Publ. Mat. **62** (2018), 641–649
DOI: 10.5565/PUBLMAT6221809

A CHARACTERIZATION OF FINITE MULTIPERMUTATION SOLUTIONS OF THE YANG–BAXTER EQUATION

D. Bachiller, F. Cedó, and L. Vendramin

Abstract: We prove that a finite non-degenerate involutive set-theoretic solution (X,r) of the Yang–Baxter equation is a multipermutation solution if and only if its structure group $G(X,r)$ admits a left ordering or equivalently it is poly-$\mathbb{Z}$.

2010 Mathematics Subject Classification: Primary: 16T25, 20F16, 20F60.

Key words: Yang–Baxter equation, set-theoretic solution, brace, ordered groups, poly-(infinite cyclic) group.

Introduction

According to Drinfeld [**9**], a set-theoretic solution of the Yang–Baxter equation is a pair (X,r), where X is a set and $r\colon X\times X\to X\times X$ is a bijective map such that

$$(r\times \mathrm{id}_X)(\mathrm{id}_X\times r)(r\times \mathrm{id}_X)=(\mathrm{id}_X\times r)(r\times \mathrm{id}_X)(\mathrm{id}_X\times r).$$

The seminal papers [**10**] and [**15**] initiated the study of non-degenerate involutive set-theoretic solutions of the Yang–Baxter equation. Etingof, Schedler, and Soloviev introduced the structure group $G(X,r)$ of a solution (X,r) as the group presented with set of generators X and with relations $xy=uv$ whenever $r(x,y)=(u,v)$. This group turned out to be very important to understand set-theoretic solutions. As proved by Gateva-Ivanova and Van den Bergh, the structure group $G(X,r)$ of a finite non-degenerate involutive set-theoretic solution of the Yang–Baxter equation is a Bieberbach group, i.e. a finitely generated torsion-free abelian-by-finite group.

In [**10**] multipermutation solutions were introduced. This is an important notion that was intensively studied [**2, 3, 4, 5, 12, 14, 20**]. In [**17**, Proposition 4.2] Jespers and Okniński proved that the structure

The two first-named authors are partially supported by grants FEDER-MINECO MTM2014-53644-P and MTM2017-83487-P. The third-named author is partially supported by PICT-2014-1376, MATH-AmSud 17MATH-01, ICTP, ERC advanced grant 320974, and the Alexander von Humboldt Foundation.

group of a finite multipermutation solution is poly-$\mathbb{Z}$. The main result of this paper is to prove the converse: a finite solution (X, r) such that $G(X, r)$ is poly-$\mathbb{Z}$ is a multipermutation solution.

To prove our result we use the language of braces introduced by Rump in [**19**]. Braces are algebraic structures that generalize radical rings. This fact allows us to use tools and techniques from ring theory to study set-theoretic solutions of the Yang–Baxter equation.

In [**11**, Theorem 23] Farkas proved that a Bieberbach group is poly-$\mathbb{Z}$ if and only if it admits a left ordering. Since Chouraqui proved in [**6**, Theorem 1] that the structure group of a finite non-degenerate involutive set-theoretic solution of the Yang–Baxter equation is a Garside group, our result in particular yields an infinite family of Garside groups that are not left orderable.

1. Preliminaries

A *set-theoretic solution* of the Yang–Baxter equation is a pair (X, r), where X is a set and $r\colon X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r).$$

A solution (X, r) is said to be *involutive* if $r^2 = \mathrm{id}_{X^2}$ and it is said to be *non-degenerate* if

$$r(x, y) = (\sigma_x(y), \gamma_y(x)),$$

where σ_x and γ_x are permutations of X for all $x \in X$. The *structure group* $G(X, r)$ of a non-degenerate solution (X, r) is defined as the group presented with set of generators X and with relations $xy = uv$ whenever $r(x, y) = (u, v)$. In [**6**, Theorem 1] Chouraqui proved that the structure group of a non-degenerate involutive set-theoretic solution of the Yang–Baxter equation is a Garside group. A simpler proof of this result was recently given by Dehornoy in [**8**].

Example 1.1. Let X be a conjugacy class of a finite group G such that the subgroup generated by X is non-abelian. Then the map

$$r\colon X \times X \to X \times X, \quad r(x, y) = (xyx^{-1}, x),$$

is a non-degenerate solution of the Yang–Baxter equation. This solution is not involutive. We claim that $G(X, r)$ is not a Garside group. Let $G(X, r)$ act on X by conjugation. Then the center of $G(X, r)$ is the kernel of this action and hence it has finite index in $G(X, r)$. This implies that all conjugacy classes of $G(X, r)$ are finite. Thus the derived subgroup of $G(X, r)$ is a finite group by a theorem of Schur [**18**, Theorem 7.57]. In particular, $G(X, r)$ has torsion elements and hence $G(X, r)$ is not a Garside group.

Remark 1.2. In [**6**] it is conjectured that structure groups of finite non-degenerate solutions are Garside groups. Example 1.1 shows that this conjecture is not true.

Convention 1.3. *A* solution (X,r) *of the YBE will always be a non-degenerate involutive set-theoretic solution of the Yang–Baxter equation.*

A left *brace* is an abelian group $(A,+)$ with another group structure with multiplication $A\times A\to A$, $(a,b)\mapsto ab$, such that

$$a(b+c)+a=ab+ac,\quad a,b,c\in A.$$

It is known that in any left brace A the neutral elements of the groups $(A,+)$ and $(A,\cdot)$ coincide. If A is a left brace, then the map $\lambda\colon (A,\cdot)\to \operatorname{Aut}(A,+)$ given by $\lambda_a(b)=ab-a$ is a group homomorphism. It follows from the definition that $ab=a+\lambda_a(b)$ and $a+b=a\lambda_a^{-1}(b)$ for all $a,b\in A$.

An *ideal* I of a left brace A is a normal subgroup I of the multiplicative group of A such that $\lambda_a(y)\in I$ for all $a\in A$ and $y\in I$. The *socle* of a left brace A is defined as the set

$$\operatorname{Soc}(A)=\{a\in A:\lambda_a=\mathrm{id}\}=\{a\in A:a+b=ab\text{ for all }b\in A\}.$$

The socle of A is an ideal of A.

Rump proved that each left brace A produces a solution of the YBE

$$r_A\colon A\times A\to A\times A,\quad r_A(a,b)=(\lambda_a(b),\lambda^{-1}_{\lambda_a(b)}(a)).$$

One of the main results of [**10**] is the following: If (X,r) is a solution of the YBE, then there exists a bijective 1-cocycle $G(X,r)\to\mathbb{Z}^{(X)}$, where $\mathbb{Z}^{(X)}$ is the free abelian group on X. From this it immediately follows that the canonical map $\iota\colon X\to G(X,r)$ is injective. Now using the language of braces the existence of a bijective 1-cocycle can be written as follows: If (X,r) is a solution, there exists a unique left brace structure over the structure group $G(X,r)$ such that the additive group of $G(X,r)$ is isomorphic to $\mathbb{Z}^{(X)}$, the multiplicative structure is that of $G(X,r)$, and such that

$$r_{G(X,r)}(\iota\times\iota)=(\iota\times\iota)r.$$

The *permutation group* $\mathcal{G}(X,r)$ of a solution (X,r) of the YBE is defined as the group generated by $\{\sigma_x : x\in X\}$, where $r(x,y)=(\sigma_x(y),\gamma_y(x))$. It is known that the map $x\mapsto\sigma_x$ extends to a homomorphism of groups $\pi\colon G(X,r)\to\mathcal{G}(X,R)$ such that $\operatorname{Ker}(\pi)=\operatorname{Soc}(G(X,r))$ and therefore $\mathcal{G}(X,r)$ has a unique structure of left brace such that the group isomorphism

$$\overline{\pi}\colon G(X,r)/\operatorname{Soc}(G(X,r))\to\mathcal{G}(X,r)$$

induced by π is an isomorphism of left braces.

Remark 1.4. Let B be a left brace. Using the operation

$$a * b = ab - a - b = (\lambda_a - \mathrm{id})(b), \quad a, b \in B,$$

Rump introduced the series

$$B = B^{(1)} \supseteq B^{(2)} \supseteq B^{(3)} \supseteq \cdots,$$

where $B^{(m+1)} = B^{(m)} * B$ is the additive group generated by

$$\{(\lambda_a - \mathrm{id})(b) : a \in B^{(m)},\, b \in B\}$$

for all $m \geq 1$. As a corollary of [**19**, Proposition 6] Rump proved that each $B^{(m)}$ is an ideal of B. Notice that this corollary refers to right braces.

For any left brace A and a subset $X \subseteq A$ we will denote by $\langle X\rangle$ the subgroup of $(A, \cdot)$ generated by X. Similarly $\langle X\rangle_+$ will denote the subgroup of $(A, +)$ generated by X.

Remark 1.5. Let (X, r) be a finite solution of the YBE and let $G = G(X, r)$. Let m be a positive integer and let $X_1, \dots, X_r$ be the orbits of X under the action of $G^{(m)}$. Then

$$\begin{aligned} G^{(m+1)} &= \langle (\lambda_a - \mathrm{id})(b) : a \in G^{(m)},\, b \in G\rangle_+ \\ &= \langle (\lambda_a - \mathrm{id})(x) : a \in G^{(m)},\, x \in X\rangle_+ \\ &= \langle y - x : x, y \in X_i,\, 1 \leq i \leq r\rangle_+. \end{aligned}$$

The second equality follows from the fact that $(G, +)$ is generated by X and λ_a is an automorphism of $(G, +)$. The third equality is obtained using that $\lambda_a(x) \in X$ for all $x \in X$ and all $a \in G$.

2. Multipermutation solutions

Let (X, r) be a solution of the YBE. Consider the equivalence relation on X given by $x \sim y$ if and only if $\sigma_x = \sigma_y$. The *retraction* of (X, r) is defined as the solution $\mathrm{Ret}(X, r)$ induced by this equivalence relation. One defines recursively $\mathrm{Ret}^{m+1}(X, r) = \mathrm{Ret}(\mathrm{Ret}^m(X, r))$ for all m. A solution (X, r) of the YBE is said to be a *multipermutation* solution of level m if m is the minimal positive integer such that $\mathrm{Ret}^m(X, r)$ has only one element. A solution (X, r) of the YBE is said to be *irretractable* if $\mathrm{Ret}(X, r) = (X, r)$.

Recall that a group G is said to be *poly*-$\mathbb{Z}$ if it has a subnormal series

$$\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that each quotient G_i/G_{i-1} is isomorphic to $\mathbb{Z}$. A group G is said to be *left orderable* if there is a total order $<$ on G such that for any $x, y, z \in G$, $x < y$ implies $zx < zy$.

The main result of the paper is the following theorem.

Theorem 2.1. *Let (X, r) be a finite non-degenerate involutive set-theoretic solution of the Yang–Baxter equation. Then the following statements are equivalent:*

(1) *(X, r) is a multipermutation solution.*
(2) *$G(X, r)$ is left orderable.*
(3) *$G(X, r)$ is poly-$\mathbb{Z}$.*

Proof: Let $G = G(X, r)$ and $\mathcal{G} = \mathcal{G}(X, r)$. By [**15**, Theorem 1.6], G is a Bieberbach group. Hence the equivalence between (2) and (3) follows from [**11**, Theorem 23]. The implication (1) $\Rightarrow$ (3) is [**17**, Proposition 4.2]; see also [**7**] for another proof of (1) $\Rightarrow$ (2). Let us prove (2) $\Rightarrow$ (1). For that purpose let us assume that (X, r) is not a multipermutation solution. By [**13**, Theorem 5.15], the solution (G, r_G) is not a multipermutation solution. This implies that the solution $(\mathcal{G}, r_{\mathcal{G}})$ is not a multipermutation solution. Using [**3**, Proposition 6] one obtains that $G^{(m)} \neq \{0\}$ and $\mathcal{G}^{(m)} \neq \{0\}$ for all m. Since $\mathcal{G}$ is finite, there exists m such that $\mathcal{G}^{(m+1)} = \mathcal{G}^{(m)} \neq 0$. By [**11**, Theorem 23], to prove that G is not left orderable it suffices to prove that the non-trivial subgroup $H = G^{(m+1)}$ of $(G, \cdot)$ has trivial center. Let $z \in Z(H)$. Since $\mathrm{Soc}(G)$ has finite index in G and G is torsion free, without loss of generality we may assume that $z \in \mathrm{Soc}(G)$. Notice that if $h \in H$, then

$$\lambda_h(z) = hz - h = zh - h = z + h - h = z. \tag{2.1}$$

Let $X_1, \dots, X_r$ be the orbits of X under the action of $\mathcal{G}^{(m)}$. These orbits are the orbits of X under the action of $G^{(m)}$ through the map λ. Since $(G, +)$ is the free abelian group with basis X, the element z can be uniquely written as $z = z_1 + \dots + z_r$, where each $z_i \in \langle X_i \rangle_+$. From the uniqueness of the decomposition of z and (2.1) one obtains that $\lambda_h(z_i) = z_i$ for all $i \in \{1, \dots, r\}$ and $h \in H$. Now write each z_i as

$$z_i = \sum_{t \in X_i} n_t t,$$

where each $n_x \in \mathbb{Z}$. Remark 1.5 implies that $\sum_{t \in X_i} n_t = 0$. This decomposition is unique since $(G, +)$ is the free abelian group with basis X. Let $x, y \in X_i$ be such that $x \neq y$. Then there exists $g \in G^{(m)}$ such that $\lambda_g(x) = y$. From $\mathcal{G}^{(m+1)} = \mathcal{G}^{(m)}$ it follows

$$G^{(m)} = G^{(m+1)} + (\mathrm{Soc}(G) \cap G^{(m)}) = H + (\mathrm{Soc}(G) \cap G^{(m)}).$$

Thus $g = g_1 + g_2$, where $g_1 \in H$ and $g_2 \in \mathrm{Soc}(G) \cap G^{(m)}$. Since $g_2 \in \mathrm{Soc}(G)$, $g = g_2 g_1$. Therefore

$$y = \lambda_g(x) = \lambda_{g_2 g_1}(x) = \lambda_{g_2} \lambda_{g_1}(x) = \lambda_{g_1}(x).$$

Since $z_i = \lambda_{g_1}(z_i) = \sum_{t\in X_i} n_t\lambda_{g_1}(t)$, we conclude that $n_x = n_y$. Since $\sum_{t\in X_i} n_t = 0$, it follows that $n_t = 0$ for all $t \in X_i$ and all $i \in \{1,\dots,r\}$. Therefore $z = 0 = 1$ and the result follows. $\square$

Example 2.2. Let $X = \{1,2,3,4,5,6,7,8,9,a,b,c,d,e,f,g\}$ and let

$$\begin{aligned}
\sigma_1 &= \mathrm{id}, & \sigma_2 &= (37)(48)(bf)(cg),\\
\sigma_3 &= (25)(3b4f)(7c8g)(9dea), & \sigma_4 &= (25)(3g4c)(7f8b)(9dea),\\
\sigma_5 &= (38)(47)(bg)(cf), & \sigma_6 &= (34)(78)(bc)(fg),\\
\sigma_7 &= (25)(3c7b)(4g8f)(9dea), & \sigma_8 &= (25)(3f7g)(4b8c)(9dea),\\
\sigma_9 &= (38)(47)(9e)(ad), & \sigma_a &= (34)(78)(9e)(ad)(b,f)(c,g),\\
\sigma_b &= (25)(3f4b)(7g8c)(9aed), & \sigma_c &= (25)(3c4g)(7b8f)(9aed),\\
\sigma_d &= (9e)(ad)(bg)(cf), & \sigma_e &= (37)(48)(9e)(ad)(bc)(fg),\\
\sigma_f &= (25)(3g7f)(4c8b)(9aed), & \sigma_g &= (25)(3b7c)(4f8g)(9aed).
\end{aligned}$$

Then $r(x,y) = (\sigma_x(y), \sigma^{-1}_{\sigma_x(y)}(x))$ is an irretractable solution of the YBE. Hence its structure group $G(X,r)$ is not left orderable. In this case

$$\mathcal{G}(X,r) = \{\sigma_x : x \in X\}.$$

Example 2.3. Let $X = \{1,2,3,4,5,6,7,8,9,a,b,c,d,e,f,g\}$ and let

$$\begin{aligned}
\sigma_1 &= \mathrm{id}, & \sigma_2 &= (9e)(ad)(bg)(cf),\\
\sigma_3 &= (34)(78)(9e)(ad)(bf)(cg), & \sigma_4 &= (34)(78)(bc)(fg),\\
\sigma_5 &= (9a)(bc)(de)(fg), & \sigma_6 &= (9d)(ae)(bf)(cg),\\
\sigma_7 &= (34)(78)(9d)(ae)(bg)(cf), & \sigma_8 &= (34)(78)(9a)(de),\\
\sigma_9 &= (56)(78)(de)(fg), & \sigma_a &= (56)(78)(9dae)(bfcg),\\
\sigma_b &= (34)(56)(9dae)(bgcf), & \sigma_c &= (34)(56)(bc)(de),\\
\sigma_d &= (56)(78)(9ead)(bgcf), & \sigma_e &= (56)(78)(9a)(bc),\\
\sigma_f &= (34)(56)(9a)(fg), & \sigma_g &= (34)(56)(9ead)(bfcg).
\end{aligned}$$

Then $r(x,y) = (\sigma_x(y), \sigma^{-1}_{\sigma_x(y)}(x))$ is an irretractable solution of the YBE. Hence its structure group $G(X,r)$ is not left orderable. In this case

$$\mathcal{G}(X,r) = \{\sigma_x : x \in X\} \simeq \mathbb{Z}/2 \times \mathbb{D}_8,$$

where $\mathbb{D}_8$ denotes the dihedral group of size 8.

Remark 2.4. The solutions of Examples 2.2 and 2.3 correspond to the associated solutions to the only left braces of size 16 with trivial socle. This was checked with GAP and the list of small braces of [**16**].

Example 2.5. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o\}$. Let

$$\begin{aligned}
\sigma_1 &= \mathrm{id},\\
\sigma_2 &= (4ag)(5bh)(6ci)(7jd)(8ke)(9lf),\\
\sigma_3 &= (4ga)(5hb)(6ic)(7dj)(8ek)(9fl),\\
\sigma_4 &= (23)(4ogc)(5nhb)(6mia)(7j)(8l)(9k)(ef),\\
\sigma_5 &= (23)(4cmi)(5bnh)(6aog)(7d)(8f)(9e)(kl),\\
\sigma_6 &= (23)(46)(89)(ac)(dj)(el)(fk)(go)(hn)(im),\\
\sigma_7 &= (4m)(5n)(6o)(ag)(bh)(ci),\\
\sigma_8 &= (4gm)(5hn)(6io)(7jd)(8ke)(9lf),\\
\sigma_9 &= (4am)(5bn)(6co)(7dj)(8ek)(9fl),\\
\sigma_a &= (23)(4cgo)(5bhn)(6aim)(7j)(8l)(9k)(ef),\\
\sigma_b &= (23)(4o)(5n)(6m)(7d)(8f)(9e)(ac)(gi)(kl),\\
\sigma_c &= (23)(4iao)(5hbn)(6gcm)(89)(dj)(el)(fk),\\
\sigma_d &= (4a)(5b)(6c)(gm)(hn)(io),\\
\sigma_e &= (7jd)(8ke)(9lf)(amg)(bnh)(coi),\\
\sigma_f &= (4mg)(5nh)(6oi)(7dj)(8ek)(9fl),\\
\sigma_g &= (23)(46)(7j)(8l)(9k)(ao)(bn)(cm)(ef)(gi),\\
\sigma_h &= (23)(4imc)(5hnb)(6goa)(7d)(8f)(9e)(kl),\\
\sigma_i &= (23)(4oai)(5nbh)(6mcg)(89)(dj)(el)(fk),\\
\sigma_j &= (4g)(5h)(6i)(am)(bn)(co),\\
\sigma_k &= (4ma)(5nb)(6oc)(7jd)(8ke)(9lf),\\
\sigma_l &= (7dj)(8ek)(9fl)(agm)(bhn)(cio),\\
\sigma_m &= (23)(4i)(5h)(6g)(7j)(8l)(9k)(ac)(ef)(mo),\\
\sigma_n &= (23)(46)(7d)(8f)(9e)(ai)(bh)(cg)(kl)(mo),\\
\sigma_o &= (23)(4c)(5b)(6a)(89)(dj)(el)(fk)(gi)(mo).
\end{aligned}$$

Then $r(x, y) = (\sigma_x(y), \sigma^{-1}_{\sigma_x(y)}(x))$ is an irretractable solution of the YBE. Hence its structure group $G(X, r)$ is not left orderable. In this case

$$\mathcal{G}(X, r) = \{\sigma_x : x \in X\} \simeq \mathbb{S}_4.$$

The unique subgroup of index two of $\mathcal{G}(X, r)$ is an ideal of $\mathcal{G}(X, r)$. Hence $\mathcal{G}(X, r)$ is not a simple brace.

Remark 2.6. There are two braces of size 24 with trivial socle. One is the simple brace found in [**1**] and the other one is that of Example 2.5.

References

[1] D. Bachiller, Extensions, matched products, and simple braces, *J. Pure Appl. Algebra* **222(7)** (2018), 1670–1691. DOI: 10.1016/j.jpaa.2017.07.017.

[2] D. Bachiller, F. Cedó, E. Jespers, and J. Okniński, A family of irretractable square-free solutions of the Yang–Baxter equation, *Forum Math.* **29(6)** (2017), 1291–1306. DOI: 10.1515/forum-2015-0240.

[3] F. Cedó, T. Gateva-Ivanova, and A. Smoktunowicz, On the Yang–Baxter equation and left nilpotent left braces, *J. Pure Appl. Algebra* **221(4)** (2017), 751–756. DOI: 10.1016/j.jpaa.2016.07.014.

[4] F. Cedó, E. Jespers, and J. Okniński, Retractability of set theoretic solutions of the Yang–Baxter equation, *Adv. Math.* **224(6)** (2010), 2472–2484. DOI: 10.1016/j.aim.2010.02.001.

[5] F. Cedó, E. Jespers, and J. Okniński, Braces and the Yang–Baxter equation, *Comm. Math. Phys.* **327(1)** (2014), 101–116. DOI: 10.1007/s00220-014-1935-y.

[6] F. Chouraqui, Garside groups and Yang–Baxter equation, *Comm. Algebra* **38(12)** (2010), 4441–4460. DOI: 10.1080/00927870903386502.

[7] F. Chouraqui, Left orders in Garside groups, *Internat. J. Algebra Comput.* **26(7)** (2016), 1349–1359. DOI: 10.1142/S0218196716500570.

[8] P. Dehornoy, Set-theoretic solutions of the Yang–Baxter equation, RC-calculus, and Garside germs, *Adv. Math.* **282** (2015), 93–127. DOI: 10.1016/j.aim.2015.05.008.

[9] V. G. Drinfeld, On some unsolved problems in quantum group theory, in: *"Quantum Groups"* (Leningrad, 1990), Lecture Notes in Math. **1510**, Springer, Berlin, 1992, pp. 1–8. DOI: 10.1007/BFb0101175.

[10] P. Etingof, T. Schedler, and A. Soloviev, Set-theoretical solutions to the quantum Yang–Baxter equation, *Duke Math. J.* **100(2)** (1999), 169–209. DOI: 10.1215/S0012-7094-99-10007-X.

[11] D. R. Farkas, Crystallographic groups and their mathematics, *Rocky Mountain J. Math.* **11(4)** (1981), 511–551. DOI: 10.1216/RMJ-1981-11-4-511.

[12] T. Gateva-Ivanova, A combinatorial approach to the set-theoretic solutions of the Yang–Baxter equation, *J. Math. Phys.* **45(10)** (2004), 3828–3858. DOI: 10.1063/1.1788848.

[13] T. GATEVA-IVANOVA, Set-theoretic solutions of the Yang–Baxter equation, braces, and symmetric groups, Preprint (2015). arXiv: 1507.02602.

[14] T. GATEVA-IVANOVA AND P. CAMERON, Multipermutation solutions of the Yang–Baxter equation, *Comm. Math. Phys.* **309(3)** (2012), 583–621. DOI: 10.1007/s00220-011-1394-7.

[15] T. GATEVA-IVANOVA AND M. VAN DEN BERGH, Semigroups of I-type, *J. Algebra* **206(1)** (1998), 97–112. DOI: 10.1006/jabr.1997.7399.

[16] L. GUARNIERI AND L. VENDRAMIN, Skew braces and the Yang–Baxter equation, *Math. Comp.* **86(307)** (2017), 2519–2534. DOI: 10.1090/mcom/3161.

[17] E. JESPERS AND J. OKNIŃSKI, Monoids and groups of I-type, *Algebr. Represent. Theory* **8(5)** (2005), 709–729. DOI: 10.1007/s10468-005-0342-7.

[18] J. J. ROTMAN, *"An Introduction to the Theory of Groups"*, Fourth edition, Graduate Texts in Mathematics **148**, Springer-Verlag, New York, 1995. DOI: 10.1007/978-1-4612-4176-8.

[19] W. RUMP, Braces, radical rings, and the quantum Yang–Baxter equation, *J. Algebra* **307(1)** (2007), 153–170. DOI: 10.1016/j.jalgebra.2006.103.040.

[20] L. VENDRAMIN, Extensions of set-theoretic solutions of the Yang–Baxter equation and a conjecture of Gateva-Ivanova, *J. Pure Appl. Algebra* **220(5)** (2016), 206–2076. DOI: 10.1016/j.jpaa.2015.10.018.

D. Bachiller and F. Cedó:
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
Spain
E-mail address: dbachiller@mat.uab.cat
E-mail address: cedo@mat.uab.cat

L. Vendramin:
IMAS-CONICET and Departamento de Matemática
FCEN, Universidad de Buenos Aires
Pabellón 1, Ciudad Universitaria
C1428EGA, Buenos Aires
Argentina
E-mail address: lvendramin@dm.uba.ar

Primera versió rebuda el 31 de gener de 2017,
darrera versió rebuda el 28 de juny de 2017.