Publ. Mat. **61** (2017), 51–82
DOI: 10.5565/PUBLMAT_61117_02

AUTOMORPHISM GROUPS OF SIMPLICIAL COMPLEXES OF INFINITE-TYPE SURFACES

Jesús Hernández Hernández and Ferrán Valdez

Abstract: Let S be an orientable surface of infinite genus with a finite number of boundary components. In this work we consider the curve complex $\mathcal{C}(S)$, the nonseparating curve complex $\mathcal{N}(S)$, and the Schmutz graph $\mathcal{G}(S)$ of S. When all topological ends of S carry genus, we show that all elements in the automorphism groups $\mathrm{Aut}(\mathcal{C}(S))$, $\mathrm{Aut}(\mathcal{N}(S))$, and $\mathrm{Aut}(\mathcal{G}(S))$ are *geometric*, *i.e.* these groups are naturally isomorphic to the *extended* mapping class group $\mathrm{MCG}^*(S)$ of the infinite surface S. Finally, we study rigidity phenomena within $\mathrm{Aut}(\mathcal{C}(S))$ and $\mathrm{Aut}(\mathcal{N}(S))$.

2010 Mathematics Subject Classification: 20F65.

Key words: Curve complex, infinite type surface.

1. Introduction

Let S be an orientable surface whose fundamental group is finitely generated. The extended mapping class group $\mathrm{MCG}^*(S)$ acts on the curve complex $\mathcal{C}(S)$ by simplicial automorphisms and hence we have a well defined map $\Psi_{\mathcal{C}(S)}\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{C}(S))$. The following is a foundational well-known result of Ivanov:

Theorem 1.1 ([**Iva1**]). *If the genus of S is at least 2, then the map $\Psi_{\mathcal{C}(S)}$ is an isomorphism.*

Indeed, this result is used for example to determine the full group of isometries of the Teichmüller space $\mathcal{T}_g$, $g \geq 2$ with respect to both the Teichmüller and Weil–Peterson metrics [**Roy**], [**MW**], or to establish the quasi-isometric rigidity of mapping class groups [**BKMM**].

The main purpose of this article is to extend results in the lines of Theorem 1.1 to the realm of infinite (topological) type surfaces. A surface is said to be of *infinite topological type* if its fundamental group is not finitely generated. Recall that any *orientable* surface is completely determined, up to homeomorphism, by its genus $g(S) \in \mathbb{N} \cup \{\infty\}$, number of boundary components and a nested pair of topological spaces $\mathrm{Ends}^*(S) \subset \mathrm{Ends}(S)$. Roughly speaking, $\mathrm{Ends}(S)$ is the space formed by all the *topological ends* of S and $\mathrm{Ends}^*(S)$ is formed by those ends that carry (infinite) genus. We will focus our attention on infinite genus surfaces S without planar ends, for which the boundary ∂S has finitely many connected components (possibly none). Sometimes will abbreviate this condition as $\mathrm{Ends}^*(S) = \mathrm{Ends}(S)$. Our first result shows that Ivanov's theorem remains valid for a large class of infinite-type surfaces.

Theorem 1.2. *Let S be an orientable infinite genus surface with finitely many boundary components and without planar ends. Then the natural map $\Psi_{\mathcal{C}(S)}\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{C}(S))$ is an isomorphism.*

As a matter of fact Ivanov made the following metaconjecture [**Iva2**]: *every object naturally associated to a surface S and having a sufficiently rich structure has* $\mathrm{MCG}^*(S)$ *as its groups of automorphisms. Moreover, this can be proved by a reduction to the theorem about the automorphisms of $\mathcal{C}(S)$.*

We check the first statement of this metaconjecture for the following simplicial objects:

1. The *nonseparating curve complex* $\mathcal{N}(S)$. This is the *abstract* simplicial subcomplex of $\mathcal{C}(S)$ formed by all nonseparating curves, that is, all the (isotopy classes of) essential curves α such that $S \setminus \alpha$ is connected. This was first introduced by Schmutz Schaller in [**Sch**].
2. The *Schmutz graph* $\mathcal{G}(S)$. Introduced by Schmutz Schaller in [**Sch**], this is the simplicial graph whose vertex set is the same as the vertex set of $\mathcal{N}(S)$, and two vertices span an edge whenever their geometric intersection number is 1. It is also known as *a modified complex of nonseparating curves* (see [**FM**]), for it can be thought as a 1-dimensional simplicial complex.

More precisely:

Theorem 1.3. *Let S be an orientable infinite genus surface with finitely many boundary components and without planar ends. Then the natural map $\Psi_X\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(X)$ is an isomorphism for $X = \mathcal{N}(S)$ or $\mathcal{G}(S)$.*

Recall that this result is valid when S is a finite type topological surface, see [**Sch**] and [**Irm**] for details. However, it is important to stress that, contrary to the finite topological case, our proof of this result *does not fall back on Theorem 1.2*, that is, we do not achieve it by a reduction to an analog of Ivanov's theorem. On the other hand, though the techniques that we use to prove both Theorem 1.2 and 1.3 rely heavily on the hypothesis that S has no planar ends, we suspect that these results remain valid for surfaces with arbitrarily many planar ends.

In addition to the study of the action of $\mathrm{MCG}^*(S)$ on abstract simplicial complexes, we study rigidity phenomena within the curve complex and the nonseparating curve complex.

Theorem 1.4. *Let S_1 and S_2 be orientable infinite genus surfaces with finitely many boundary components and without planar ends. If $\mathcal{C}(S_1)$ and $\mathcal{C}(S_2)$ are isomorphic, then S_1 is homeomorphic to S_2.*

As we will see in Subsection 4.2 this result is not valid if we allow the infinite genus surface S to have planar ends. In Subsection 4.2 we will also see that the tools used in the proof of this theorem work for nonseparating curves. Hence, we have the following:

Corollary 1.5. *Let S_1 and S_2 be orientable infinite genus surfaces with finitely many boundary components, without planar ends and $\phi : \mathcal{N}(S_1) \to \mathcal{N}(S_2)$ an isomorphism. Then S_1 is homeomorphic to S_2.*

Reader's guide. We address first the classification of all *orientable* surfaces and we define the simplicial complexes and the mapping class group studied in this text in the context of infinite-type surfaces in Section 2. We introduce in Section 3 the adjacency graph of a pants decomposition and show (Theorem 3.3) how this graph captures the topology of any infinite genus surface S without planar ends. The main results of this paper are proven in Section 4. More precisely:

1. The injectivity of the natural map $\Psi_{\mathcal{C}(S)}$ is proven in Subsection 4.1. When S is of finite type this fact follows from Alexander's method. The key argument is to show that, up to taking a finite power, every mapping class fixing every element in $\mathcal{C}(S)$ *fixes* an embedded graph defined by a filling multicurve. Since the "up to taking a finite power" part of the argument does not work for infinite-type surfaces, we get around by introducing a variant of Alexander's method adapted to a suitable exhaustion of S by compact subsurfaces.

2. Rigidity results, namely Theorem 1.4 and Corollary 1.5, are proven in Subsection 4.2. The key ingredient in this case is to capture the topology of the infinite-type surface using pants decompositions (Theorem 3.3).

3. Ivanov's original proof of $\Psi_{\mathcal{C}(S)}$'s surjectivity heavily relies on the fact that S has finite topological type, specially on the so called chain-connectedness property of the arc complex $B(S)$ (see Lemma 2 and its proof in [**Iva1**] for details). In Subsection 4.3 we get around these difficulties using the following strategy. First, following ideas of Schmutz Schaller [**Sch**], we prove that $\mathrm{Aut}(\mathcal{G}(S))$ and $\mathrm{Aut}(\mathcal{N}(S))$ are isomorphic (Theorem 4.8). Here the techniques are identical to the finite topological type case. Then we prove that the natural map $\Psi_{\mathcal{G}(S)}\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{G}(S))$ is surjective (Theorem 4.9). The key ingredient is to use Dehn–Thurston coordinates adapted to infinite-type surfaces without planar ends. We complete the proof of Theorem 1.2 using the surjectivity of $\Psi_{\mathcal{G}(S)}$ and Lemma 4.10. Finally, we use the techniques introduced in the proof of Lemma 4.10 and Theorem 4.8 to finish the proof of Theorem 1.3.

At the end in Section 5 we show that, contrary to the compact case, rigidity results like Theorem 1.4 and Corollary 1.5 cannot be extended to injective simplicial maps when S is an infinite genus surface.

We refer the reader to [**FK**], [**FN**], and [**Fuj**] for previous work on groups formed by mapping classes of infinite-type surfaces. We want to stress, however, that the cited authors focus their work on several subgroups of what we here call the mapping class group (*e.g.* those with asymptotic qualities for a specific surface or quasiconformal automorphisms of a Riemann surface) and on their action on the Teichmüller space. On the other hand, the *ray graph*[1] (which is an analog of the complex of curves) of the infinite surface obtained by removing the Cantor set from the plane, and the action of the corresponding mapping class group on this graph, have been studied recently by J. Bavard, see [**Bav**].

Acknowledgements. We want to thank Camilo Ramírez Maluendas for the question that lead to the creation of this article. We are grateful to Hamish Short and Javier Aramayona for carefully reading preliminary versions of this text. The first author would like to thank Daniel Juan Pineda for his support during the realization of this project. The second author was generously supported by LAISLA, CONACYT CB-2009-01 127991 and PAPIIT projects IN100115, IN103411, and IB100212 during the realization of this project. Finally, we want to thank the anonymous referee for helpful comments and suggestions.

[1]Introduced by D. Calegari in `https://lamington.wordpress.com/2014/10/24/mapping-class-groups-the-next-generation/`.

2. Preliminaries

2.1. Topological invariants for infinite-type surfaces. Let X be a locally compact, locally connected, connected Hausdorff space.

Definition 2.1 ([**Fre**]). Let $U_1 \supseteq U_2 \supseteq \cdots$ be an infinite sequence of non-empty connected open subsets of X such that for each $i \in \mathbb{N}$ the boundary ∂U_i is compact and $\cap_{i\in\mathbb{N}}\overline{U_i} = \emptyset$. Two such sequences $U_1 \supseteq U_2 \supseteq \cdots$ and $U_1' \supseteq U_2' \supseteq \cdots$ are said to be equivalent if for every $i \in \mathbb{N}$ there exist j, k such that $U_i \supseteq U_j'$ and $U_i' \supseteq U_k$. The corresponding equivalence class is called a *topological end* of X.

The set of ends $\mathrm{Ends}(X)$ of X can be endowed with a topology in the following way. For any set U in X whose boundary is compact, we define U^* to be the set of all ends $[U_1 \supseteq U_2 \supseteq \cdots]$ for which there is a representative such that $U_n \subset U$ for n sufficiently large. With respect to this topology, $\mathrm{Ends}(X)$ is a compact, closed, totally disconnected space without interior points (see for example [**Ray**, Theorem 1.5]).

Henceforth, unless is stated otherwise, *all surfaces under consideration are orientable.* The *genus* of a surface is the maximum of the genera of its compact subsurfaces. A surface is said to be *planar* if all of its compact subsurfaces are of genus zero. We define $\mathrm{Ends}^*(S) \subset \mathrm{Ends}(S)$ as the set of all ends which are not planar. As stated in the following theorem, any orientable surface is determined, up to homeomorphism, by its genus, boundary and space of ends. Henceforth all surfaces in this text are connected.

Theorem 2.2. *Let S and S' be two orientable surfaces of the same genus. Then S and S' are homeomorphic if and only if they have the same number of boundary components, and* $\mathrm{Ends}^*(S) \subset \mathrm{Ends}(S)$ *and* $\mathrm{Ends}^*(S') \subset \mathrm{Ends}(S')$ *are homeomorphic as nested topological spaces.*

The proof of this theorem for the case when S and S' have no boundary can be found in [**Ric**]. The case for surfaces with boundary was proven in [**PM**].

2.2. Complexes and graphs of curves. There are several curve complexes that one can associate to a surface of finite genus with finitely many boundary components and punctures. In this section we extend the definitions of these complexes to noncompact surfaces of infinite topological type and explore some of their basic properties.

Abusing language and notation, we will call *curve*, a topological embedding $S^1 \hookrightarrow S$, the isotopy class of this embedding and its image on S. A curve is said to be *essential* if it is neither homotopic to a point nor to a boundary component or to the boundary of a neighborhood of a puncture. Hereafter all curves are considered essential unless otherwise stated. Two curves are *disjoint* if they are distinct and their geometric intersection number is 0. Recall that if α and β are two isotopy classes of curves in S, their geometric intersection number is defined as $i(a, b) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}$.

Definition 2.3 (Multicurves)**.** A multicurve is either a set of just one curve, or a pairwise disjoint and locally finite set of curves of S. We allow multicurves to consist of an infinite set of curves. If M is a multicurve of S, the surface obtained by cutting S along pairwise disjoint representatives of the elements of M will be denoted by S_M.

Infinite *countable* multicurves arise in surfaces with infinitely generated fundamental group. Take for example the Loch Ness Monster, that is, a surface with infinite genus and one end. If S is a compact surface of genus g with n boundary components, the *complexity* of S, denoted by $\kappa(S)$, is equal to $3g - 3 + n$. This is the cardinality of a maximal multicurve in S.

Definition 2.4 (The Curve complex)**.** The Curve complex of S, $\mathcal{C}(S)$, is the *abstract* simplicial complex whose vertices are the isotopy classes of essential curves in S, and whose simplices are multicurves of finite cardinality. We denote the set of vertices of $\mathcal{C}(S)$ by $\mathcal{V}(\mathcal{C}(S))$.

Remark 2.5. We want to stress that $\mathcal{C}(S)$ is an abstract simplicial complex. Moreover, this simplicial complex is *flag*, that is, every complete subgraph on $r + 1$ vertices contained in the 1-skeleton is the 1-skeleton of an r-simplex. This property has two important consequences: first, a flag complex is completely determined by its 1-skeleton and second, $\mathrm{Aut}(\mathcal{C}(S))$, the group of *simplicial* automorphisms of $\mathcal{C}(S)$, and $\mathrm{Aut}(\mathcal{C}^1(S))$, the group of automorphism of $\mathcal{C}^1(S)$, the 1-skeleton of $\mathcal{C}(S)$, are isomorphic. For these reasons we will restrict our study of $\mathcal{C}(S)$ to $\mathcal{C}^1(S)$. We refer the reader to [**Koz**] and [**AS**] for details on abstract and flag complexes.

Recall that an essential curve is said to be *separating* if the surface obtained by cutting S along its image is disconnected. It is said to be nonseparating otherwise.

Definition 2.6 (The nonseparating curve complex)**.** The nonseparating curve complex of S, $\mathcal{N}(S)$, is the subcomplex of $\mathcal{C}(S)$ whose vertices are the isotopy classes of essential *nonseparating* curves in S. We denote the set of vertices of $\mathcal{N}(S)$ by $\mathcal{V}(\mathcal{N}(S))$.

Definition 2.7 (The Schmutz graph)**.** The Schmutz graph of S, $\mathcal{G}(S)$, is the simplicial graph whose vertices are the isotopy classes of essential nonseparating curves in S, and two vertices span an edge if the geometric intersection number of the corresponding isotopy classes of curves is 1.

Proposition 2.8. *Let S be a surface of infinite genus. Then $\mathcal{C}(S)$, $\mathcal{N}(S)$, and $\mathcal{G}(S)$ are connected. In particular $\mathcal{C}^1(S)$ and $\mathcal{N}^1(S)$ have diameter two while $\mathcal{G}(S)$ has diameter at most four.*

Proof: Given any two distinct curves α and β (either in $\mathcal{V}(\mathcal{C}(S))$ or in $\mathcal{V}(\mathcal{N}(S))$), we can always find a compact (finite genus) subsurface S' such that contains α and β. Hence we can take an essential nonseparating curve γ on S contained in $S \setminus S'$ and not isotopic to α and β. Therefore $\mathcal{C}^1(S)$ and $\mathcal{N}^1(S)$ are connected, $\mathrm{diam}(\mathcal{C}^1(S)) = \mathrm{diam}(\mathcal{N}^1(S)) = 2$.

If α and β are two distinct nonseparating curves, as in the paragraph above, we can always find a curve γ such that $i(\alpha,\gamma) = i(\gamma,\beta) = 0$; then we can always find curves δ_1 and δ_2 such that $i(\alpha,\delta_1) = i(\delta_1,\gamma) = i(\gamma,\delta_2) = i(\delta_2,\beta) = 1$. Hence $\mathcal{G}(S)$ is connected, $\mathrm{diam}(\mathcal{G}(S)) \leq 4$. □

Remark 2.9. Two as diameter for $\mathcal{C}^1(S)$ and $\mathcal{N}^1(S)$ is optimal, but four as diameter for $\mathcal{G}(S)$ is not necessarily optimal.

2.3. Mapping class groups. Through this article, we will be working with the mapping class group of a surface S. When S is compact, this group has different (equivalent) definitions, see for example [**FM**, §2.1]. In this paper we will be working with the following definition.

Definition 2.10 (Mapping class group)**.** Let S be a surface. We denote by $\mathrm{Homeo}^+(S,\partial S)$ the group of orientation-preserving homeomorphisms of S that restrict to the identity on the boundary, and by $\mathrm{Homeo}(S)$ the group of *all* homeomorphisms of S. The mapping class group of S is the group $\mathrm{Homeo}^+(S)/{\sim}$, where $\sim$ represents the isotopy relation relative to the boundary. We denote it by $\mathrm{MCG}(S)$. The extended mapping class group of S is the group $\mathrm{MCG}^*(S) := \mathrm{Homeo}(S)/{\sim}$, where $\sim$ represents the isotopy relation but in this case isotopies *are not relative to the boundary.*

The group $\mathrm{MCG}^*(S)$ is incredibly big. As evidence for this we have the following lemma and corollaries.

Lemma 2.11. *Let S be a surface, of either infinite or finite type, and F a subsurface of S such that $S \setminus F$ has genus at least* 1 *and the boundary components of F are either boundary components of S or essential curves of S. Then there exists a subgroup of* $\mathrm{MCG}^*(S)$ *isomorphic to* $\mathrm{MCG}(F)$, *with infinite index in* $\mathrm{MCG}^*(S)$.

Proof: The subgroup of $\mathrm{MCG}^*(S)$ formed by those orientation-preserving elements $[h] \in \mathrm{MCG}^*(S)$ that have a representative h with support on F, is isomorphic to $\mathrm{MCG}(F)$. This subgroup will have index greater or equal to the number of different elements in $\mathrm{MCG}^*(S)$ that have its support in the interior of the complement of F, thus it will have infinite index. □

Corollary 2.12. *Let S be an infinite genus surface and $S_{g,n}$ be a compact surface of genus g and n boundary components. Then the homomorphism $\mathrm{MCG}^*(S_{g,n}) \to \mathrm{MCG}^*(S)$ induced by the inclusion $S_{g,n} \hookrightarrow S$ is injective.*

Corollary 2.13. *Let S be an infinite genus surface and $\{(g_i, n_i)\}_{i\in\mathbb{N}} \subset (\mathbb{N} \times \mathbb{Z}^+) \setminus \{(0,1)\}$ a sequence. Then $\mathrm{MCG}^*(S)$ contains a subgroup isomorphic to $\prod_{i\in\mathbb{N}} \mathrm{MCG}(S_{g_i,n_i})$.*

3. Ends of adjacency graphs and surfaces

In this section we prove that, under the hypotheses $\mathrm{Ends}(S)=\mathrm{Ends}^*(S)$, one can determine topologically $\mathrm{Ends}(S)$ using the adjacency graph of a pants decomposition of S.

Recall that a genus 0 surface with three boundary components is usually called a *pair of pants* or simply *pants*.

Definition 3.1 (Pants decomposition and the adjacency graph)**.** A *pants decomposition* is a maximal multicurve P, with respect to set-inclusion. We say $\alpha, \beta \in P$ are adjacent with respect to P if they bound the same pair of pants in $S_P = S \setminus P$. The adjacency graph of P, $\mathcal{A}(P)$, is the simplicial graph whose vertex set is P and two vertices span an edge if and only if they are adjacent with respect to P. We say two nonseparating curves form a *peripheral pair* if they bound, along with a boundary component of S, a pair of pants.

Remark that according to our definition curves forming the boundary of S *do not* form part of the pants decomposition P for they are not essential. Since a pants decomposition is a multicurve, then it is a locally finite set.

If P is a pants decomposition, S_P is the disjoint union of surfaces homeomorphic to a pair of pants, for otherwise we contradict maximality. We remark that while $\mathcal{A}(P)$ can be realized abstractly as a graph in $\mathcal{C}^1(S)$, it is not an induced subgraph.

Remark 3.2. A separating curve α is said to be an *outer separating* curve if by cutting S along α one of the resulting connected components is a pair of pants. A nonouter separating curve is a separating curve which is not an outer separating curve. It can be easily checked that the only cut points of an adjacency graph $\mathcal{A}(P)$ correspond to nonouter separating curves, and nonouter separating curves always correspond to cut points of any adjacency graph in which they are vertices. Also, we can easily check the vertices corresponding to outer separating curves always have degree less than or equal to two.

Theorem 3.3. *Let S be an orientable infinite genus surface without planar ends and P be a pants decomposition of S. Then* $\mathrm{Ends}(\mathcal{A}(P))$ *is homeomorphic to* $\mathrm{Ends}(S)$.

Proof: Let $\mathcal{P}$ be a fixed hyperbolic structure on a pair of pants such that the length of every boundary component is equal to some fixed constant. Let $\partial\mathcal{P} = \{\mathfrak{C}_j\}_{j=1}^3$, choose $x_j \in \mathfrak{C}_j$ on each boundary component of $\mathcal{P}$ and denote by γ_{ij} a geodesic arc in $\mathcal{P}$ joining x_i to x_j. We define a convenient hyperbolic structure on S as follows. Let $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ be countably many copies of $\mathcal{P}$ and $S_P = \sqcup_{i\in\mathbb{N}}P_i$ be the disjoint union of pants defined by removing P from S. For each $i \in \mathbb{N}$, let σ_i be a bijection between $\partial P_i := \{C_j^i\}_{j=1}^3$ and $\partial\mathcal{P}_i := \{\mathfrak{C}_j^i\}_{j=1}^3$. If the pairs of pants P_l and P_j share a boundary component $C_j^l = C_{j'}^j$ in S then we glue $\mathcal{P}_l$ and $\mathcal{P}_j$ using an isometry $\iota\colon \sigma_l(C_j^l) \to \sigma_j(C_{j'}^j)$ that sends the chosen point $x_j \in \sigma_l(C_j^l)$ to the chosen point $x_{j'} \in \sigma_j(C_{j'}^j)$ and performs no twist on the geodesic arcs γ_{ji} and $\gamma_{j'i}$ in $\sigma_l(C_j^l)$ and $\sigma_j(C_{j'}^j)$ having x_j and $x_{j'}$ as extremities.

Let Σ be S armed with the resulting hyperbolic structure. Remark that, since all $\mathcal{P}_i$ are isometric to *the same* hyperbolic pair of pants, in Σ the distance between the geodesics corresponding to adjacent curves, with respect to P, are uniformly upper- and lower-bounded, and the lengths of the geodesics corresponding to curves in P are also upper- and lower- bounded. In Figure 1 we illustrate how to induce a quasi-dense quasi-isometric embedding from $\mathcal{A}(P)$ to Σ, which implies that $\mathrm{Ends}(\mathcal{A}(P))$ is homeomorphic to $\mathrm{Ends}(\Sigma)$. In this figure the points depicted on the boundary correspond to the points $x_j \in \mathfrak{C}_j$ and the arcs are the geodesic arcs γ_{ij} chosen at the beggining of this proof. In other words, the points x_j and geodesic arcs γ_{ij} are the image under the embedding of vertices and edges of $\mathcal{A}(P)$. Since Σ is homeomorphic to S, we have that $\mathrm{Ends}(S)$ is homeomorphic to $\mathrm{Ends}(\Sigma)$, which gives us the desired result. □

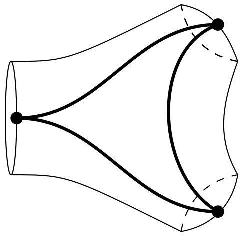

FIGURE 1. A natural embedding of $\mathcal{A}(P)$ into S.

Remark 3.4. Figure 1 also illustrates how to define a homeomorphism between $\mathrm{Ends}(\mathcal{A}(P))$ and $\mathrm{Ends}(\Sigma)$, giving another way to prove the preceding result. On the other hand, we can think of punctures on a surface as planar ends, and hence the preceding result is not true if we allow the surface S to have them.

4. Proof of main results

4.1. Injectivity. In this section we prove the injectivity of the natural map $\Psi_{\mathcal{C}(S)}$:

Lemma 4.1. *Let S be an orientable infinite genus surface such that without planar ends. The natural map*

$$\Psi_{\mathcal{C}(S)}\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{C}(S)) \tag{1}$$

is injective.

The proof of this result will rely on the following lemma and a variant of the Alexander method (see [**FM**, §2.3] for details on this method).

Lemma 4.2. *Let S be an infinite genus surface possibly with marked points and possibly a finite number of boundary components. Let $\gamma_1, \ldots, \gamma_n$ be a collection of simple closed curves and simple proper arcs in S satisfying the following three properties:*

1. *The γ_i are in pairwise minimal position. That is, for $i \neq j$, the (geometric) intersection of γ_i with γ_j is minimal within their homotopy classes.*
2. *The γ_i are pairwise nonisotopic.*
3. *For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$, or $\gamma_j \cap \gamma_k$ is empty.*

If $\gamma_1', \ldots, \gamma_n'$ is another such collection so that γ_i is isotopic to γ_i' for each i, then there is an isotopy of S that takes γ_i' to γ_i for all i simultaneously, and hence takes $\cup\gamma_i$ to $\cup\gamma_i'$.

A collection of curves $\gamma_1, \dots, \gamma_n$ satisfying items 1–3 in the preceding lemma will be called an *Alexander system* in S. The proof of this lemma is exactly the same as the proof of Lemma 2.9 in [**FM**].

Proof of Lemma 4.1: Let $h\colon S \to S$ be an homeomorphism such that $h(\alpha)$ is isotopic to α for all $\alpha \in \mathcal{V}(\mathcal{C}(S))$. For every infinite genus surface such that $\mathrm{Ends}(S) = \mathrm{Ends}^*(S)$ we can find a family of compact subsurfaces $\{K_i\}_{i\in\mathbf{N}}$ such that:

- $S = \cup_{i\in\mathbf{N}} K_i$.
- $K_i \subset K_j$ if $i < j$.
- K_i has genus at least 3 for all $i \in \mathbf{N}$.
- $K_j \setminus K_i$ admits at least one curve nonisotopic to any boundary curve of K_j for $i < j$.
- Every boundary component of K_i that is not a boundary component of S is an essential separating curve of S.

For each $i \in \mathbf{N}$ let us write ∂K_i for the boundary of K_i, $\partial_S K_i$ for all curves in ∂K_i that are part of the boundary of S, and $\partial_i K_i$ for $\partial K_i \setminus \partial_S K_i$. Given such a family $\{K_i\}_{i\in\mathbb{N}}$ of compact subsurfaces we can find $\{\Gamma_i\}_{i\in\mathbb{N}}$ a collection of finite subsets of $\mathcal{V}(\mathcal{C}(S))$ such that:

- Every boundary component of K_i that is not a boundary component of S is in Γ_j for $i < j$ and is disjoint from every other curve in $\cup_{i\in\mathbf{N}}\Gamma_i$.
- Γ_0 fills K_0 and $\Gamma_j \setminus \Gamma_{j-1}$ fills $K_j \setminus K_{j-1}$ for all $j > 0$. In addition $\Gamma_i \subset \Gamma_j$ for $i < j$.
- If we cut $K_j \setminus K_i$ along $\Gamma_j \setminus \Gamma_i$, then we obtain either discs or annuli with one boundary component in ∂K_k, for $i < j$ and some k with $i \leq k \leq j$.
- For all $\gamma \in \Gamma_j \setminus \Gamma_i$ and $\gamma' \in \Gamma_i$, we have that $i(\gamma, \gamma') = 0$. Moreover, if we define for each $i \in \mathbf{N}$

$$\Gamma'_i = \Gamma_i \cup \partial_i K_i, \tag{2}$$

  then, for all $\gamma \in \Gamma_j \setminus \Gamma'_i$ and $\gamma' \in \Gamma'_i$, we have $i(\gamma, \gamma') = 0$.
- Γ_i, Γ'_i and $\Gamma'_{i+1} \setminus \Gamma'_i$ are Alexander systems in S, for all $i \in \mathbf{N}$.

Figure 2 shows an example of the family $\{K_i\}_{i\in\mathbf{N}}$ and its corresponding collection of nonseparating curves $\{\Gamma_j\}_{j\in\mathbf{N}}$.

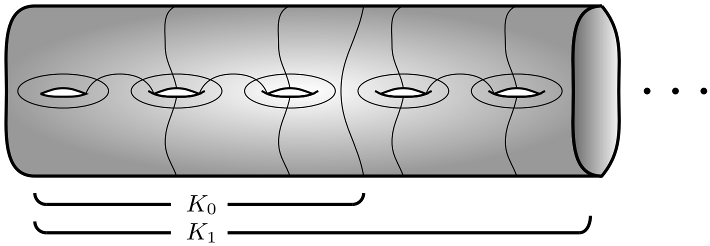

FIGURE 2. Example for $K_0, K_1, \ldots$ and $\Gamma_0, \Gamma_1, \ldots$.

Lemma 4.3. *There exist a family of homotopies* $H_i \colon S\times[0,1] \to S$ *such that:*

1. $H_i\big|_{S\times\{0\}}$ *is the identity for* $i \in \mathbf{N}$.
2. $H_i\big|_{K_i\times\{1\}} = h\big|_{K_i}$ *for* $i \in \mathbf{N}$.
3. $H_i\big|_{K_i\times[0,1]} = H_j\big|_{K_i\times[0,1]}$ *for* $i < j$.

The proof of this lemma is rather technical, the main difficulty being to prove that $H_i\big|_{K_i\times[0,1]} = H_j\big|_{K_i\times[0,1]}$ for $i < j$. We leave it for later. We will use the lemma to finish the proof of Lemma 4.1.

For every $x \in S$ there exists $i \in \mathbf{N}$ such that $x \in K_i$ and $x \notin \partial_i K_i$. Define $H\colon S\times[0,1] \to S$ as $H(x,t) = H_i(x,t)$. From item 3 in Lemma 4.2 we deduce that H is well-defined. The function H is clearly continuous, $H\big|_{S\times\{0\}}$ is the identity and $H\big|_{S\times\{1\}} = h$. Thus H is an homotopy from the identity to h. This, modulo the proof of Lemma 4.3, finishes the proof of Lemma 4.1. □

Proof of Lemma 4.3: The idea of the proof is a variant of the Alexander method (see [**FM**, §2.3] for details on this method). Roughly speaking, the most important aspect of this proof (and variant with respect to the finite type method) is that the sequence of homotopies $\{H_i\}_{i\in\mathbb{N}}$ is synchronized, allowing the final homotopy to be defined. By hypothesis, for every $\gamma \in \mathcal{C}(S)$, the curves γ and $h(\gamma)$ are isotopic.

We use an inductive construction for the isotopies, using $i = 0$ as a base case and then showing how to construct H_{i+1} having already constructed H_i.

By Lemma 4.2 there exists an isotopy $\tilde{H}_0\colon S\times[0,1] \to S$, that takes γ to $h(\gamma)$ for all $\gamma \in \Gamma'_0$ simultaneously. Let us define $f_0 := \tilde{H}_0\big|_{S\times\{1\}}$ and $g_0 := h^{-1} \circ f_0$. Remark that:

- Due to $\tilde{H}_0$, f_0 is isotopic to the identity in S, and so g_0 is isotopic to h^{-1} in S.
- g_0 fixes all the curves in Γ'_0.
- g_0 is an homeomorphism such that $g_0(K_0) = K_0$.

We claim that h has to be orientation-preserving. To see this remark first that since h fixes every isotopy class of essential curve in S, up to composing h with an isotopy, we can find an embedded subsurface (with boundary) $S' \hookrightarrow S$ of genus bigger than 3 such $h\big|_{S'} \colon S' \to S'$ is a homeomorphism that fixes the isotopy class of every essential curve in S'. If h reverses orientation, so does $h\big|_{S'}$ for S is connected. But this is impossible since, any homeomorphism of a finite type surface, which is not a sphere with at most three holes, that preserves the isotopy class of every essential curve must be orientation preserving, see Proposition 2.6 in [**McP**]. This implies that g_0 sends each connected component in $S \setminus \Gamma'_0$ to itself. This fact follows from the argument explained in the second paragraph of the proof of Proposition 2.8, p. 62–63, in [**FM**]. We omit it to shorten the proof.

By hypotheses Γ_0 fills K_0 and $\Gamma_j \setminus \Gamma_{j-1}$ fills $K_j \setminus K_{j-1}$ for all $j > 1$. Hence:

$$S \setminus \Gamma'_i = \left(\bigsqcup_{k=1}^{n_i} A_k \right) \sqcup \left(\bigsqcup_{k=1}^{m_i} D_k \right) \sqcup S_i, \tag{3}$$

where each D_k is homeomorphic to a disc, each A_k is homeomorphic to an annulus and $S_i = S \setminus K_i$ is an infinite genus surface. Furthermore:

1. The boundary of each disc D_k is formed by segments contained in Γ_i.
2. The boundary of each annulus A_k is contained in Γ'_i.

From Alexander's lemma (see [**FM**, Lemma 2.1, p. 47]) we deduce that g_0 restricted to D_k is isotopic to $\mathrm{Id}\big|_{D_k}$. The restriction of g_0 to A_k is also isotopic to the identity for else this restriction (and thus a restriction of h^{-1}) will be a non-trivial Dehn twist and we could then find a curve $\gamma \in \mathcal{C}(S)$ intersecting the interior of A_k which is not fixed by h^{-1}. From these facts we conclude that g_0 is isotopic to the identity in $K_0 \operatorname{rel} \partial K_0$. Hence we can define a global isotopy $L_0 \colon S \times [0,1] \to S$ such that:

- $L_0\big|_{K_0 \times \{0\}} = g_0\big|_{K_0}$.
- $L_0\big|_{K_0 \times \{1\}}$ is the identity in K_0.
- $L_0\big|_{(S \setminus K_0) \times [0,1]}$ is the identity in $S \setminus K_0$.

This implies that $h \circ L_0$ is a global isotopy that restricts on K_0 to an isotopy from f_0 to h. The (homotopy) composition of $\tilde{H}_0$ with $h \circ L_0$ defines the isotopy H_0 that satisfies the conditions of Lemma 4.3.

Now, let H_i be an isotopy satisfying the conditions of Lemma 4.3. To construct H_{i+1} we first recall that:

- $\partial K_i = \Gamma'_i \setminus \Gamma_i$ is a set of separating curves.
- $\Gamma'_i \subset \Gamma'_{i+1}$.
- Γ'_i and $\Gamma'_{i+1} \setminus \Gamma'_i$ are Alexander systems.

Using these facts and Lemma 4.2, we obtain a global isotopy $\tilde{H}_{i+1}$ that takes γ to $h(\gamma)$ for all $\gamma \in \Gamma'_{i+1}$ and satisfies:

$$\tilde{H}_j\big|_{\Gamma'_i \times [0,1]} = \tilde{H}_{i+1}\big|_{\Gamma'_i \times [0,1]}, \quad \text{for } j < i+1. \tag{4}$$

Let $f_{i+1} := \tilde{H}_{i+1}\big|_{S \times \{1\}}$ and $g_{i+1} := h^{-1} \circ f_{i+1}$. Due to the same arguments as with g_0, there exists a global isotopy $h \circ L_{i+1}$ that restricts on K_{i+1} to an isotopy from f_{i+1} to h. Moreover, since all the discs and annuli that appear in $S \setminus \Gamma'_i$ also appear in (and are disjoint from) the discs and annuli in $S \setminus \Gamma'_{i+1}$, we can have that $h \circ L_i(x,t) = h \circ L_{i+1}(x,t)$ for $x \in K_i$ and $t \in [0,1]$. Thus the (homotopy) composition of $\tilde{H}_{i+1}$ and $h \circ L_{i+1}$ defines the isotopy H_{i+1} that satisfies all the conditions of Lemma 4.3. □

4.2. Rigidity. In this section we give the proof of Theorem 1.4 and Corollary 1.5. This requires some auxiliary facts and lemmas, that we state and prove in the following paragraphs.

Through this section S_1 and S_2 will denote (connected, orientable) infinite genus surfaces with a finite number of boundary components and $\phi\colon \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ a simplicial isomorphism. We remark that the image via ϕ of any pants decomposition of S_1 is a pants decomposition of S_2. Moreover, if P is a pants decomposition of S_1, then $\alpha, \beta \in P$ are adjacent with respect to P if and only if $\phi(\alpha)$ and $\phi(\beta)$ are adjacent with respect to $\phi(P)$. The sufficiency of this statement can be found in [**Sha**] and the necessity follows from the fact that we are dealing with an isomorphism of the curve complex. Therefore $\phi\colon \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ induces a map

$$\varphi\colon \mathcal{A}(P) \to \mathcal{A}(\phi(P)) \tag{5}$$

as follows: $\alpha \mapsto \varphi(\alpha) := \phi(\alpha)$. Moreover, φ is an isomorphism. For this reason cut points of $\mathcal{A}(P)$ go to cut points under ϕ and this isomorphism sends:

1. Nonouter separating curves to nonouter separating curves.
2. Nonseparating curves to nonseparating curves.
3. Outer separating curves to outer separating curves.

The proof of items 1 and 2 can be found in [**Sha**], whereas item 3 follows from 1, 2, and the fact that ϕ is an isomorphism. The following lemmas can be deduced from the work of Irmak (see [**Irm**]), but since we use them several times later, we present elementary and simple proofs.

Lemma 4.4. *Let S_1 and S_2 be infinite genus surfaces and let $\phi\colon \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ be an isomorphism. If $\mathcal{X} \subset \mathcal{C}(S_1)$ bounds a pair of pants on S_1, then its image $\phi(\mathcal{X})$ bounds a pair of pants in S_2.*

Proof: Recall that a subset $\mathcal{X} \subset \mathcal{C}(S_1)$ bounds a pair of pants on S_1 if $S_1 \setminus \mathcal{X} = S_1' \sqcup \mathcal{P}$, where $\mathcal{P}$ is a pair of pants and $\mathcal{X} = \partial\mathcal{P}$. We divide the proof according to the cardinality of $\mathcal{X}$. Remark first that $\mathcal{X}$ has cardinality at least 2 since S_1 is orientable.

Suppose now that $\mathcal{X} = \{\alpha, \beta\}$ and let P be a pants decomposition of S_1 containing both curves. In this situation we can suppose without loss of generality that $\{\alpha, \beta\}$ are contained in a surface homeomorphic to $S_{1,1}$ and the degree of α as a vertex of $\mathcal{A}(P)$ is equal to one. This situation is depicted on the left-hand side of Figure 3. In particular α is nonseparating. Then $\phi(\alpha)$ is a nonseparating curve adjacent to $\phi(\beta)$ and whose degree as vertex of $\mathcal{A}(\phi(P))$ is 1. The only possibility is that $\{\phi(\alpha), \phi(\beta)\}$ bound a pair of pants as in the left-hand side of Figure 3 as well.

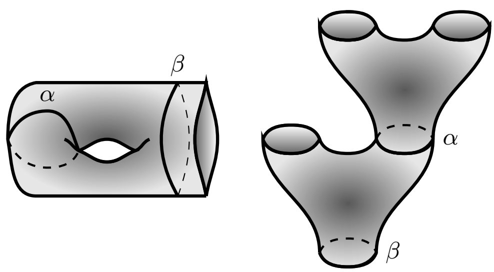

Figure 3. The two options for $\deg(\alpha) = 1$.

The last case to consider is $\mathcal{X} = \{\alpha, \beta, \gamma\}$. Since it is impossible to bound a pair of pants using two separating curves and one nonseparating curve we only have the following subcases according to the number of separating curves:

1. Three separating curves. In this case, $\phi(\alpha)$, $\phi(\beta)$, and $\phi(\gamma)$ are three different separating curves, since separating curves go to separating curves as mentioned before. If these curves did not bound a pair of pants on S_2 we would have, as in Figure 4, a pair of pants bounded by $\phi(\alpha)$ and $\phi(\beta)$ but not bounded by $\phi(\gamma)$, another pair of pants bounded by $\phi(\beta)$ and $\phi(\gamma)$ but not bounded by $\phi(\alpha)$ and another pair of pants bounded by $\phi(\gamma)$ and $\phi(\alpha)$ but not bounded by $\phi(\beta)$. But then none of these curves would be separating, leading us to a contradiction. Hence, $\phi(\alpha)$, $\phi(\beta)$, and $\phi(\gamma)$ bound a pair of pants on S_2.
2. One separating curve. Let α and γ be nonseparating curves and let β be a separating curve. Then $\phi(\alpha)$ and $\phi(\gamma)$ are nonseparating curves and $\phi(\beta)$ is a separating curve, given the properties of ϕ mentioned before. If these curves did not bound a pair of pants on S_2, then we would have a pair of pants bounded by $\phi(\alpha)$ and $\phi(\beta)$ but not bounded by $\phi(\gamma)$, and another pair of pants bounded by $\phi(\beta)$ and $\phi(\gamma)$ but not bounded by $\phi(\alpha)$; but since $\phi(\beta)$ is a separating curve there cannot exist a pair of pants bounded by both $\phi(\alpha)$ and $\phi(\gamma)$, given that they are on different connected components of $S_2 \setminus \{\phi(\beta)\}$, which leads us to a contradiction ($\phi(\alpha)$ and $\phi(\gamma)$ must be adjacent in $\mathcal{A}(\phi(P))$. Then $\phi(\alpha)$, $\phi(\beta)$, and $\phi(\gamma)$ bound a pair of pants on S_2.
3. Three nonseparating curves. Given that α, β, and γ are nonseparating curves, we can always find a pants decomposition P such that all their neighbours in $\mathcal{A}(P)$ are nonseparating, α and β have degree three in $\mathcal{A}(P)$, γ has degree four in $\mathcal{A}(P)$, and α and γ only have one common neighbour β in $\mathcal{A}(P)$. For an example consider Figure 5. Then, $\phi(\alpha)$ and $\phi(\beta)$ have degree three, $\phi(\gamma)$ has degree four, and all their neighbours in $\phi(P)$ are nonseparating. If $\phi(\alpha)$, $\phi(\beta)$, and $\phi(\gamma)$ do not bound a pair of pants on S_2 then there exist a pair of pants bounded by $\phi(\alpha)$, $\phi(\beta)$, and $\delta_1 \neq \phi(\gamma)$, another pair of pants bounded by $\phi(\beta)$, $\phi(\gamma)$, and $\delta_2 \neq \phi(\alpha)$, and another pair of pants bounded by $\phi(\alpha)$, $\phi(\gamma)$, and $\delta_3 \neq \phi(\beta)$. Since $\phi(\beta)$ is the only common neighbour of $\phi(\alpha)$ and $\phi(\gamma)$, then δ_3 is not an essential curve, which means it is isotopic to a boundary component; but this leads us to a contradiction, since $\phi(\gamma)$ would then have degree at most 3.

□

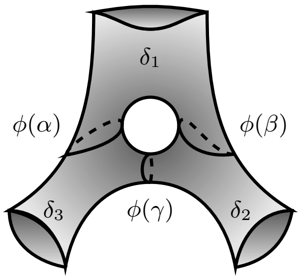

Figure 4. If $\phi(\alpha)$, $\phi(\beta)$ and $\phi(\gamma)$ do not bound a pair of pants.

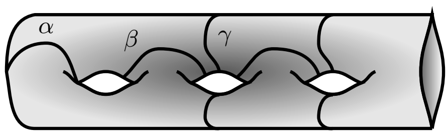

Figure 5. Three nonseparating curves bounding a pair of pants.

Remark 4.5. If P is a pants decomposition and $\alpha \in P$ is a nonseparating curve of degree 2 in $\mathcal{A}(P)$ such that its neighbours are also nonseparating curves, then α forms part of two peripheral pairs, namely one with each neighbour (otherwise either of its neighbours or α itself would become separating).

Lemma 4.6. *Let S_1 and S_2 be infinite genus surfaces and let $\phi\colon \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ be an isomorphism. If α and β form a peripheral pair, then their images form a peripheral pair. In particular, S_1 and S_2 have the same number of boundary components.*

Proof: If S_1 admits at least 2 peripheral pairs such that their curves are pairwise disjoint as in Figure 6, then we can always find a pants decomposition P of S_1 such that, in $\mathcal{A}(P)$, all the neighbours of β are nonseparating, $\deg(\alpha) = 3$ and $\deg(\beta) = 2$. Then all the neighbours of $\phi(\beta)$ are nonseparating, and $\phi(\beta)$ has degree 2, hence it has to form a peripheral pair with $\phi(\alpha)$ by the previous remark.

If for any two peripheral pairs in S_1 at least one curve of each pair intersect each other, we can always find a pants decomposition P of S_1 such that, in $\mathcal{A}(P)$, all the neighbours of α and β are nonseparating, $\deg(\alpha) = \deg(\beta) = 3$, and there is *only one* pair of pants in $S_1 \setminus P$ that is bounded by α and β at the same time, namely the one formed by α

and β being a peripheral pair. Then $\phi(P)$ is a pants decomposition with all the neighbours of $\phi(\alpha)$ and $\phi(\beta)$ being nonseparating, $\deg(\phi(\alpha)) = \deg(\phi(\beta)) = 3$ and there exists a pair of pants in S_2 bounded by $\phi(\alpha)$, $\phi(\beta)$, and δ. We now prove, by contradiction, that δ is not an essential curve. If δ were an essential curve different from $\phi(\alpha)$ and $\phi(\beta)$, due to Lemma 4.4 applied to ϕ^{-1} there would exist a pair of pants induced by P in S_1 that is bounded by α, β, and $\phi^{-1}(\delta)$, which is not possible. So δ cannot be an essential curve different to both $\phi(\alpha)$ and $\phi(\beta)$. If $\delta = \phi(\alpha)$ or $\delta = \phi(\beta)$, then either $\phi(\beta)$ or $\phi(\alpha)$, respectively, becomes separating, which is not possible since α and β are nonseparating. Then δ is not essential, which implies it is isotopic to a boundary component and so $\phi(\alpha)$ and $\phi(\beta)$ form a peripheral pair.

This result implies that S_2 has at least as many boundary components as S_1, and applying the same result to ϕ^{-1} we get that they have the same number of boundary components, even if this number is infinite. □

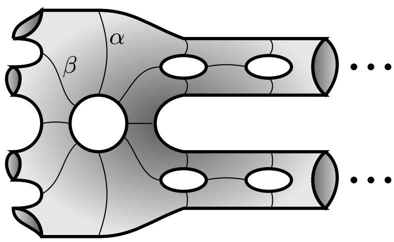

FIGURE 6. Example of a convenient pants decomposition.

Now that we have proved these auxiliary lemmas, we are ready to prove that any isomorphism between $\mathcal{C}(S_1)$ and $\mathcal{C}(S_2)$ implies that S_1 is homeomorphic to S_2 (Theorem 1.4), and that the same can be said about isomorphisms between $\mathcal{N}(S_1)$ and $\mathcal{N}(S_2)$ (Corollary 1.5).

Proof of Theorem 1.4: Let S_1 and S_2 be orientable infinite genus surfaces with finitely many boundary components and without planar ends. We suppose that $\mathcal{C}(S_1)$ and $\mathcal{C}(S_2)$ are isomorphic. Let us show that S_1 and S_2 are homeomorphic. Let P be a pants decomposition of S_1. From the fact that (5) is an isomorphism and from Theorem 3.3, we have $\mathrm{Ends}(S_1) \cong \mathrm{Ends}(\mathcal{A}(P)) \cong \mathrm{Ends}(\mathcal{A}(\phi(P))) \cong \mathrm{Ends}(S_2)$. From the surface classification theorem for infinite surfaces by Richards [**Ric**] and Prishlyak and Mischenko [**PM**], it is sufficient to prove that S_1 and S_2 have the same number of boundary components to guarantee that they are homeomorphic. But this is guaranteed by Lemma 4.6. □

Remark 4.7. Theorem 1.4 cannot be extended to infinite genus surfaces with punctures. Indeed, let S be an infinite genus surface with $n > 0$ boundary components and without planar ends. Let S' be the infinite genus surface obtained from S by glueing one punctured disc to S along a boundary component. Clearly S and S' are not homeomorphic, but $\mathcal{C}(S) \cong \mathcal{C}(S')$.

Proof of Corollary 1.5: Let S_1 and S_2 be orientable infinite genus surfaces with finitely many boundary components and without planar ends. We suppose that $\mathcal{N}(S_1)$ and $\mathcal{N}(S_2)$ are isomorphic. Let us show that S_1 and S_2 are homeomorphic. The statement is immediate, for all arguments given in the proof of Theorem 1.4 remain valid if we change $\mathcal{C}(S)$ for $\mathcal{N}(S)$ and take all pants decompositions to be formed just by nonseparating curves. □

4.3. Surjectivity. In this section we finish the proof of Theorem 1.2 and give a full proof of Theorem 1.3. As explained in the introduction, we begin by the following results:

Theorem 4.8. *Let S be an infinite genus surface. Then* $\mathrm{Aut}(\mathcal{G}(S)) \cong \mathrm{Aut}(\mathcal{N}(S))$.

Theorem 4.9. *Let S be an infinite genus surface without planar ends. The natural map:*

$$\Psi_{\mathcal{G}(S)}\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{G}(S)) \tag{6}$$

is surjective.

This two results imply that the natural map:

$$\Psi_{\mathcal{N}(S)}\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{N}(S)) \tag{7}$$

is surjective. Using the surjectivity of this map, we complete the proof of Theorem 1.2 using the following:

Lemma 4.10. *Let S be an infinite genus surface without planar ends. The natural map:*

$$\Psi_{\mathcal{C}(S)}\colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{C}(S)) \tag{8}$$

is surjective.

4.3.1. Proof of Theorems 4.8 and 4.9. The proofs of Theorems 4.8 and 4.9 require some auxiliary lemmas given in [**Irm**] and [**Sch**] but adapted to the context of infinite-type surfaces. When the proofs of these lemmas can be easily deduced from the cited works we just state them without a proof. When this is not the case elementary and simple proofs are provided. We recall first, following Schmutz Schaller [**Sch**], the different components that a curve might have.

Definition 4.11 (Curve components)**.** Let α and β be nonseparating curves such that $i(\alpha,\beta)\geq 2$. Let β_1 be a connected component of β in $S_\alpha = S\backslash\alpha$. If the surface resulting from cutting S_α along β_1 is connected, then β_1 is called a nonseparating component of β (with respect to α). Otherwise, β_1 is called a separating component of β (with respect to α). If β_1 connects the two different boundary components of S_α induced by α, then β_1 is called a two-sided component. Otherwise it is called a one-sided component.

Lemma 4.12 ([**Sch**])**.** *Let S be an infinite genus surface and $\alpha,\beta\in\mathcal{V}(\mathcal{N}(S))$ such that $i(\alpha,\beta)\geq 2$. If β has a nonseparating component β_1 with respect to α, then there exists $\gamma,\gamma'\in\mathcal{V}(\mathcal{N}(S))\setminus\{\alpha,\beta\}$ such that $N(\alpha,\beta)\subset(N(\gamma)\cup N(\gamma'))$. Moreover, if β_1 is one-sided, then α, γ, γ' are mutually disjoint; if β_1 is two-sided, then $\{\alpha,\gamma,\gamma'\}$ is a triple with*

$$i(\alpha,\beta)=i(\beta,\gamma)+i(\beta,\gamma')$$

and $\min\{i(\beta,\gamma),i(\beta,\gamma')\}>0$.

Lemma 4.13 (*Ibid.*)**.** *Let S_1 and S_2 be infinite genus surfaces and let $\phi\colon\mathcal{G}(S_1)\to\mathcal{G}(S_2)$ be an isomorphism. Then for any disjoint curves α and β, their images under ϕ will also be disjoint.*

Proof of Theorem 4.8: Let $\phi\in\mathrm{Aut}(\mathcal{N}(S))$. Since any automorphism of $\mathcal{N}(S)$ (and $\mathcal{G}(S)$ respectively) is uniquely determined by the function on the set of vertices and $\mathcal{V}(\mathcal{N}(S))=\mathcal{V}(\mathcal{G}(S))$, then ϕ induces a bijection $\phi^*\colon\mathcal{G}(S)\to\mathcal{G}(S)$.

From the work of Irmak [**Irm**, Lemma 2.7] on the characterization of two curves that intersect once, one can deduce the following: if S_1 and S_2 are infinite genus surfaces and $\phi_1\colon\mathcal{N}(S_1)\to\mathcal{N}(S_2)$ is an isomorphism, using the fact that ϕ_1 maps disjoint curves into disjoint curves and it induces an isomorphism between $\mathcal{A}(P)$ and $\mathcal{A}(\phi_1(P))$ for any pants decomposition P, we obtain that the conditions for the characterization of intersection 1 are preserved under ϕ_1. Then for any curves α_1 and α_2 such that $i(\alpha_1,\alpha_2)=1$ we have that $i(\phi_1(\alpha_1),\phi_1(\alpha_2))=1$. This fact applied to ϕ and ϕ^{-1} implies that ϕ^* must preserve adjacency and non-adjacency. Hence we can define the function

$$\Phi\colon\ \mathrm{Aut}(\mathcal{N}(S))\to\mathrm{Aut}(\mathcal{G}(S)) \tag{9}$$

as $\phi\mapsto\phi^*$. This function is clearly an injective group homomorphism.

In the same way, for any automorphism of $\mathcal{G}(S)$ we can induce a bijection from $\mathcal{N}(S)$ to itself, and due to Lemma 4.13 this bijection will become an automorphism of $\mathcal{N}(S)$, proving the surjectivity of Φ. Therefore Φ is an isomorphism. □

Remark 4.14. From the proof of Theorem 4.8 and the proof of Corollary 1.5 we conclude that the statements of Lemmas 4.4 and 4.6 remain valid if we change $\mathcal{C}(S)$ for $\mathcal{G}(S)$.

The following four lemmas are used in the proof of Theorem 4.9. Let us recall first, following Schmutz Schaller [**Sch**], the notion of triple of curves.

Definition 4.15 (Triples of curves)**.** Let α, β, and γ be nonseparating curves of S. We will say $\{\alpha, \beta, \gamma\}$ is a triple if $i(\alpha, \beta) = i(\alpha, \gamma) = i(\beta, \gamma) = 1$ and there exists a subsurface $Y \subset S$ which contains α, β, and γ, and Y is homeomorphic to a torus with one boundary component.

Lemma 4.16. *Let S be an orientable infinite genus surface and $\alpha, \beta \in \mathcal{V}(\mathcal{N}(S))$ be such that $i(\alpha, \beta) \geq 2$. If β does not have two-sided components with respect to α, then there exists $\gamma, \gamma' \in \mathcal{V}(\mathcal{N}(S)) \setminus \{\alpha, \beta\}$ such that $\{\alpha, \gamma, \gamma'\}$ is a triple with*

$$i(\alpha, \beta) = i(\beta, \gamma) + i(\beta, \gamma')$$

and $\min\{i(\beta, \gamma), i(\beta, \gamma')\} > 0$.

Proof: Let α_1 and α_2 be the boundary components on S_α induced by α. Since β does not have two-sided components then it only has one-sided components and therefore we can choose a curve γ that intersects α once, does not intersect any one sided component of β based on α_1 and intersects β in such a way that $0 < i(\gamma, \beta) \leq \frac{1}{2} i(\alpha, \beta)$. This can be done by drawing γ disjoint from every one-sided component of β based on α_1, we keep on going "following" a convenient one-sided component of β based on α_2 until before we reach α_2. By this point, we will intersect α_2 in the corresponding point by either turning left or right, but depending on whether we turn either right or left we will intersect β either $k \leq \frac{1}{2} i(\alpha, \beta)$ times or $i(\alpha, \beta) - k$ times. So we turn accordingly. See Figure 7 for examples.

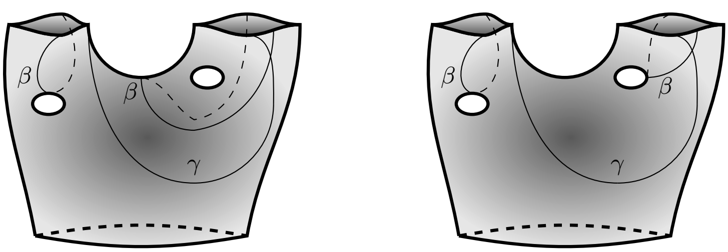

Figure 7. Examples of β and γ in S_α.

Then let N be a regular neighbourhood of α and γ; since $i(\alpha,\gamma) = 1$ then N is homeomorphic to a torus with one boundary component. Let γ' be the image of γ under a Dehn twist along α on N. See Figure 8 for the corresponding diagram.

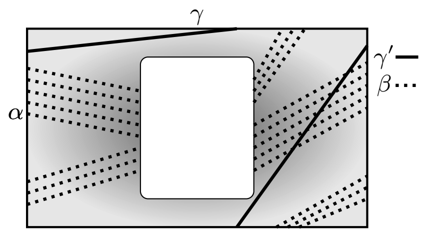

FIGURE 8. Diagram of N.

Thus $\{\alpha,\gamma,\gamma'\}$ form a triple and by construction $i(\beta,\gamma') = i(\alpha,\beta) - i(\beta,\gamma)$, with both curves intersecting β at least once. □

Lemma 4.17. *Let S be an infinite genus surface and let $\phi\colon \mathcal{G}(S) \to \mathcal{G}(S)$ be an automorphism. Then $i(\alpha,\beta) = i(\phi(\alpha),\phi(\beta))$ for all $\alpha,\beta \in \mathcal{V}(\mathcal{G}(S))$.*

Proof: Let $\alpha,\beta \in \mathcal{V}(\mathcal{G}(S))$. If $i(\alpha,\beta) = 0$, then due to Lemma 4.13 we have that $i(\phi(\alpha),\phi(\beta)) = 0$. If $i(\alpha,\beta) = 1$, then, due to ϕ being an automorphism, $i(\phi(\alpha),\phi(\beta)) = 1$. For $i(\alpha,\beta) \geq 2$, we will proceed by induction on the geometric intersection number.

Let us suppose the geometric intersection number is preserved under automorphisms for curves which intersect at most k times for a $k \geq 1$.

Let $i(\alpha,\beta) = k+1$. Due to Lemmas 4.12 and 4.16, we know there exists $\gamma,\gamma' \in \mathcal{V}(\mathcal{G}(S)) \setminus \{\alpha,\beta\}$ such that $\{\alpha,\gamma,\gamma'\}$ is a triple, $i(\alpha,\beta) = i(\beta,\gamma) + i(\beta,\gamma')$ and $\min\{i(\beta,\gamma), i(\beta,\gamma')\} > 0$.

Since $i(\beta,\gamma), i(\beta,\gamma') < k+1$, then $i(\beta,\gamma) = i(\phi(\beta),\phi(\gamma))$ and $i(\beta,\gamma') = i(\phi(\beta),\phi(\gamma'))$.

From the work of Schmutz Schaller [**Sch**, Corollary 8] one can deduce that if S is an infinite genus surface then the image under any automorphism of $\mathcal{G}(S)$ of a triple is also a triple. Indeed, due to Lemma 4.13 we can deduce that pants decompositions are mapped to pants decompositions, and by Theorem 4.8, ϕ induces an isomorphism between $\mathcal{A}(P)$ and $\mathcal{A}(\phi(P))$ for any pants decomposition P. Given any triple $\{\delta_1,\delta_2,\delta_3\}$ let Q be the one-holed torus that contains it, then we can find curves δ_4, δ_5, δ_6 such that δ_4 is disjoint from δ_5, δ_4, δ_5, and δ_6 are disjoint from every curve in the triple, δ_4 and δ_5 separate a two-holed torus that contains Q, δ_6 is disjoint from the boundary curve of Q and intersects once both δ_4 and δ_5. This implies that $\phi(\delta_1)$, $\phi(\delta_2)$, and $\phi(\delta_3)$ are contained in a double-punctured torus Q' bounded by $\phi(\delta_4)$ and $\phi(\delta_5)$, $\phi(\delta_6)$ are disjoint from every curve in $\{\phi(\delta_1), \phi(\delta_2), \phi(\delta_3)\}$ and intersect once both $\phi(\delta_4)$ and $\phi(\delta_5)$. Hence $\phi(\{\delta_1, \delta_2, \delta_3\})$ is a triple.

Therefore $\{\phi(\alpha), \phi(\gamma), \phi(\gamma')\}$ form a triple. Using again a diagram of the torus with one boundary component which contains this triple (see Figure 8), we can see that each time $\phi(\beta)$ intersects $\phi(\alpha)$ then either $\phi(\beta)$ intersects $\phi(\gamma)$ or $\phi(\beta)$ intersects $\phi(\gamma')$. Therefore $i(\phi(\beta), \phi(\gamma)) + i(\phi(\beta), \phi(\gamma')) \geq i(\phi(\alpha), \phi(\beta))$. Thus $i(\alpha, \beta) \geq i(\phi(\alpha), \phi(\beta))$. Applying the same argument on ϕ^{-1} we obtained the symmetric inequality, therefore $i(\alpha, \beta) = i(\phi(\alpha), \phi(\beta))$. □

Lemma 4.18. *Let S be an infinite genus surface and let $\phi\colon \mathcal{G}(S) \to \mathcal{G}(S)$ be an automorphism. If P is a pants decomposition of S, then there exist an homeomorphism $h \in \mathrm{MCG}^*(S)$ such that $h(\alpha) = \phi(\alpha)$ for all $\alpha \in P$.*

Proof: From Remark 4.14, we know that $\phi(P)$ is a pants decomposition and the boundaries of pair of pants in $S_{\phi(P)}$ induced by curves of $\phi(P)$ are boundaries of pair of pants in S_P induced by curves of P. Then we can define an homeomorphism of S by parts using homeomorphisms from the connected components of S_P to the corresponding connected components of $S_{\phi(P)}$; this homeomorphism by construction will agree with ϕ for every element in P. □

Remark 4.19. It is clear, using Theorem 4.8, that this lemma remains valid if we substitute $\mathcal{G}(S)$ by $\mathcal{N}(S)$.

Lemma 4.20 ([**Sch**])**.** *Let S' be a surface of genus zero and four boundary components. Let $\alpha, \beta \in \mathcal{V}(\mathcal{C}(S'))$ with $i(\alpha, \beta) = 2$.*

1. *Let $\gamma \in \mathcal{V}(\mathcal{C}(S'))$ such that $i(\alpha, \gamma) = 2$. Then there exists $h \in \mathrm{MCG}^*(S')$ such that $h(\alpha) = \alpha$ and $h(\beta) = \gamma$.*
2. *There are exactly two curves $\gamma_1, \gamma_2 \in \mathcal{V}(\mathcal{C}(S'))$ such that $i(\alpha, \gamma_i) = i(\beta, \gamma_i) = 2$ for $i = 1, 2$. Moreover, there exists $h \in \mathrm{MCG}^*(S')$ such that $h(\alpha) = \alpha$, $h(\beta) = \beta$, and $h(\gamma_1) = \gamma_2$.*

Remark 4.21. The homeomorphism of item 1 in the preceding lemma is just a Dehn twist about α, where as the homeomorphism from item 2 is an orientation-reversing involution that leaves invariant each connected component in the boundary of $S_{0,4}$.

The proof of Theorem 4.9 uses the notion of Dehn–Thurston coordinates. Therefore we recall it and discuss it briefly in the context of infinite surfaces in the following paragraphs.

Definition 4.22 (Dehn–Thurston coordinates)**.** A Dehn–Thurston coordinates system of curves is a set D of curves that parametrizes every curve $\alpha \in \mathcal{V}(\mathcal{C}(S))$ using the geometric intersection number, i.e. for $\alpha, \beta \in \mathcal{V}(\mathcal{C}(S))$ if $i(\alpha, \gamma) = i(\beta, \gamma)$ for all $\gamma \in D$, then $\alpha = \beta$.

For compact surface, it is well known that Dehn–Thurston coordinate systems exist, see [**PH**]. For noncompact surfaces such a system of curves can be realized in the following way. Let $\{\alpha_i\}_{i\in\mathbf{N}}$ be a pants decomposition, $\{\beta_i\}_{i\in\mathbf{N}}$ be curves such that $i(\alpha_i, \beta_i) = 2$ and $i(\alpha_i, \beta_j) = 0$ for $i \neq j$, and $\{\gamma_i\}_{i\in\mathbf{N}}$ be curves such that $i(\alpha_i, \gamma_i) = i(\beta_i, \gamma_i) = 2$ and $i(\alpha_i, \gamma_j) = 0$ for $i \neq j$. Then the set of curves D formed by the union of elements in $\{\alpha_i\}_{i\in\mathbf{N}}$, $\{\beta_i\}_{i\in\mathbf{N}}$, and $\{\gamma_i\}_{i\in\mathbf{N}}$ is a Dehn–Thurston coordinate system. Indeed, any essential curve δ in S will only intersect finitely many curves in D, hence we can take any compact subsurface S', such that it contains δ and there is a (finite) subset D' of D that is a Dehn–Thurston coordinate system of S'. Any other curve in S with the same Dehn–Thurston coordinates as δ on the system D would have to be isotopic to a curve contained in S', and thus would have the same Dehn–Thurston coordinates as δ on the system D', therefore it would be isotopic to δ. We must remark that, when S is an infinite genus surface such that $\mathrm{Ends}(S) = \mathrm{Ends}^*(S)$, we can also construct the Dehn–Thurston coordinate system D with families $\{\alpha_i\}_{i\in\mathbf{N}}$, $\{\beta_i\}_{i\in\mathbf{N}}$, and $\{\gamma_i\}_{i\in\mathbf{N}}$ formed exclusively by nonseparating curves. However, not every pants decomposition formed exclusively by nonseparating curves is part of a Dehn–Thurston system.

Proof of Theorem 4.9: Given that $\mathrm{Ends}(S) = \mathrm{Ends}^*(S)$ we can construct $P = \{\alpha_i\}_{i\in\mathbf{N}}$ a pants decomposition of S formed by nonseparating curves such that it can be extended to a Dehn–Thurston system. Let $\phi\colon \mathcal{G}(S) \to \mathcal{G}(S)$ an automorphism. Due to Lemma 4.18 there exists an homeomorphism $h_1\colon S \to S$ such that $h_1(\alpha_i) = \phi(\alpha_i)$ for all $\alpha_i \in P$.

Again, since $\mathrm{Ends}(S) = \mathrm{Ends}^*(S)$ we can construct $\{\beta_i\}_{i\in\mathbf{N}}$ a collection of nonseparating curves such that $i(\alpha_i, \beta_i) = 2$ for all i and $i(\alpha_i, \beta_j) = 0$ for $i \neq j$. We can define an homeomorphism $h_2\colon S \to S$ such that $h_2(h_1(\alpha_i)) = h_1(\alpha_i) = \phi(\alpha_i)$ and $h_2(h_1(\beta_i)) = \phi(\beta_i)$ in the following way. For every $i \in \mathbf{N}$ the curves $\alpha = h_1(\alpha_i)$, $\beta = h_1(\beta_i)$, and $\gamma = \phi(\beta_i)$ satisfy the hypotheses of part 1 in Lemma 4.20 and lie in a subsurface S_i homeomorphic to $S_{0,4}$ that does not contain any element in $h_1(P) \setminus \{h(\alpha_i)\}$, S_i contains $h_1(\beta_i)$ and $\phi(\beta_i)$, and its boundary components are isotopic to the curves adjacent to $h_1(\alpha_i)$ with respect to $h_1(P)$. Let $h_{2,i}\colon S_i \to S_i$ be the homeomorphism from item 1 in Lemma 4.20. This homeomorphism is just a Dehn twist about α, therefore it preserves orientation and its support $K_i \subset S_i$ satisfies that $K_i \cap K_j = \emptyset$ for $i \neq j$ for all $i, j \in \mathbf{N}$. Hence h_2 can be defined by parts using $\{h_{2,i}\}_{i\in\mathbf{N}}$.

Let $\{\gamma_i\}_{i\in\mathbf{N}}$ be a collection of curves such that $i(\alpha_i, \gamma_i) = i(\beta_i, \gamma_i) = 2$ and $i(\alpha_i, \gamma_j) = 0$ for $i \neq j$. We can define an homeomorphism $h_3\colon S \to S$ such that $h_3(h_2(h_1(\alpha_i))) = h_2(h_1(\alpha_i)) = \phi(\alpha_i)$, $h_3(h_2(h_1(\beta_i))) = h_2(h_1(\beta_i)) = \phi(\beta_i)$, and $h_3(h_2(h_1(\gamma_i))) = \phi(\gamma_i)$ in the following way. For every $i \in \mathbf{N}$, let now $\alpha = h_1(h_2(\alpha_i))$, $\beta = h_1(h_2(\beta_i))$, $\gamma_1 = h_1(h_2(\gamma_i))$, and $\gamma_2 = \phi(\gamma_i)$. Analogously to the preceding case, these curves satisfy the hypotheses of part 2 in Lemma 4.20. Let $h_{3,i}\colon R_i \to R_i$ be the (orientation-reversing) homeomorphism from part 2 in Lemma 4.20, where R_i is homeomorphic to $S_{0,4}$ and contains the curves α, β, γ_1, and γ_2. It is not difficult to see that if $i \neq j$ and $R_i \cap R_j \neq \emptyset$, then $R_{i,j} = R_i \cap R_j \cong S_{0,3}$. Moreover $h_{3,i}$ and $h_{3,j}$ coincide in $R_{i,j}$ and hence we can define h_3 by parts using $\{h_{3,i}\}_{i\in\mathbf{N}}$.

Let $h = h_3 \circ h_2 \circ h_1$. Since $P' = P \cup \{\beta_i\} \cup \{\gamma_i\}$ form a Dehn–Thurston coordinates system of curves, then $h(P')$ is a Dehn–Thurston coordinates system of curves, and by construction $h(\varepsilon) = \phi(\varepsilon)$ for all $\varepsilon \in P'$. Therefore, due to Lemma 4.17, for all $\delta \in \mathcal{V}(\mathcal{G}(S))$ and all $\varepsilon \in P'$:

$$(10) \qquad i(\phi(\delta), \phi(\varepsilon)) = i(\delta, \varepsilon) = i(h(\delta), h(\varepsilon)) = i(h(\delta), \phi(\varepsilon)),$$

then $\phi(\delta) = h(\delta)$ for all $\delta \in \mathcal{V}(\mathcal{G}(S))$, which implies $\Psi_{\mathcal{G}(S)}$ is surjective. □

Corollary 4.23. *Let S_1 and S_2 be infinite genus surfaces without planar ends, and let $\phi\colon \mathcal{G}(S_1) \to \mathcal{G}(S_2)$ be an isomorphism. Then S_1 and S_2 are homeomorphic and ϕ is induced by a mapping class in* $\mathrm{MCG}^*(S_1)$.

Proof: Let $\phi\colon \mathcal{G}(S_1) \to \mathcal{G}(S_2)$ be an isomorphism. By the same argument as in Lemma 4.13, applied to ϕ and ϕ^{-1}, we have that ϕ preserves disjointness and nondisjointness. Thus it induces an isomorphism $\phi\colon \mathcal{N}(S_1) \to \mathcal{N}(S_2)$. By Corollary 1.5 we obtain that S_1 is homeomorphic to S_2. The rest of the proof follows from Theorems 4.8 and 4.9. □

4.3.2. Proof of Lemma 4.10. Any $\phi \in \mathrm{Aut}(\mathcal{C}(S))$ sends nonseparating curves to nonseparating curves, hence $\phi|_{\mathcal{N}(S)} \in \mathrm{Aut}(\mathcal{N}(S))$ and then due to Theorem 4.9 there exists $h \in \mathrm{MCG}^*(S)$ such that $\phi|_{\mathcal{N}(S)}(\alpha) = h(\alpha)$ for all $\alpha \in \mathcal{V}(\mathcal{N}(S))$. Hence we only need to check that ϕ and h coincide in the separating curves of S. Let α be a separating curve of S; we consider three cases.

1. If both connected components of S_α have positive genus, then we can find a pants decomposition P such that $\alpha \in P$, $P \setminus \{\alpha\} \subset \mathcal{V}(\mathcal{N}(S))$ and $\deg(\alpha) = 4$ in $\mathcal{A}(P)$; let β_1, γ_1, β_2, and γ_2 be the

neighbours of α in $\mathcal{A}(P)$ such that β_i and γ_i are in the same connected component of S_α for $i = 1, 2$. Let also δ_1 and δ_2 be nonseparating curves such that $i(\alpha, \delta_i) = 0$ and $i(\beta_i, \delta_i) = i(\gamma_i, \delta_i) = 1$ for $i = 1, 2$. See Figure 9 for an example.

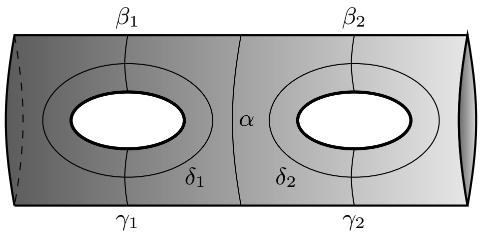

FIGURE 9. Catching α in a $S_{0,4}$.

By construction and Lemma 4.4, $\phi(\alpha)$ and $h(\alpha)$ are contained in the $S_{0,4}$ subsurface bounded by $\phi(\beta_1)$, $\phi(\gamma_1)$, $\phi(\beta_2)$, and $\phi(\gamma_2)$ (recall that $\phi(\beta_i) = h(\beta_i)$ and $\phi(\gamma_i) = h(\gamma_i)$ for $i = 1, 2$ since they are nonseparating curves). Even more, since $i(\alpha, \delta_i) = 0$ for $i = 1, 2$ then $\phi(\alpha)$ and $h(\alpha)$ must be contained in the annulus formed by cutting the aforementioned $S_{0,4}$ subsurface along the arcs of $\phi(\delta_i) = h(\delta_i)$ for $i = 1, 2$; therefore $\phi(\alpha) = h(\alpha)$.

2. If α is an outer curve, then let P be a pants decomposition such that the peripheral pairs of P bounding the same boundary components as α, are consecutive to one another (similar to the proof of Lemma 4.6), and α intersects only one curve in P (namely β); let also γ be a nonseparating curve that intersects each curve in the peripheral pairs bounding the same boundary component as α only once while being disjoint from α. Figure 10 illustrates this situation.

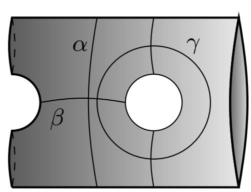

FIGURE 10. Catching α again in a $S_{0,4}$.

Due to ϕ being an isomorphism, $\phi(\alpha)$ will intersect $\phi(\beta)$ and be disjoint of every other curve in P. Using that and Lemma 4.6, we know that $\phi(\alpha)$ and $h(\alpha)$ are contained in the $S_{0,4}$ subsurface bounded by two boundary components of S and the images of the adjacent curves in $\mathcal{A}(P)$ of β; even more, $\phi(\alpha)$ and $h(\alpha)$ must be contained in the pair of pants resulting from cutting the aforementioned $S_{0,4}$ subsurface that contains them along the arc of $\phi(\gamma) = h(\gamma)$. Since there is only one curve in this pair of pants which is an essential curve of S, then $\phi(\alpha) = h(\alpha)$.

3. Let S_1 and S_2 be the two connected components of S_α and suppose that S_1 has genus zero and $n' \geq 3$ boundary components. We can find the following: a finite sequence $\{\beta_i\}_{i=1}^{n'-1}$ composed of outer curves, such that $i(\beta_i, \alpha) = 0$ for $i = 1, \ldots, n'-1$, $i(\beta_i, \beta_{i+1}) = 2$ for $i = 1, \ldots, n'-2$, and $i(\beta_i, \beta_j) = 0$ for $j \notin \{i-1, i+1\}$; a pants decomposition P (composed solely of nonseparating curves) of the infinite genus connected component of $S \setminus \{\alpha\}$; and finally, a curve γ which intersects once the curves δ_1 and δ_2 forming the peripheral pair that bounds the boundary of S_2 induced by α. Figure 11 illustrates this situation.

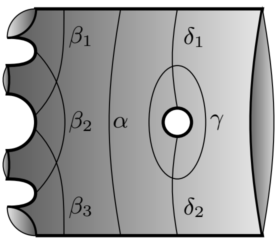

FIGURE 11. Catching α in an annulus.

Given that any isomorphism of $\mathcal{C}(S)$ sends outer curves to outer curves, part 2 of this proof, the fact that $\phi(\alpha)$ and $h(\alpha)$ must both be essential curves and they must be different from every element of $\phi(\{\beta_i\}_{i=1}^{n'}) \cup \phi(P) \cup \{\phi(\gamma)\}$; we can conclude that $\phi(\alpha)$ and $h(\alpha)$ must be contained in the annulus obtained by cutting S along $\phi(\{\beta_i\}_{i=1}^{n'}) \cup \phi(P) \cup \{\phi(\gamma)\}$. The boundary components of this annulus are formed by arcs of $\phi(\beta_i)$ for $i = 1, \ldots, n'-1$, $\phi(\gamma)$, $\phi(\delta_1)$, and $\phi(\delta_2)$. Therefore $\phi(\alpha) = h(\alpha)$.

□

4.3.3. Proof of Theorem 1.3. From Theorem 4.9 we know that the natural map:

$$\Psi_{\mathcal{N}(S)} \colon \mathrm{MCG}^*(S) \to \mathrm{Aut}(\mathcal{N}(S)) \tag{11}$$

is surjective. Let us suppose $h_1, h_2 \in \mathrm{MCG}^*(S)$ are such that $h_1 \neq h_2$ and $\Psi_{\mathcal{N}(S)}(h_1) = \Psi_{\mathcal{N}(S)}(h_2)$. Then since $\Psi_{\mathcal{C}(S)}$ is injective we have that $\Psi_{\mathcal{C}(S)}(h_1) \neq \Psi_{\mathcal{C}(S)}(h_2)$ even though their restrictions to $\mathcal{N}(S)$ are the same. This implies that $\Psi_{\mathcal{C}(S)}(h_1)$ and $\Psi_{\mathcal{C}(S)}(h_2)$ differ in some separating curves. But given that the restrictions of $\Psi_{\mathcal{C}(S)}(h_1)$ and $\Psi_{\mathcal{C}(S)}(h_2)$ to $\mathcal{N}(S)$ are the same, we can use the same technique as in the proof of Lemma 4.10, for catching the separating curves in an annulus (or a pair of pants), which means $\Psi_{\mathcal{C}(S)}(h_1)(\alpha) = \Psi_{\mathcal{C}(S)}(h_2)(\alpha)$ for every separating curve α. Thus we have reached a contradiction and therefore $\Psi_{\mathcal{N}(S)}$ is injective, hence it is an isomorphism. We finish the proof by remarking that $\Psi_{\mathcal{G}(S)} = \Phi \circ \Psi_{\mathcal{N}(S)}$, where Φ is the isomorphism between $\mathrm{Aut}(\mathcal{N}(S))$ and $\mathrm{Aut}(\mathcal{G}(S))$ defined in (9). □

Remark 4.24. Using Theorem 1.2 we can deduce that, for an infinite genus surface S such that $\mathrm{Ends}(S) = \mathrm{Ends}^*(S)$, every automorphism φ of $\mathrm{MCG}^*(S)$ sending Dehn twists to Dehn twist must be an inner automorphism. The proof of this fact is taken verbatim from the proof of Theorem 2 in [**Iva1**]. However, it is still unknown if, as in the compact case, every automorphism of $\mathrm{MCG}^*(S)$ sends Dehn twists to Dehn twists.

5. Counterexamples

In this section we show that Theorem 1.4 is not valid if the morphism between curve complexes is not an isomorphism. For that, let us first recall the notion of *superinjective map.*

Definition 5.1 (Superinjectivity)**.** A simplicial map $f\colon \mathcal{C}(S_1) \to \mathcal{C}(S_2)$ is called superinjective if for any two vertices α and β in $\mathcal{C}(S_1)$ such that $i(\alpha, \beta) \neq 0$ we have that $i(f(\alpha), f(\beta)) \neq 0$.

Every superinjective map is injective. For compact surfaces, we have the following theorem concerning superinjective maps.

Theorem 5.2 ([**Irm**])**.** *Let S be a closed, connected, orientable surface of genus at least* 3*. A simplicial map, $f\colon \mathcal{C}(S) \to \mathcal{C}(S)$, is superinjective if and only if f is induced by an homeomorphism of S.*

The following lemma shows that this result is not true for a large class of surfaces of infinite genus and, in this sense, Theorem 1.4 is optimal.

Lemma 5.3. *Let S be a surface such that* $\mathrm{Ends}^*(S) \neq \emptyset$*. Then there exists a simplicial superinjective map $f\colon \mathcal{C}(S) \to \mathcal{C}(S)$ which is not surjective.*

Proof: This proof makes reference to Figure 12. Let $\alpha \in \mathcal{V}(\mathcal{C}(S))$ be a separating curve. Without loss of generality we can think that α is contained in a subsurface S_i in $[S_1 \supseteq S_2 \supseteq \cdots] \in \text{Ends}^*(S)$ where i is large enough.

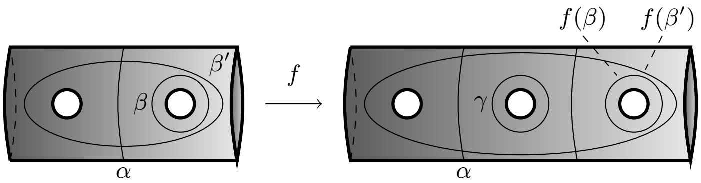

Figure 12. A superinjective but not surjective simplicial map.

We describe f topologically. Let S_1 and S_2 be the two connected components of S_α. Cut S along α and then glue in a copy of $S_{1,2}$. This operation produces a new surface $S' = S_1 \cup S_2 \cup S_{1,2}$. Remark that S is homeomorphic to S' and that there is a natural inclusion map $f_i \colon S_i \hookrightarrow S'$, for $i = 1, 2$. If $\beta \in \mathcal{V}(\mathcal{C}(S_i))$, then we define $f(\beta) = f_i(\beta)$ for $i = 1, 2$. On the other hand, if β' intersects the curve α we define $f(\beta')$ as depicted in Figure 12. Clearly, f is superinjecive but no essential curve properly contained in the copy of $S_{1,2}$ that we introduced is in the image of f. Hence f is not surjective and, in particular, f cannot be induced by a class in $\text{MCG}^*(S)$. $\square$

We think that this result can be optimized in the following way.

Conjecture 5.4. *Let S be a surface such that* $\text{Ends}^*(S) \neq \emptyset$ *and* $\{\alpha_1, \ldots, \alpha_n\} \subset \mathcal{C}(S)$ *be simplex. Then there exists a simplicial superinjective map $f \colon \mathcal{C}(S) \to \mathcal{C}(S)$ whose image does not intersect $\{\alpha_1, \ldots, \alpha_n\}$.*

The following result shows that the statement of Theorem 1.4 is not valid for superinjective maps.

Lemma 5.5. *There exist uncountably many examples of pairs of non-homeomorphic infinite genus surfaces S_1 and S_2 for which there exists a superinjective map $f \colon \mathcal{C}(S_1) \to \mathcal{C}(S_2)$.*

Proof: The arguments are similar to those of the proof of Lemma 5.3. Let S_1 be the Loch Ness monster and $\alpha \in \mathcal{C}(S_1)$ be a separating curve. Let S be your favorite infinite genus surface and suppose that S has at least two boundary components. We describe f topologically. Cut S_1 along α and then glue in a copy of S as indicated in Figure 13. This produces S_2. The rest of the proof is analogous to the proof of Lemma 5.3. □

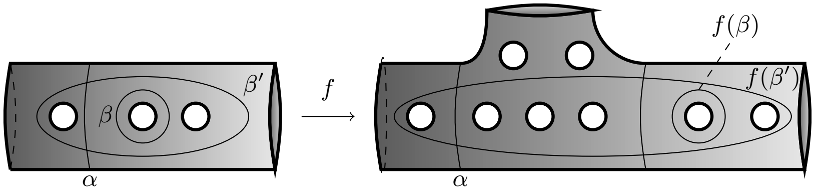

Figure 13. A superinjective map between two non-homeomorphic surfaces.

References

[AS] J. Aramayona and J. Souto, Automorphisms of the graph of free splittings, *Michigan Math. J.* **60(3)** (2011), 483–493. DOI: 10.1307/mmj/1320763044.

[Bav] J. Bavard, Hyperbolicité du graphe des rayons et quasi-morphismes sur un gros groupe modulaire, *Geom. Topol.* **20(1)** (2016), 491–535. DOI: 10.2140/gt.2016.20.491.

[BKMM] J. Behrstock, B. Kleiner, Y. Minsky, and L. Mosher, Geometry and rigidity of mapping class groups, *Geom. Topol.* **16(2)** (2012), 781–888. DOI: 10.2140/gt.2012.16.781.

[FM] B. Farb and D. Margalit, *"A Primer on Mapping Class Groups"*, Princeton Mathematical Series **49**, Princeton University Press, Princeton, NJ, 2012.

[Fre] H. Freudenthal, Über die Enden topologischer Räume und Gruppen, *Math. Z.* **33(1)** (1931), 692–713. DOI: 10.1007/BF01174375.

[Fuj] E. Fujikawa, Modular groups acting on infinite dimensional Teichmüller spaces, in: *"In the Tradition of Ahlfors and Bers, III"*, Contemp. Math. **355**, Amer. Math. Soc., Providence, RI, 2004, pp. 239–253. DOI: 10.1090/conm/355/06455.

[FK] L. Funar and C. Kapoudjian, An infinite genus mapping class group and stable cohomology, *Comm. Math. Phys.* **287(3)** (2009), 784–804. DOI: 10.1007/s00220-009-0728-1.

[FN] L. Funar and M. Nguyen, On the automorphisms group of the asymptotic pants complex of an infinite surface of genus zero (2013). arXiv:1308.6143.

[Irm] E. IRMAK, Complexes of nonseparating curves and mapping class groups, *Michigan Math. J.* **54(1)** (2006), 81–110. DOI: 10.1307/mmj/1144437439.

[Iva1] N. V. IVANOV, Automorphism of complexes of curves and of Teichmüller spaces, *Internat. Math. Res. Notices* **1997(14)** (1997), 651–666. DOI: 10.1155/S1073792897000433.

[Iva2] N. V. IVANOV, Fifteen problems about the mapping class groups, in: *"Problems on Mapping Class Groups and Related Topics"*, Proc. Sympos. Pure Math. **74**, Amer. Math. Soc., Providence, RI, 2006, pp. 71–80. DOI: 10.1090/pspum/074/2264532.

[Koz] D. KOZLOV, *"Combinatorial Algebraic Topology"*, Algorithms and Computation in Mathematics **21**, Springer, Berlin, 2008. DOI: 10.1007/978-3-540-71962-5.

[MW] H. MASUR AND M. WOLF, The Weil–Petersson isometry group, *Geom. Dedicata* **93(1)** (2002), 177–190. DOI: 10.1023/A:1020300413472.

[McP] J. D. MCCARTHY AND A. PAPADOPOULOS, Simplicial actions of mapping class groups, in: *"Handbook of Teichmüller Theory. Volume III"*, IRMA Lect. Math. Theor. Phys. **17**, Eur. Math. Soc., Zürich, 2012, pp. 297–423. DOI: 10.4171/103-1/6.

[PH] R. C. PENNER AND J. L. HARER, *"Combinatorics of Train Tracks"*, Princeton University Press, Princeton, NJ, 1992.

[PM] A. O. PRISHLYAK AND K. I. MISCHENKO, Classification of noncompact surfaces with boundary, *Methods Funct. Anal. Topology* **13(1)** (2007), 62–66.

[Ray] F. RAYMOND, The end point compactification of manifolds, *Pacific J. Math.* **10** (1960), 947–963.

[Ric] I. RICHARDS, On the classification of noncompact surfaces, *Trans. Amer. Math. Soc.* **106** (1963), 259–269. DOI: 10.1090/S0002-9947-1963-0143186-0.

[Roy] H. L. ROYDEN, Automorphisms and isometries of Teichmüller space, in: *"Advances in the Theory of Riemann Surfaces"* (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies **66**, Princeton Univ. Press, Princeton, N.J, 1971, pp. 369–383.

[Sha] K. J. SHACKLETON, Combinatorial rigidity in curve complexes and mapping class groups, *Pacific J. Math.* **230(1)** (2007), 217–232. DOI: 10.2140/pjm.2007.230.217.

[Sch] P. Schmutz Schaller, Mapping class groups of hyperbolic surfaces and automorphism groups of graphs, *Compositio Math.* **122(3)** (2000), 243–260. DOI: 10.1023/A:1002672721132.

Jesús Hernández Hernández:
Aix-Marseille Université
39 rue F. Joliot Curie
13453 Marseille Cedex 13
France
E-mail address: jhdezhdez@gmail.com

Ferrán Valdez:
Centro de Ciencias Matemáticas
UNAM
Campus Morelia
58190, Morelia, Michoacán
México
E-mail address: ferran@matmor.unam.mx

Primera versió rebuda el 2 de febrer de 2015,
darrera versió rebuda el 4 de maig de 2016.