with respect to $\lambda \in]0,1]$ in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Hence, for any $\Omega' \Subset \Omega$, there exists a constant $C_{\Omega'}$ independent of λ such that

$$|u_\lambda(x) - u_\lambda(x_0)| \leq C_{\Omega'} |x - x_0|.$$

Passing to the limit we obtain, up to a subsequence, the convergence of $\lambda u_\lambda(x_0)$ to a constant u_0 and of λv_λ to 0. We finally prove that

v_λ converges to a blow-up solution of (2.19). First observe that v_λ satisfies the following equation in Ω:

$$-\Delta_H v_\lambda + H(\nabla v_\lambda)^q + \lambda v_\lambda + \lambda u_\lambda(x_0) = f. \tag{4.17}$$

Hence, using again the arguments of the proof of the previous results, we can pass to the limit in (4.17), obtaining that v_λ converges to a solution $v \in W^{2,2}_{\rm loc}(\Omega)$ of the problem (2.19).

Now we prove a lower bound for v. Let $z = \frac{C_1}{d^\alpha}$, with $C_1 \in]0, C_0[$ fixed. Then in a sufficiently small inner tubular neighborhood of $\partial\Omega$, namely $\Omega \setminus \Omega_{\delta_0}$, we have that

$$-\Delta_H z + H(\nabla z)^q + \lambda z \le f - \lambda u_\lambda(x_0).$$

On the other hand, v_λ is bounded from below in Ω_{δ_0}, namely there exists a constant $M \ge 0$ such that

$$v_\lambda \ge -M \quad \text{in } \Omega_{\delta_0}.$$

Adapting the methods used in Theorem 2.2 it is possible to obtain that

$$v_\lambda \ge -M + z = -M + \frac{C_1}{d^\alpha} \quad \text{in } \Omega. \tag{4.18}$$

Passing to the limit, also v satisfies (4.18).

Now we show that for any couple $(\tilde u_0, \tilde v)$ of problem (2.19), with $\tilde v$ such that blows up at the boundary, $\tilde v$ diverges as in (2.15). To this aim, it is possible to consider $\overline{w}_{\varepsilon,\delta}$ as in (4.5) which is supersolution of the ergodic equation (2.19) in $\Omega_\delta \setminus \Omega_{\delta_0}$, for some $0 < \delta < \delta_0 = \delta_0(\varepsilon)$. Hence, by the comparison principle, and letting δ go to zero, we can conclude that

$$-C \le \tilde v \le \overline{w}_{\varepsilon,0} + \max_{d=\delta_0(\varepsilon)} |\tilde v| = (C_0+\varepsilon)d^{-\alpha} + \max_{d=\delta_0(\varepsilon)} |\tilde v| \quad \text{in } \Omega. \tag{4.19}$$

Hence, $\tilde v$ is such that $-\Delta_H \tilde v + H(\nabla \tilde v)^q + \tilde v = g$, with $g = f - \tilde u_0 + \tilde v$. The bounds in (4.19) and the condition (2.18) assure that $g \in L^\infty_{\rm loc}(\Omega)$ and also satisfies (2.18). By Theorem 2.2 we get that $\tilde v$ satisfies (2.15).

Now we show that if $(\tilde u_0, \tilde v) \in \mathbb{R} \times W^{2,2}_{\rm loc}(\Omega)$ is a couple which solves (2.19) and $\tilde v$ blows up at the boundary, then $\tilde u_0 = u_0$ and $\tilde v = v + C$, for some constant $C \in \mathbb{R}$.

As regards the uniqueness of the ergodic constant u_0, the proof runs similarly as in [**LL**], supposing by contradiction that $u_0 < \tilde u_0$. Let us choose $\varepsilon > 0$, and $0 < \theta < 1$. First, observe that obviously v satisfies

$$-\Delta_H v + H(\nabla v)^q + \varepsilon v = f + \varepsilon v - u_0 \quad \text{a.e. in } \Omega. \tag{4.20}$$

On the other hand, we have from the 1-homogeneity of H that

$$-\Delta_H(\theta\tilde v) + H(\nabla(\theta\tilde v))^q + \varepsilon\theta\tilde v \le f + C(1-\theta) + \varepsilon\theta\tilde v - \theta\tilde u_0.$$

Moreover, since v and $\tilde{v}$ diverge as $d^{-\alpha}$ near to the boundary of Ω, then $\theta\tilde{v} \le C_\theta + v$. So, from the above inequality it follows that

$$\begin{aligned} -\Delta_H(\theta\tilde{v})+H(\nabla(\theta\tilde{v}))^q+\varepsilon\theta\tilde{v} &\le f+\varepsilon v - u_0+(u_0-\theta\tilde{u}_0)+\varepsilon C_\theta + C(1-\theta) \\ &\le f+\varepsilon v - u_0, \end{aligned}$$

where last inequality holds for θ sufficiently near to 1 and for $\varepsilon = \varepsilon(\theta)$ sufficiently small. Hence, $\theta\tilde{v}$ is a subsolution of (4.20). By Theorem 2.2, $\theta\tilde{v} \le v$; as $\theta \to 1$, $\tilde{v} \le v$. This is in contradiction with the fact that any function of the type $\tilde{v}+c_1$, with $c_1 \in \mathbb{R}$ solves the ergodic problem with the same constant $\tilde{u}_0$. $\square$

Proof of Theorem 2.5: The hypothesis $q = 2$ allows to perform a suitable change of variable. Let $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$ be a solution of the ergodic equation (2.19) with $v = \infty$ on $\partial\Omega$. Then the function

$$w = e^{-v}$$

belongs to $W^{1,\infty}_0(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$. Let us verify that $|\nabla w| \in L^\infty$. Due the condition (2.18), we have that $C_0 = 1$ in (2.15), and then $|v| \le \log(d_H^{-1})$. Moreover, using also (2.21) we can apply Theorem 3.2, obtaining that $|\nabla v|\, d_H$ is bounded. Hence

$$|\nabla w| = |\nabla v| e^{-v} \le C.$$

Now we observe that using the properties of H it holds that the function w is a $W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}(\Omega)$ solution of

$$\begin{cases} -\Delta_H w + f(x)\, w = u_0\, w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ w > 0 & \text{in } \Omega. \end{cases} \tag{4.21}$$

The ergodic constant u_0 is a critical point of the Rayleigh quotient

$$\mathcal{R}[\psi] = \frac{\int_\Omega H(\nabla\psi)^2\, dx + \int_\Omega f(x)\psi^2\, dx}{\int_\Omega \psi^2\, dx}.$$

As a matter of fact, we claim that u_0 is the minimum eigenvalue, namely

$$u_0 = \min_{\substack{\psi \in W^{1,2}_0(\Omega) \\ u \ne 0}} \mathcal{R}[\psi],$$

and u_0 is the only eigenvalue associated to a positive eigenfunction. The claim follows observing, first of all, that being f bounded from below, $f \ge -C$, the Rayleigh quotient $\mathcal{R}[\psi]$ satisfies

$$\mathcal{R}[\psi] \ge -C.$$

Then the existence of the minimum value of $\mathcal{R}[\psi]$ easily follows by using standard arguments of Calculus of Variations. Moreover the simplicity of u_0 and the fact that it is the unique eigenvalue associated to a positive eigenfunction follows by adapting the proof contained, for example, in [**DG2**] and [**KLP**]. Hence problem (4.21) admits, up to a multiplicative constant, a unique solution. This implies that if v_1 and v_2 solve (2.19), then v_1 and v_2 differ by a constant. □

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