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AUTOMORPHISMS OF CERTAIN NIEMEIER LATTICES AND ELLIPTIC FIBRATIONS

MARIE JOSÉ BERTIN

Université Pierre et Marie Curie (Paris 6),
Institut de Mathématiques, 4 Place Jussieu,
75005 PARIS, France

ODILE LECACHEUX

Université Pierre et Marie Curie (Paris 6),
Institut de Mathématiques, 4 Place Jussieu,
75005 PARIS, France

Abstract. Nishiyama introduced a lattice theoretic classification of the elliptic fibrations on a $K3$ surface. In a previous paper we used his method to exhibit 52 elliptic fibrations, up to isomorphisms, of the singular $K3$ surface of discriminant -12. We prove here that the list is complete with a 53th fibration, thanks to a remark of Elkies and Schütt. We characterize the fibration both theoretically and with a Weierstrass model.

Keywords: K3 surfaces, Niemeier lattices, Elliptic fibrations

1. Introduction

In a previous paper [BGHLMSW], the authors gave a classification, up to automorphisms, of the elliptic fibrations on the singular $K3$ surface X whose transcendental lattice is isometric to $\langle 6\rangle \oplus \langle 2\rangle$. This classification was derived from the Kneser-Nishiyama method. Each elliptic fibration was given with the Dynkin diagrams characterizing its reducible fibers, the rank and torsion of its Mordell-Weil group. Hence 52 elliptic fibrations were obtained.

Later on, Elkies and Schütt informed us that we missed an elliptic fibration. More precisely, Elkies said how he discovered the lack [El]: "while tabulating some information about the lattices in this genus (positive-definite even lattice of rank 18 and discriminant 12)... I had already done the smaller discriminants), including

E-mail addresses: marie-jose.bertin@imj-prg.fr, odile.lecacheux@imj-prg.fr.
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the sizes of their automorphism groups, and calculated their total mass (=sum of $1/|\mathrm{Aut}(G)|$) which added up to less than the prediction of the mass formula. The discrepancy was a fraction $1/N$ so I guessed that just one lattice, with N automorphisms, was missing, and eventually figured out where I lost the 53rd lattice."

This paper intends to complete the gap.

Let us recall briefly the context. Given $\mathcal{E}$ an elliptic fibration on X, we define its trivial lattice by $\mathrm{T}(\mathcal{E}) := U \oplus (W_{\mathcal{E}})_{\mathrm{root}}$ where $W_{\mathcal{E}}$ denotes its frame lattice, that is the orthogonal complement of U in the Neron-Severi group $NS(X)$. The Mordell-Weil group of $\mathcal{E}$ is encoded in the frame

$$MW(\mathcal{E}) = W_{\mathcal{E}}/(W_{\mathcal{E}})_{\mathrm{root}}. \tag{1}$$

Thus

$$\mathrm{rk}(MW(\mathcal{E})) = \mathrm{rk}(W_{\mathcal{E}}) - \mathrm{rk}(W_{\mathcal{E}})_{\mathrm{root}} \qquad (MW(\mathcal{E}))_{\mathrm{tors}} = \overline{(W_{\mathcal{E}})_{\mathrm{root}}}/(W_{\mathcal{E}})_{\mathrm{root}}. \tag{2}$$

The Kneser-Nishiyama's method provides a determination of the frame. Starting from the transcendental lattice of X

$$T_X = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix},$$

denote T the root lattice $T = A_5 \oplus A_1$, orthogonal complement of $T_X(-1)$ in the root lattice E_8. Take a Niemeier lattice L, that is a unimodular lattice of rank 24, with root lattice L_{root}, often written $L = N(L_{\mathrm{root}})$. Consider a primitive embedding $\phi : T \hookrightarrow L$. The orthogonal complement of $\phi(T)$ in L is the frame of an elliptic fibration on X and since T is a root lattice [BGHLMSW], it suffices to consider all the primitive embeddings of T in L_{root} to obtain all the elliptic fibrations on X. Denote

$$W = (\phi(A_5 \oplus A_1))^{\perp_L} \qquad \text{and} \qquad N = (\phi(A_5 \oplus A_1))^{\perp_{L_{\mathrm{root}}}}$$

and observe that $W_{\mathrm{root}} = N_{\mathrm{root}}$. Moreover the trivial lattice of the elliptic fibration provided by ϕ satisfies $T(\mathcal{E}) = U \oplus W_{\mathrm{root}}$ and we can apply formulae (1) and (2).

Now given two points P and Q of the Mordell-Weil group, we can define a height pairing. The Mordell-Weil group, up to its torsion subgroup, equipped with this height pairing, is the Mordell-Weil lattice $MWL(X)$ which satisfies

$$MWL(X) = W/\overline{W_{\mathrm{root}}}.$$

Thus we recover more than the rank and torsion but also torsion and infinite sections of the elliptic fibration.

To list all the primitive embeddings of $A_5 \oplus A_1$ in the various Niemeier lattices, the authors of [BGHLMSW] used Nishiyama's tables [Nis] p.309 and p.323. They noticed two primitive embeddings of A_5 in D_6, not isomorphic by the Weyl group of D_6, namely

$$i_1(A_5) = (d_5, d_4, d_3, d_2, d_1) \qquad \text{and} \qquad i_2(A_5) = (d_6, d_4, d_3, d_2, d_1)$$

but p.323, Nishiyama missed the orthogonal complement in D_6 of $i_1(A_5)$. That is the origin of the gap which concerns the primitive embeddings of $A_5 \oplus A_1$ in $L = N(D_6^4)$ and $L = N(A_9^2 D_6)$.

The paper is divided in two parts. In the first part we prove that the two primitive embeddings of A_5 in D_6 give two primitive embeddings of $A_5 \oplus A_1$ in $N(D_6^4)$ isomorphic by an element of $\mathrm{Aut}(N(D_6^4))$ so lead to just one elliptic fibration up to isomorphism. On the contrary, these embeddings i_1 and i_2 give rise to two non isomorphic primitive embeddings in $N(A_9^2 D_6)$ thus exactly to two elliptic fibrations

and not only one as listed in [BGHLMSW]. Hence we obtain the 53th fibration denoted by #40 bis. We also explain the determination of the Mordell-Weil lattices.

In the second part we show how to derive the corresponding elliptic fibrations from the fibration #50 of [BGHLMSW] with Weierstrass equation (10) and its associated graph. We set also the correspondence between the results found in the first part of the paper and those coming from the graph.

Acknowledgement 1. *We are grateful to N. Elkies and M. Schütt for pointing out a missing fibration in the classification* [BGHLMSW].

2. Some facts concerning Niemeier lattices and their automorphisms

Concerning the definitions and properties of the irreducible root lattices A_n, D_n, E_n and their dual lattices we refer to [BGHLMSW] or [BL] and use Bourbaki's notations, as in the Dynkin diagram of D_6 (see section 3).

Let L a Niemeier lattice i.e. a unimodular lattice of rank 24. We define its root lattice $L_{\rm root} = \{\alpha \in L / < \alpha, \alpha >= -2\}$ where $< ., . >$ denotes the $\mathbb{Z}$-bilinear form on L. We recall that a Niemeier lattice L is, up to an isomorphism, entirely determined by its root lattice $L_{\rm root}$; thus it is denoted $L = N(L_{\rm root})$. It can be realized as a sublattice of the dual lattice $(L_{\rm root})^*$ of $L_{\rm root}$. Thus $N(L_{\rm root})/L_{\rm root}$ is a finite abelian group, called the "glue code" or the set of "glue vectors". Writing $L_{\rm root} = L_1 \oplus L_2 ... \oplus L_k$ where the L_i are irreducible root lattices of type A_n, D_n or E_n, a typical glue vector of L can be written [CS],

$$z = [y_1, y_2, ..., y_k] \tag{3}$$

where y_i is a member of the dual lattice L_i^*. Any y_i can be altered by adding a vector of L_i so we may suppose that y_i belongs to a standard system of representatives for the cosets of L_i in L_i^*. It is usual to choose the glue vectors to be of minimal length in their cosets.

The various vectors z of (3) must have integral inner products with each other and be closed under addition modulo $L_1 \oplus ... \oplus L_k$. This process is called "gluing" the components L_1, ... L_k.

2.1. The automorphism group $\mathbf{Aut}(L_{\mathbf{root}})$.

In the sequel we denote $X \rtimes Y$ a split extension of a group Y by a group X. We recall that

$$\mathrm{Aut}(L_m) = W(L_m) \rtimes G_1(L_m)$$

where $W(L_m)$ is the Weyl group of L_m and $G_1(L_m)$ the subgroup of $\mathrm{Aut}(L_m)$ consisting of all Dynkin diagram automorphisms of L_m.

Set $G_0(L_{\rm root}) := \prod_{m=1}^{k} W(L_m)$, $\quad G_1(L_{\rm root}) := \prod_{m=1}^{k} G_1(L_m)$ and $K(L_{\rm root})$ the following subgroup of $\mathrm{Aut}(L_{\rm root})$

$$K(L_{\rm root}) := \{\tau \in \mathrm{Aut}(L_{\rm root}) / \tau(L_m) = L_m \ \ \forall\ m,\ \ 1 \le m \le k\}.$$

The group $G_0(L_{\rm root})$ is called the Weyl group of $L_{\rm root}$ and is a normal subgroup of $K(L_{\rm root})$. The group $G_1(L_{\rm root})$ is a subgroup of $K(L_{\rm root})$ and we have the relation

$$K(L_{\rm root}) = \prod_{m=1}^{k} \mathrm{Aut}(L_m) = G_0(L_{\rm root}) \rtimes G_1(L_{\rm root}).$$

For each $1 \le i < j \le k$ such that $L_i \simeq L_j$, denote t_{ij} the transposition between the entries i and j and set

$$G_2(L_{\rm root}) := \langle t_{ij} / 1 \le i < j \le k \quad L_i \simeq L_j \rangle$$

the subgroup of $\mathrm{Aut}(L_{\mathrm{root}})$ of all permutations of the concerned entries. Finally we get

$$\mathrm{Aut}(L_{\mathrm{root}}) = K(L_{\mathrm{root}}) \rtimes G_2(L_{\mathrm{root}}) = (G_0(L_{\mathrm{root}}) \rtimes G_1(L_{\mathrm{root}})) \rtimes G_2(L_{\mathrm{root}}).$$

2.2. **The automorphism group Aut(L).** Since the spanning set $\Delta = \{\alpha \in L/\ \langle \alpha, \alpha\rangle = -2\}$ of L_{root} is stable under the action of $\mathrm{Aut}(L)$, it follows that L_{root} is stable under $\mathrm{Aut}(L)$ and we get a group homomorphism

$$\begin{array}{ccc} \mathrm{Aut}(L) & \to & \mathrm{Aut}(L_{\mathrm{root}}) \\ \tau & \mapsto & \tau|_{L_{\mathrm{root}}} \end{array}.$$

Set $G_0(L) := G_0(L_{\mathrm{root}})$; it is a normal subgroup of $\mathrm{Aut}(L)$. Define the subgroup of $\mathrm{Aut}(L)$, $G_1(L) := \mathrm{Aut}(L) \cap G_1(L_{\mathrm{root}})$. They satisfy the relation

$$K(L_{\mathrm{root}}) \cap \mathrm{Aut}(L) = G_0(L) \rtimes G_1(L).$$

Defining the subgroup $H(L)$ of $\mathrm{Aut}(L)$ by $H(L) := \mathrm{Aut}(L) \cap (G_1(L_{\mathrm{root}}) \rtimes G_2(L_{\mathrm{root}}))$, it follows $\mathrm{Aut}(L) = G_0(L) \rtimes H(L)$. Define the subgroup $G_2(L)$ of $G_2(L_{\mathrm{root}})$ by

$$G_2(L) := \{\tau \in G_2(L_{\mathrm{root}})/\tau_1\tau \in H(L) \ \text{ for some } \tau_1 \in G_1(L_{\mathrm{root}})\}.$$

From this definition we get a surjective homomorphism π_2

$$\begin{array}{rccc} \pi_2 : & H(L) & \to & G_2(L) \\ & \tau & \mapsto & \tau_2 \end{array}$$

and the exact sequence

$$1 \to G_1(L) \to H(L) \to G_2(L) \to 1. \tag{4}$$

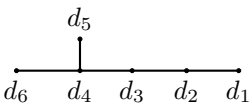
Because $\mathrm{Aut}(L)$ is a subgroup of $\mathrm{Aut}(L_{\mathrm{root}})$, we get the induced action of $\mathrm{Aut}(L)$ on the "glue code" L/L_{root}. Moreover this action is the identity if and only if the element τ of $\mathrm{Aut}(L)$ belongs to $G_0(L)$. Finally we observe that $H(L)$ is identical to the subgroup of $G_0(L_{\mathrm{root}}) \rtimes G_1(L_{\mathrm{root}})$ consisting of the elements preserving the "glue code".

For more details explaining how $\mathrm{Aut}(L)$ is obtained from $\mathrm{Aut}(L_{\mathrm{root}})$ and how we can construct an automorphism of L, we refer to [IS1] and [IS2].

3. The Niemeier lattice $N(D_6^4)$

Recall first the glue vectors of D_6. They are denoted [0], [1], [2], [3] by Conway and Sloane [CS] and δ_6, $\bar{\delta_6}$, $\tilde{\delta_6}$ in [BGHLMSW] with the following correspondence

$$\begin{array}{lll} [1] & = \delta_6 & = \frac{1}{2}(d_1 + 2d_2 + 3d_3 + 4d_4 + 2d_5 + 3d_6) \\ [2] & = \bar{\delta_6} & = d_1 + d_2 + d_3 + d_4 + \frac{1}{2}(d_5 + d_6) \\ [3] & = \tilde{\delta_6} & = \frac{1}{2}(d_1 + 2d_2 + 3d_3 + 4d_4 + 3d_5 + 2d_6), \end{array}$$



and satisfy $[1] + [3] = [2]$.

Also $\mathrm{Aut}(D_6) = W(D_6) \rtimes G_1(D_6)$ with $G_1(D_6) \simeq \mathbb{Z}/2\mathbb{Z}$ which interchanges the glue vectors [1] and [3].

Moreover

$$N(D_6^4) = \mathbb{Z}\{D_6 \oplus D_6 \oplus D_6 \oplus D_6, \text{glue code}\}.$$

The glue code, i.e. the set of glue vectors is given by all the even permutations of $[0,1,2,3]$ where i denotes, by abuse of notation, the glue vector $[i]$. Thus $\mathcal{A}_4$ is contained in $\mathrm{Aut}(N(D_6^4))$. More explicitly the glue code is

$$\begin{array}{cccc} [0,0,0,0], & [0,1,2,3], & [0,3,1,2], & [0,2,3,1], \\ [1,1,1,1], & [1,0,3,2], & [1,3,2,0], & [1,2,0,3], \\ [2,2,2,2], & [2,0,1,3], & [2,3,0,1], & [2,1,3,0], \\ [3,3,3,3], & [3,0,2,1], & [3,1,0,2], & [3,2,1,0]. \end{array} \tag{5}$$

Lemma 1. *Up to an isomorphism of the Weyl group $W(D_6)$, there are two primitive embeddings of A_5 in D_6, namely*

$$i_1(A_5) = (d_5, d_4, d_3, d_2, d_1)$$
$$i_2(A_5) = (d_6, d_4, d_3, d_2, d_1).$$

These two embeddings are interchanged by the element $g \in G_1(D_6)$ interchanging d_5 and d_6. Moreover g acts on the glue vectors of D_6:

$$g([1]) = [3], \quad g([2]) = [2], \quad g([3]) = [1].$$

Proof. It follows straightforward from the definitions. □

Theorem 1. *Let y be any glue vector of $N(D_6^4)$, $y = [a,b,c,d]$. Define the application of g on the glue code as $g(y) = [g(a), g(b), g(c), g(d)]$. Denote by τ any transposition of two components. Then $\tau \circ g \in Aut(N(D_6^4))$.*

Proof. Consider any permutation of two elements, for example take for τ the transposition of the two last components. Observe first that τ and g commute; it follows $(\tau \circ g)^2 = \mathrm{Id}$. This allows us to present the action of $\tau \circ g$ on the glue code as below:

$$\begin{array}{ccccccc} [0,0,0,0] & [0,1,2,3] & [0,2,3,1] & [1,1,1,1] & [1,3,2,0] \\ \tau\circ g \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ [0,0,0,0] & [0,3,1,2] & [0,2,3,1] & [3,3,3,3] & [3,1,0,2] \end{array}$$

$$\begin{array}{ccccccc} [2,2,2,2] & [2,0,1,3] & [2,1,3,0] & [1,0,3,2] & [1,2,0,3] \\ \tau\circ g \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ [2,2,2,2] & [2,0,1,3] & [2,3,0,1] & [3,0,2,1] & [3,2,1,0]. \end{array}$$

Since $\tau \circ g$ is bijective on the glue code it belongs to $\mathrm{Aut}(N(D_6^4))$. The same conclusion is obtained if τ is an arbitrary transposition.

Remark 1. *The well-known isomorphism $G_1(N(D_6^4)) \rtimes G_2(N(D_6^4)) \simeq \mathcal{S}_4$* [CS] *can be given explicitly as*

$$\begin{array}{ccc} \mathcal{S}_4 & \to & Aut(N(D_6^4)) \\ \sigma & \mapsto & \sigma \circ g^{e(\sigma)} \end{array}$$

where $e(\sigma) = 0$ if σ is even and 1 otherwise.

Remark 2. *Moreover if τ permutes the two last components, $\tau \circ g$ fixes the glue vectors having their two first components made with 0 or 2, permutes the glue vectors beginning by 0 on one side and the glue vectors beginning by 2 on the other side; also it transforms the glue vectors beginning by 1 into the glue vectors beginning by 3.*

□

Corollary 1. *The two primitive embeddings of $A_5 \oplus A_1$ in $N(D_6^4)$ given by $(i_1(A_5), d_6, 0, 0)$ and $(i_2(A_5), d_5, 0, 0)$ are isomorphic by an element of $Aut(N(D_6^4))$.*

Proof. We take for τ the transposition of the two last components. We get that $\tau \circ g$ interchanges the two embeddings and by the previous theorem belongs to $\mathrm{Aut}(N(D_6^4))$. □

4. The Niemeier lattice $N(A_9^2D_6)$

Aside the glue vectors of D_6 defined in the previous section, the glue group of A_9 is cyclic, generated by α, see for example [BL] or [CS]:

$$\alpha = \frac{1}{10}[9a_1 + 8a_2 + 7a_3 + 6a_4 + 5a_5 + 4a_6 + 3a_7 + 2a_8 + a_9].$$

By abuse of notation we write 1 for the class of α in A_9^*/A_9 and more generally i for the class of $i\alpha$. We recall that

$$\mathrm{Aut}(A_9) = W(A_9) \rtimes G_1(A_9),$$

where $W(A_9)$ denotes the Weyl group and $G_1(A_9)$ consists in the automorphisms of the Dynkin diagram of A_9 forming a group of order 2 exchanging a_i and a_{10-i} for all $1 \leq i \leq 9$ and therefore i and $10-i$ according to the above convention. This automorphism acting on the first (resp. second) factor A_9 of L_{root} will be denoted γ_1 (resp. γ_2). It follows

$$G_1(L_{\mathrm{root}}) = G_1(A_9^{(1)}A_9^{(2)}D_6) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
$$G_2(L_{\mathrm{root}}) \simeq \mathbb{Z}/2\mathbb{Z} = \langle h \rangle,$$

where h exchanges the two copies of A_9.

Set $\gamma = \gamma_1\gamma_2$, $h_1 = \gamma_1 g$ and $h_2 = \gamma_2 g$.

Proposition 1. (1) *The subgroup $G_1(L) = Aut(L) \cap G_1(L_{root}) \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by γ.*

(2) *The automorphism h of $G_2(A_9^{(1)}A_9^{(2)}D_6)$ is an automorphism of $G_2(L) = G_2(N(A_9^{(1)}A_9^{(2)}D_6))$; moreover h_1h and h_2h belong to $Aut(L)$. Hence the subgroup $G_2(L)$ is generated by h.*

(3) *The subgroup $H(L) = (G_1(L_{root}) \rtimes G_2(L_{root})) \cap Aut(L)$ is generated by h_1h and h_2h.*

Proof. Recall, [CS], that the glue code is generated by

$$[2, 4, 0], \quad [5, 0, 1], \quad [0, 5, 3],$$

and that $G_1(L) \simeq \mathbb{Z}/2\mathbb{Z}$ and $G_2(L) \simeq \mathbb{Z}/2\mathbb{Z}$.

(1) We verify that γ belongs to $G_1(L_{\mathrm{root}})$, preserves the glue code and is of order 2.

(2) According to 2.2, it suffices to exhibit an element $h_1 \in G_1(L_{\mathrm{root}})$ such that $h_1h \in (G_1(L_{\mathrm{root}}) \rtimes G_2(L_{\mathrm{root}})) \cap \mathrm{Aut}(L)$, i.e. preserving the glue code of L. We verify easily $h_1h([2, 4, 0]) = [6, 2, 0] = 3 \times [2, 4, 0]$, $h_1h([5, 0, 1]) = [0, 5, 3]$ and $h_1h([0, 5, 3]) = [5, 0, 1]$. Thus $h \in \mathrm{Aut}(L)$ and generates $G_2(L)$ since $G_2(L) \simeq \mathbb{Z}/2\mathbb{Z}$.

(3) This follows from the previous item and the isomorphisms $G_1(L) \simeq \mathbb{Z}/2\mathbb{Z}$ and $G_2(L) \simeq \mathbb{Z}/2\mathbb{Z}$ [CS].

□

TABLE 1. Contributions for the height pairing

fiber	IV^*	III^*	I_n $n>1$	I_n^*
Dynkin diagram	E_6	E_7	A_{n-1}	D_{n+4}
$i=j$	$4/3$	$3/2$	$i(n-i)/n$	$\begin{cases} 1 & i=1 \\ 1+n/4 & i=2,3 \end{cases}$
$i<j$	$2/3$	–	$i(n-j)/n$	$\begin{cases} 1/2 & i=1 \\ 1/2+n/4 & i=2,3 \end{cases}$

Corollary 2. *The two primitive embeddings of A_5 in D_6, namely i_1 and i_2, correspond to at most two elliptic fibrations of X, non isomorphic by an automorphism of $N(A_9^{(1)}A_9^{(2)}D_6)$.*

Proof. From the proposition we deduce that the fibration obtained with the embeddings $A_1 = a_1$ in $A_9^{(1)}$ and $i_1(A_5)$ in D_6 is isomorphic by the automorphism h_2h to $A_1 = a_9$ embedded in $A_9^{(2)}$ and $i_2(A_5)$ in D_6. Similarly, the fibration obtained with the embeddings $A_1 = a_1$ in $A_9^{(2)}$ and $i_1(A_5)$ in D_6 is isomorphic by the automorphism of h_1h to $A_1 = a_9$ embedded in $A_9^{(1)}$ and $i_2(A_5)$ in D_6.

□

5. FROM PRIMITIVE EMBEDDINGS TO MORDELL-WEIL LATTICES

Let X the $K3$-surface of discriminant -12 studied in [BGHLMSW]. To each primitive embedding of $A_5 \oplus A_1$ in L_{root} for L Niemeier lattice, corresponds an elliptic fibration of X. Define $W = (A_5 \oplus A_1)^{\perp_L}$ and $N = (A_5 \oplus A_1)^{\perp_{L_{\text{root}}}}$. First observe that $W_{\text{root}} = N_{\text{root}}$. Then the configuration of singular fibers in the corresponding elliptic fibration is encoded in the trivial lattice $T(X)$ of the elliptic fibration given by

$$T(X) = U \oplus W_{\text{root}}.$$

The torsion group is given by $\overline{W_{\text{root}}}/W_{\text{root}}$.

The Mordell-Weil lattice $MWL(X)$, that is the Mordell-Weil group modulo its torsion subgroup equipped with the height pairing is given by

$$MWL(X) = W/\overline{W_{\text{root}}},$$

where the bar means the primitive closure. The height pairing of two points P and Q of the Mordell-Weil group is given by the Shioda's formulae

$$\langle P, Q\rangle = 2 + \bar{P}.\bar{O} + \bar{Q}.\bar{O} - \bar{P}.\bar{Q} - \sum_v \text{contr}_v(P,Q) \tag{6}$$

and the height of P by

$$h(P) = \langle P, P\rangle = 4 + 2\bar{P}.\bar{O} - \sum_v \text{contr}_v(P) \tag{7}$$

where O denotes the zero, the bar their associated sections and v runs through the singular fibers. If $\Theta_{v,i}$ is a component of the singular fiber Θ_v and if P (resp. Q) intersects $\Theta_{v,i}$ (resp. $\Theta_{v,j}$), $i<j$, we recall the table of their contributions, Table 1.

Recall that the single components of an I_n^* fiber, $n > 0$, are distinguished into the near component Θ_1 which intersects the same double component as the zero component and the far components Θ_2, Θ_3.

5.1. **Defining sections of our fibrations.** In each class of W/N we choose a representative in order to form either a torsion or an infinite section of the fibration. The section V is defined as

$$V = kF + mO + \omega,$$

F being the generic fiber, O the zero section, ω a well chosen glue vector in a coset of W/N. Since V has to satisfy $V.F = 1$, it follows $m = 1$. The rational integer k can be obtained from the relation $V.V = -2$, since $\omega.\omega$ is even. Finally the glue vector ω is chosen so that V cuts each singular fiber in exactly one point. Then we test if the section cuts or not the zero section in order to apply the height formula (7). Sections with height 0 are torsion sections. Moreover we have to determine infinite sections with a height matrix giving the discriminant of the $K3$ surface, that is in our case 12, according to the formula [ScSh]

$$\text{disc}(NS(X)) = (-1)^{\text{rank}E(K)}\text{disc}(T(X))\text{disc}(MWL(X))/(\#E(K)_{\text{tors}})^2. \tag{8}$$

6. The elliptic fibration from $L = N(D_6^4)$

Take the unique, up to $\text{Aut}(L)$, primitive embedding of $A_5 \oplus A_1$ in L given by $\phi(A_5 \oplus A_1) = (i_1(A_5), d_6, 0, 0)$. We get

$$(i_1(A_5))^{\perp_{D_6}} = z_6 = 2\delta_6 = 2[1],$$

$$(A_1)^{\perp_{D_6}} = \langle d_5\rangle \oplus \langle x_3 := d_5 + d_6 + 2d_4 + d_3, d_3, d_2, d_1\rangle = A_1 \oplus D_4,$$

$$N := ((i_1(A_5) \oplus A_1)^{\perp_{L_{\text{root}}}} = (\langle z_6\rangle, A_1 \oplus D_4, D_6, D_6)$$

and $N_{\text{root}} = (0, A_1 \oplus D_4, D_6, D_6)$. Since $\det N = 12 \times 4^3$, $\det W = 12$, it follows

$$W/N \simeq (\mathbb{Z}/2\mathbb{Z})^3 .$$

An elliptic fibration is characterized by its torsion sections, infinite sections and where these sections cut the singular fibers of the fibration. All these data are encoded in W/N and so we shall first compute these groups.

Observing that [2] and [3] do not belong to $i_1(A_5)^{\perp_{D_6^*}}$, the elements of the glue code (5) belonging to W/N are only those beginning by 0 or 1, precisely

$$\begin{array}{cccc} [0,0,0,0] & [0,1,2,3] & [0,3,1,2] & [0,2,3,1] \\ [1,1,1,1] & [1,0,3,2] & [1,3,2,0] & [1,2,0,3]. \end{array}$$

Among them only those beginning by 0 belongs to $\overline{W}_{\text{root}}$. Thus torsion sections can be realized only from the glue vectors

$$[0,0,0,0], [0,1,2,3], [0,3,1,2], [0,2,3,1].$$

Moreover we must choose in them elements belonging to $\overline{W}_{\text{root}}$. Since, in the coset [3], $\tilde{\delta}_6$ satisfies

$$2\tilde{\delta}_6 = d_1 + 2d_2 + d_3 + 2x_3 + d_5 \in D_4 \oplus A_1$$

and in coset [2],

$$2\bar{\delta}_6 = 2d_1 + 2d_2 + d_3 + x_3 \in D_4$$

TABLE 2. Contributions

	Contr. on D_4	Contr. on A_1	Contr. on D_6	Contr. on D_6
$\bar{\delta}_6 \in [2]$	1	0	1	1
$\tilde{\delta}_6 \in [3]$	1	1/2	1+1/2	1+1/2
$\delta - d_3 - d_4 - d_6 \in [1]$	1	1/2	1+1/2	1+1/2
$\delta \in [1]$	1	0	1+1/2	1+1/2

TABLE 3. Contributions and heights of the sections from $N(D_6^4)$

		Contr. D_4	Contr. A_1	Contr. D_6	Contr. D_6	ht.
Q_1	$0 + 2F + [0, 2, 3, 1]$	1	0	3/2	3/2	0
Q_3	$0 + 2F + [0, 3, 1, 2]$	1	1/2	3/2	1	0
Q_2	$0 + 2F + [0, 1-d_3-d_4-d_6, 2, 3]$	1	1/2	1	3/2	0
W_1	$0 + 2F + [1, 0, 3, 2]$	0	0	3/2	1	3/2
W_1+Q_1	$0 + 2F + [1, 2, 0, 3]$	1	0	0	3/2	3/2
W_1+Q_3	$0 + 2F + [1, 3, 2, 0]$	1	1/2	1	0	3/2
W_1+Q_2	$0 + 3F + [1, 1-d_3-d_4-d_6, 1, 1]$	1	1/2	3/2	3/2	3/2

it is possible to write torsion sections from $[0, 3, 1, 2]$, $[0, 2, 3, 1]$ and $[0, 0, 0, 0]$. It remains to find in the coset $[1]$ an element with the same property, that is $\delta - d_3 - d_4 - d_6$, since

$$2\delta_6 - 2d_3 - 2d_4 - 2d_6 = d_1 + 2d_2 + x_3 + d_5 \in D_4 \oplus A_1.$$

The Mordell-Weil lattice being $W/\overline{W_{\text{root}}}$, the infinite sections can be realized from the classes

$$[1, 0, 3, 2], [1, 2, 0, 3], [1, 3, 2, 0], [1, 1 - d_3 - d_4 - d_6, 1, 1].$$

The various contributions to the singular fibers can be derived from Table 1.

Taking in account the different values $\delta_6^2 = \tilde{\delta}_6^2 = (\delta_6 - d_3 - d_4 - d_6)^2 = -3/2$ and $\bar{\delta}_6^2 = -1$, we can draw a table with the various contributions to height for the different sections in Table 3.

It is easily derived that the torsion group of the elliptic fibration is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the Mordell-Weil lattice is generated by a section of height $3/2$, in concordance with the formula (8),

$$-12 = \text{disc}NS(X) = -4 \times 4 \times 4 \times 2 \times \frac{3}{2} \times \frac{1}{4^2}.$$

Thus we have proved the following result.

Proposition 2. *The elliptic fibration on the K3-surface X derived from Niemeier lattice $L = N(D_6^4)$ has singular fibers of type A_1 (I_2), D_4 (I_0^*), D_6 (I_2^*), D_6 (I_2^*). Its Mordell-Weil group has rank 1 and torsion part isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Its Mordell-Weil lattice is generated by an infinite section of height $3/2$.*

7. THE ELLIPTIC FIBRATIONS FROM $L = N(A_9^2 D_6)$

Let L be the Niemeier lattice with $L = N(A_9^{(1)} A_9^{(2)} D_6)$. By [CS] we know that L is obtained from the following glue vectors

$$L/L_{\text{root}} = \langle [2,4,0], [5,0,1], [0,5,3] \rangle,$$

where 1 denotes the coset in A_9^*/A_9 of $\alpha = \frac{1}{10}(9a_1 + 8a_2 + 7a_3 + 6a_4 + 5a_5 + 4a_6 + 3a_7 + 2a_8 + a_9)$. From Corollary 2 we know that we have at most two elliptic fibrations coming from the Niemeier lattice $L = N(A_9^2 D_6)$ non isomorphic by an automorphism of L. We shall prove that we have effectively two.

7.1. **First embedding in D_6.** We embed A_1 in $A_9^{(1)}$ by $\phi(A_1) = \langle a_1^{(1)} \rangle$ and A_5 in D_6 by $i_1(A_5) = (d_5, d_4, d_3, d_2, d_1)$. As computed in [BGHLMSW], we obtain

$$N = (\phi(A_1) \oplus i_1(A_5))^{\perp_{L_{\text{root}}}} = [\langle a_1 + 2a_2, a_3, ..., a_9 \rangle, A_9, \langle z_6 \rangle]$$

with $z_6 = d_1 + 2d_2 + 3d_3 + 4d_4 + 2d_5 + 3d_6$ and $\det(\langle a_1 + 2a_2, a_3, ..., a_9 \rangle) = 2 \times 10$; thus $\det(N) = 2 \times 10 \times 10 \times 6$. It follows $N_{\text{root}} = [\langle a_3, ..., a_9 \rangle, A_9, 0] \simeq A_7^{(1)} \oplus A_9^{(2)}$ and $W/N = \langle [2,4,0], [5,0,1] \rangle \simeq \mathbb{Z}/10\mathbb{Z}$. Since there is no integer k satisfying $k([2,4,0]) \in N_{\text{root}}$ and no integer k' with $k'([5,0,1]) \in N_{\text{root}}$, we deduce that $\overline{W_{\text{root}}}/W_{\text{root}} = (0)$ so the corresponding elliptic fibration has trivial torsion and rank 2.

Now we want to determine the Mordell-Weil lattice of the fibration, in our case

$$\text{MWL}(X) = W/\overline{W_{\text{root}}} \simeq W/W_{\text{root}}.$$

The infinite sections are derived from elements of the glue code of W/N, namely from

$$\begin{array}{lllll} [2,4,0], & [4,8,0], & [6,2,0], & [8,6,0], & [0,0,0] \\ [5,0,1], & [7,4,1], & [9,8,1], & [1,2,1], & [3,6,1]. \end{array}$$

We define sections as explained in 5.1 so we search in each coset j an element α_j satisfying $\alpha_j.a_j = 1$ and $\alpha_j.a_i = 0$. We obtain a unique solution

$$-\alpha_j := j\alpha - (j-1)a_1 - (j-2)a_2 ... - a_{j-1}.$$

We observe that $\alpha_j \in W$ for all j but $j = 1$. Thus we choose in the coset of α_1 an element in W and cutting $A_7 = \langle a_3, a_4, ..., a_9 \rangle$ in exactly one point, namely $-\bar{\alpha_1} = \alpha - a_1 - a_2$. The elements $(\alpha_1, \alpha_2, ..., \alpha_9)$ are in fact the dual elements $(a_1^*, a_2^*, ..., a_9^*)$. So their Gram matrix is minus the inverse matrix of the Gram matrix of the a_i, namely

TABLE 4. Height and pairing-First embedding

		I_8	I_{10}	$<V_i,V_1>$	$<V_i,V_2>$	$ht(V_i)$	$order$	
V_1	$O+2F+[\alpha_9,\alpha_8,1]$	7	8	$\frac{61}{40}$	$\frac{1}{20}$	$\frac{61}{40}$	10	V_1
V_2	$O+2F+[\alpha_8,\alpha_6,0]$	6	6	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{10}$	5	V_2
V_3	$O+3F+[\alpha_7,\alpha_4,1]$	5	4	$\frac{63}{40}$	$\frac{3}{20}$	$\frac{69}{40}$	10	V_1+V_2
V_4	$O+2F+[\alpha_6,\alpha_2,0]$	4	2	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{4}{10}$	5	$2V_2$
V_5	$O+2F+[\alpha_5,0,1]$	3	0	$\frac{13}{8}$	$\frac{1}{4}$	$\frac{17}{8}$	2	V_1+2V_2
V_6	$O+2F+[\alpha_4,\alpha_8,0]$	2	8	$\frac{3}{20}$	$\frac{3}{10}$	$\frac{9}{10}$	5	$3V_2$
V_7	$O+3F+[\alpha_3,\alpha_6,1]$	1	6	$\frac{67}{40}$	$\frac{7}{20}$	$\frac{109}{40}$	10	V_1+3V_2
V_8	$O+2F+[\alpha_2,\alpha_4,0]$	0	4	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{8}{5}$	5	$4V_2$
V_9	$O+2F+[\alpha_1-a_1-a_2,\alpha_2,1]$	1	2	$\frac{59}{40}$	$\frac{-1}{20}$	$\frac{61}{40}$	10	V_1-V_2
V_{11}	$O+2F+[-a_1-2a_2-a_3,0,0]$	2	0	$\frac{-1}{4}$	$\frac{-1}{2}$	$\frac{5}{2}$	0	$-5V_2$
V_{12}	$O+3F+[0,0,2\delta_6]$	0	0	3	0	6	0	$2V_1-V_2$

(9)
$$\begin{pmatrix}
\frac{9}{10} & \frac{4}{5} & \frac{7}{10} & \frac{3}{5} & \frac{1}{2} & \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \\
\frac{4}{5} & \frac{8}{5} & \frac{7}{5} & \frac{6}{5} & 1 & \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\
\frac{7}{10} & \frac{7}{5} & \frac{21}{10} & \frac{9}{5} & \frac{3}{2} & \frac{6}{5} & \frac{9}{10} & \frac{3}{5} & \frac{3}{10} \\
\frac{3}{5} & \frac{6}{5} & \frac{9}{5} & \frac{12}{5} & 2 & \frac{8}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\
\frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 2 & \frac{3}{2} & 1 & \frac{1}{2} \\
\frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{8}{5} & 2 & \frac{12}{5} & \frac{9}{5} & \frac{6}{5} & \frac{3}{5} \\
\frac{3}{10} & \frac{3}{5} & \frac{9}{10} & \frac{6}{5} & \frac{3}{2} & \frac{9}{5} & \frac{21}{10} & \frac{7}{5} & \frac{7}{10} \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 & \frac{6}{5} & \frac{7}{5} & \frac{8}{5} & \frac{4}{5} \\
\frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{7}{10} & \frac{4}{5} & \frac{9}{10}
\end{pmatrix}$$

We read directly on the above matrix

$$\alpha_1^2=\alpha_9^2=-\frac{9}{10},\alpha_2^2=\alpha_8^2=-\frac{8}{5},\alpha_3^2=\alpha_7^2=-\frac{21}{10},\alpha_4^2=\alpha_6^2=-\frac{12}{5},\alpha_5^2=-\frac{5}{2}$$

and we compute $\bar{\alpha_1}^2=-\frac{9}{10}$.

Hence we obtain the nine non zero sections V_i, $1\leq i\leq 9$, quoted in the Table 4. Using the entries of the matrix (9) we obtain their contributions to the singular fibers, their heights and the various $\langle V_i,V_1\rangle$ and $\langle V_i,V_2\rangle$, according to formulae (6) and (7). Moreover the determinant of the height matrix of V_1,V_2 is equal to $\frac{3}{20}$ fitting with the formula (8). These data allow in turn to express V_j for $j\geq 3$ as a linear combination of V_1 and V_2. For example, looking for a relation $V_3=aV_1+bV_2$, we compute $\langle V_3,V_k\rangle=a\langle V_1,V_k\rangle+b\langle V_2,V_k\rangle$ with $k=1,2$. Thus we get two equations in a,b and solving the system it follows $a=b=1$.

Finally the order in the Table 4 refers to the order of the element in W/N of the corresponding section.

Theorem 2. *The Mordell-Weil lattice can be generated by the section V_2 and another section whose class in W/N is of order 10 or 2 (V_1, V_3, V_7, V_9 or V_5). It also can be generated by V_1 and V_3 or V_9.*

The rational quadratic forms associated to these various height matrices are all equivalent to the quadratic form $Q(x,y) = \frac{1}{40}(61x^2 + 4xy + 4y^2)$.

The sublattice of index 10, N/N_{root} of W/N_{root}, is generated by $V_{11} = -5V_2$ and V_{12} with $< V_{11}, V_{12} >= 0$.

Proof. We observe that the nine first sections are not in the same class modulo N_{root}.

The rational quadratic form $Q(x,y)$ associated to the height matrix of (V_1, V_2) is $Q(x,y) = \frac{1}{40}(61x^2 + 4xy + 4y^2)$.

Other properties are simple transcriptions of the base change which can be derived from the last column of Table 4. For example the rational quadratic form associated to the height matrix of (V_1, V_9) is equivalent to $Q(x,y)$ since

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{61}{40} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{61}{40} & \frac{59}{40} \\ \frac{59}{40} & \frac{61}{40} \end{pmatrix}$$

Finally we verify that the height matrix of (V_{11}, V_{12}), namely $\begin{pmatrix} \frac{5}{2} & 0 \\ 0 & 6 \end{pmatrix}$, has determinant $15 = 10^2\frac{3}{20}$. Moreover there exists a sublattice of index 2 generated by V_2 and V_{12} with $< V_2, V_{12} >= 0$.

□

7.2. **Second embedding in D_6.** We embed A_1 in $A_9^{(1)}$ by $\phi(A_1) = \langle a_1^{(1)} \rangle$ and A_5 in D_6 by $i_2(A_5) = (d_6, d_4, d_3, d_2, d_1)$. We obtain

$$N = (\phi(A_1) \oplus i_1(A_5))^{\perp_{L_{\text{root}}}} = [\langle a_1 + 2a_2, a_3, ..., a_9 \rangle, A_9, \langle \tilde{z_6} \rangle]$$

with $\tilde{z_6} = d_1 + 2d_2 + 3d_3 + 4d_4 + 3d_5 + 2d_6$ and $\det(\langle a_1 + 2a_2, a_3, ..., a_9 \rangle) = 2 \times 10$; thus $\det(N) = 2 \times 10 \times 10 \times 6$. It follows $N_{\text{root}} = [\langle a_3, ..., a_9 \rangle, A_9, 0] \simeq A_7^{(1)} \oplus A_9^{(2)}$ and $W/N = \langle [2,4,0], [0,5,3] \rangle \simeq \mathbb{Z}/10\mathbb{Z}$. Since there is no integer k satisfying $k([2,4,0]) \in N_{\text{root}}$ and no integer k' with $k'([0,5,3]) \in N_{\text{root}}$, we deduce that $\overline{W_{\text{root}}}/W_{\text{root}} = (0)$ so the corresponding elliptic fibration has trivial torsion and rank 2.

Theorem 3. *The Mordell-Weil lattice can be generated by the section Z_2 and another section whose class in W/N is of order 10 or 2 (Z_1, Z_3, Z_7, Z_9 or Z_5). It also can be generated by Z_1 and Z_3 or Z_9. The rational quadratic forms associated to these various height matrices are all equivalent to the quadratic form $\frac{1}{10}(x^2 + 15y^2)$.*

The sublattice of index 10, N/N_{root}, is generated by $Z_{11} = -5Z_2$ and Z_{12}.

Proof. The proof is similar to the previous proof.

□

Corollary 3. *The Mordell-Weil lattices for the first i_1 and second i_2 embeddings are not isomorphic. Thus they lead to two distinct elliptic fibrations.*

TABLE 5. Height and pairing-Second embedding

		I_8	I_{10}	$<Z_1,Z_i>$	$<Z_2,Z_i>$	$ht(Z_i)$	$o.$	
Z_1	$O+3F+[\alpha_4,\alpha_3,3]$	2	3	$\frac{12}{5}$	$\frac{3}{10}$	$\frac{12}{5}$	10	Z_1
Z_2	$O+2F+[\alpha_8,\alpha_6,0]$	6	6	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	5	Z_2
Z_3	$O+2F+[\alpha_2,\alpha_9,3]$	0	9	$\frac{27}{10}$	$\frac{2}{5}$	$\frac{31}{10}$	10	Z_1+Z_2
Z_4	$O+2F+[\alpha_6,\alpha_2,0]$	4	2	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	5	$2Z_2$
Z_5	$O+2F+[0,\alpha_5,3]$	0	5	$\frac{3}{2}$	0	$\frac{3}{2}$	2	Z_1-3Z_2
Z_6	$O+2F+[\alpha_4,\alpha_8,0]$	2	8	$\frac{9}{10}$	$\frac{3}{10}$	$\frac{9}{10}$	5	$3Z_2$
Z_7	$O+2F+[\alpha_8,\alpha_1,3]$	6	1	$\frac{9}{5}$	$\frac{1}{10}$	$\frac{8}{5}$	10	Z_1-2Z_2
Z_8	$O+2F+[\alpha_2,\alpha_4,0]$	0	4	$\frac{6}{5}$	$\frac{4}{10}$	$\frac{8}{5}$	5	$4Z_2$
Z_9	$O+3F+[\alpha_6,\alpha_7,3]$	4	7	$\frac{21}{10}$	$\frac{1}{5}$	$\frac{19}{10}$	10	Z_1-Z_2
Z_{11}	$O+2F+[-a_1-2a_2-a_3,0,0]$	2	0	$\frac{-1}{4}$	$\frac{-1}{2}$	$\frac{5}{2}$	0	$-5Z_2$
Z_{12}	$O+3F+\left[0,0,\tilde{\delta}_6\right]$	0	0	3	0	6	0	$2Z_1-6Z_2$

Proof. According to the previous theorems, the Mordell-Weil lattice for the first (resp. second) embedding can be generated by the sections V_1 and V_2 (resp. Z_2 and Z_5) with height matrix $\begin{pmatrix} \frac{61}{40} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{10} \end{pmatrix}$ (resp. $\begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$).

As we can prove easily that these two matrices are not equivalent, since there is no matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \frac{a^2}{10}+\frac{3}{2}b^2 & \frac{ac}{10}+\frac{3}{2}bd \\ \frac{ac}{10}+\frac{3}{2}bd & \frac{c^2}{10}+\frac{3}{2}d^2 \end{pmatrix} = \begin{pmatrix} \frac{61}{40} & \frac{1}{20} \\ \frac{1}{20} & \frac{1}{10} \end{pmatrix},$$

for there are no integers a and b satisfying $4(a^2+15b^2)=61$. □

8. WEIERSTRASS EQUATIONS

In this second part we obtain the Weierstrass equations of the unique, up to automorphism of the Niemeier lattice $N(D_6^4)$, elliptic fibration denoted #36 as in [BGHLMSW] and of the two elliptic fibrations, non isomorphic by an automorphism of the Niemeier lattice $N(A_9^2D_6)$, numbered #40 as in [BGHLMSW] and #40 bis. These fibrations are given with their torsion and infinite sections and their Mordell-Weil lattices so we can easily see the parallelism between the theoretic results of the first part and the new ones coming from the Weierstrass equations.

8.1. **Background and method.** We start from fibration #50 of ([BGHLMSW]) with Weierstrass equation

$$E_u : y^2+(u^2+3)yx+(u^2-1)^2y=x^3, \tag{10}$$

which is the universal elliptic curve with torsion structure $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3\mathbb{Z}$.

The points $A_2 = (-\frac{1}{4}(u^2-1)^2, 0)$, $A_{22} = (-(u+1)^2, (u+1)^3)$ and $A_{23} = (-(u-1)^2, (u-1)^3)$ are 2-torsion points and the point $P_3 = (0,0)$ is a 3-torsion point.

The singular fibers are of type I_6 for $u = 1, -1, \infty$ and I_2 for $3, -3, 0$.

The components of an I_n fiber are numbered cyclically, $\Theta_{i,j}$ being the $j-$th component of the singular fiber above $u = i$ and the component $\Theta_{i,0}$ intersecting the zero section.

8.2. **The graph Γ.** The vertices of the graph Γ are the twelve torsion sections and the 24 components $\Theta_{i,j}$. Two vertices are linked by an edge if they intersect. To make it easily visible, only some parts of this graph are drawn on the following figures.

Recall first that two torsion sections do not intersect.

Then, we compute for a set of generating sections, which component of singular fibers are intersected, derived for example from the method given in [Cr] or in [Si]. For the other torsion sections, we use the algebraic structure of the Néron model or the height of sections as explained below.

Recall that the height of a torsion point P is 0 involving conditions on $contr_v(P)$ since from formula (7) and Table 1 it follows $4 = \sum_v contr_v(P)$. For example, the only possible sum of contributions for the 3-torsion point P_3 is $\frac{2\times4}{6} + \frac{2\times4}{6} + \frac{2\times4}{6} + 0+0+0$, and for a two-torsion point $\frac{3\times3}{6} + \frac{3\times3}{6} + 0 + \frac{1}{2} + \frac{1}{2} + 0$. Since the sum of two $2-$torsion points is also a $2-$torsion point, only one $2-$torsion point intersects the component $\Theta_{i,0}$, for a given reducible fiber. These remarks allow us to construct Γ.

Let us now summarize useful results. The point P_3 intersects the component $\Theta_{i,2}$ (by convention $\Theta_{i,2}$ not $\Theta_{i,4}$) of the I_6 fibers and the component $\Theta_{i,0}$ of the I_2 fibers. The point A_2 intersects the components $\Theta_{\infty,0}$ and $\Theta_{0,0}$, the point A_{22} intersects the components $\Theta_{1,0}$ and $\Theta_{-3,0}$. These two points intersects $\Theta_{i,3}$ for the others I_6 fibers and $\Theta_{i,1}$ for the other I_2 fibers.

8.3. **Method for building elliptic fibrations from fibration #50.** Recall that it is sufficient to identify a divisor D on the surface that has the shape of a singular fiber from Kodaira's list and an irreducible curve C with $C.D = 1$ to find an elliptic fibration with D as a singular fiber and C as a section. The fibration is induced by the linear system $|D|$.

Moreover, if we can draw two divisors D and D' on the graph Γ with $D.D' = 0$ it is easier to determine a new fibration. We must define a function, called elliptic parameter, with divisor $D' - D$. Moreover if D and D' are subgraph of Γ we use the elliptic curve E_u. The method and computations are explicited for the fibration #36.

9. FIBRATION #36

9.1. **Weierstrass equation.** We consider the divisors drawn in black (double circle) for D' and green (dashed circle) for D on the graph (Figure 1) namely

$$D = \Theta_{-1,1} + 2\Theta_{-1,0} + \Theta_{-1,5} + 2(0) + 2\Theta_{1,0} + \Theta_{1,1} + \Theta_{1,5}$$
$$D' = (P_3) + \Theta_{\infty,5} + 2\Theta_{\infty,4} + 2\Theta_{\infty,3} + 2\Theta_{\infty,2} + \Theta_{\infty,1} + (2P_3).$$

The divisors D and D' correspond to two singular fibers of type I_2^* of the same fibration since $D.D' = 0$.

We see also that $\Theta_{-1,3}, \Theta_{1,3}$ and A_2 in blue are a part of another singular fiber.

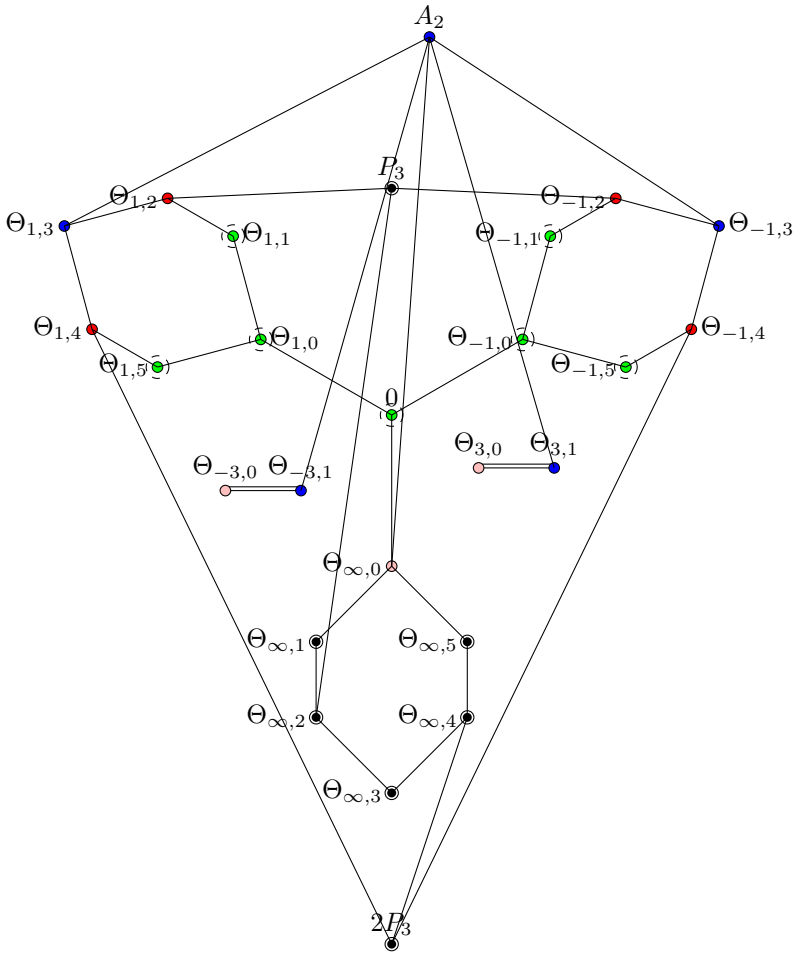


FIGURE 1. Fibration #36

Let w be a parameter for the new fibration such that $w = \infty$ on D and 0 on D'.

So the divisors D and D' correspond to the same element in the Néron-Severi group $NS(X)$. Let $D = \delta + \Delta$ and $D' = \delta' + \Delta'$ where δ, δ' are sums of sections, $\delta = 2(O)$ and $\delta' = (P_3) + (2P_3)$, while Δ, Δ' are sums of components of singular fibers. It follows from the equality $\delta = \delta'$ in the group $NS(X)/T(X)$ that $\delta - \delta' = 2(0) - (P_3) - (2P_3)$ is the divisor of a function on the elliptic curve E_u, precisely the function x. The parameter w is then equal to $x.f(u)$. We compute $f(u)$ using three blow- up to get a pole of order 1 on $\Theta_{1,1}, \Theta_{-1,5}, \Theta_{\infty,1}$ and obtain

$$w = \frac{x}{(u^2-1)^2}.$$

Eliminating x in the equation of E_u and setting $y = (u^2-1)^2 z, u = 1 + U$ it follows a quartic equation in z, U, w. All the transformations are summarized in the birational transformation $\phi : (X, Y, w) \mapsto (x, y, u)$ leading to the following Weierstrass equation E_w

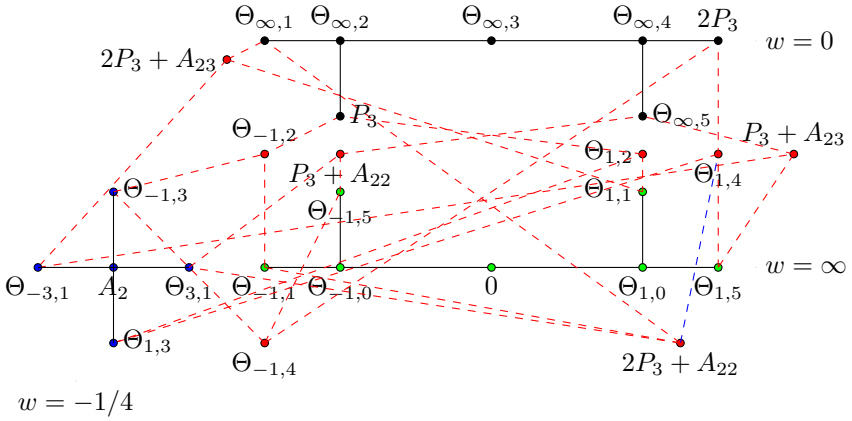


FIGURE 2. Fibration 36

$$E_w : Y^2 = X\left(X - w\left(1+4w\right)\right)\left(X + w^2\left(1+4w\right)\right) \tag{11}$$

with

$$x = \frac{w\left(1+4w\right)^2\left(X+4w^3\right)^2\left(2Y+X\left(2w+1\right)\right)^2}{\left(Y - Xw - 2w^3\left(1+4w\right)\right)^4}$$

$$y = -\frac{\left(1+4w\right)^3 X\left(X+4w^3\right)^4\left(2Y+X\left(2w+1\right)\right)^2}{\left(Y - Xw - 2w^3\left(1+4w\right)\right)^6}$$

$$u = \frac{\left(1+4w\right)\left(X+4w^3\right)}{Y - Xw - 2w^3\left(1+4w\right)} + 1.$$

The singular fibers are of type I_2^* for $w = 0, \infty$, I_0^* for $w = -1/4$, I_2 for $w = -1$.

We compute that the function $w + 1/4$ is equal to 0 on $\Theta_{\pm 3,1}$, giving thus with A_2 and $\Theta_{\pm 1,3}$ a complete description of the singular fiber I_0^*.

The component $\Theta_{0,1}$ is a component of the singular fiber I_2 obtained for $w = -1$ and does not intersect the new 0 section. The second component is the curve with the parametrization

$$u = -2\,\frac{-3+z^2}{3+z^2}$$

$$x = -9\,\frac{\left(z-1\right)^2\left(3+z\right)^2\left(z-3\right)^2\left(z+1\right)^2}{\left(3+z^2\right)^4} \qquad y = 27\,\frac{\left(3+z\right)^2\left(z-1\right)^2\left(z+1\right)^4\left(z-3\right)^4}{\left(3+z^2\right)^6}.$$

This component gives a quadratic section on E_u and can be used to construct other fibrations.

9.2. **Sections of the fibration #36.** Denote $Q_1 = (0,0)$, $Q_2 = (w(4w+1), 0)$, $Q_3 = (-w^2(4w+1), 0)$ the two-torsion sections and $W_1 = (-4w^3, -2w^3(2w+1))$ an infinite section of E_u.

On the Figures 1 and 2, in red bullets, can be viewed the following sections of the new fibration:

$$\Theta_{1,2}, \Theta_{1,4}, \Theta_{-1,2}, \Theta_{-1,4}$$

TABLE 6. Heights for sections of fibration #36

Contr. on	$\Theta_{1,4}$	$\Theta_{1,2}$	$\Theta_{-1,2}$	$\Theta_{-1,4}$	$P_3{+}A_{22}$	$2P_3{+}A_{22}$	$P_3{+}A_{23}$	$2P_3{+}A_{23}$
I_2^* $w{=}0$	0	$\frac{3}{2}$	$\frac{3}{2}$	0	1	$\frac{3}{2}$	1	$\frac{3}{2}$
I_2^* $w{=}\infty$	0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	1
I_0^* $w{=}\frac{-1}{4}$	0	0	1	1	1	1	1	1
I_2 $w{=}-1$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
height	0	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	$\frac{3}{2}$	0
	0	W_1	Q_1	$Q_1{+}W_1$	Q_2	$Q_2{+}W_1$	$Q_3{+}W_1$	Q_3

and also

$$P_3 + A_{23}, 2P_3 + A_{23}, P_3 + A_{22}, 2P_3 + A_{22}.$$

The correspondence between sections of fibrations #50 and #36 can be settled by the transformation ϕ. Recall that the components $\Theta_{i,j}$ are obtained by blowing up. For example the section $P_3 = (x = 0, y = 0)$ intersects the component $\Theta_{1,2}$, so this component defined by $x = (u-1)^2 x_2, y = (u-1)^2 y_2$ satisfies $y_2 = 0$. It follows that the point W_1 corresponds to $\Theta_{1,2}$ and the 0 section of the new fibration to $\Theta_{1,4}$. For all results see Table 6.

9.3. **Heights of sections.** The heights of sections of the new fibration are computed with the help of the graph. For example, we can see on Figure 2 that the section $2P_3 + A_{22}$ intersects $\Theta_{1,4}$ (the zero section), $\Theta_{\infty,1}$ (I_2^* for $w = 0$), $\Theta_{-1,1}$ (I_2^* for $w = \infty$), $\Theta_{-3,1}$ (I_0^* for $w = -\frac{1}{4}$) and $\Theta_{0,1}$ (I_2 for $w = -1$). The respective contributions are then computed with Table 1 and from formula (7) it follows

$$h(2P_3 + 2A_{22}) = 4 + 2 - (3/2 + 3/2 + 1 + 1/2) = 3/2.$$

Since the height of this section is equal to $\frac{3}{2}$, according to formula (8), it generates the Mordell-Weil lattice. The results are summarized on Table 6.

10. FIBRATION #40

The two divisors

$$D = A_2 + \Theta_{-1,3} + \Theta_{-1,2} + P_3 + \Theta_{\infty,2} + \Theta_{\infty,1} + (A_{23} + 2P_3) + \Theta_{-3,1}$$
$$D' = \Theta_{1,4} + \Theta_{1,5} + \Theta_{1,0} + 0 + \Theta_{-1,0} + \Theta_{-1,5} + (P_3 + A_{22}) + \Theta_{\infty,5} + \Theta_{\infty,4} + 2P_3$$

can be viewed as two singular fibers of an elliptic fibration with elliptic parameter p determined as explained in 9.1.

First we search on E_u a function g with three simple poles at $0, (2P_3)$ and $P_3{+}A_{22}$ and three zeros at P_3, A_2 and $A_{23} + 2P_3$. Taking

$$g = r + \frac{y - y_{P_3}}{x - x_{P_3}} + s\frac{y - y_{2P_3+A_{22}}}{x - x_{2P_3+A_{22}}}$$

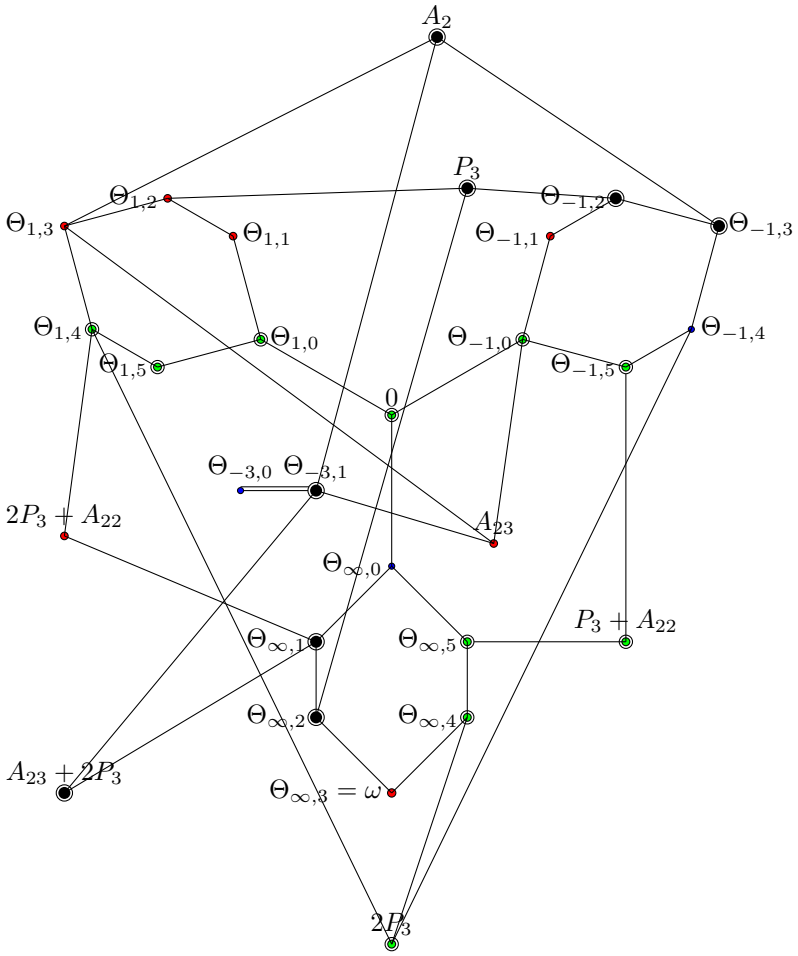


FIGURE 3. Fibration 40

and choosing r and s satisfying $g(A_2) = g(A_{23} + 2P_3) = 0$, we get $r = -u + 1, s = -\frac{u-1}{u+1}$. Finally to insure poles on D' set $p = \frac{g}{u-1}$ so

$$p = \frac{\left(2x + \left(u^2-1\right)^2\right) y - \left(u^2-1\right) x^2}{\left(u^2-1\right) x \left(x + (u+1)(u-1)^2\right)}.$$

We can remark that p can also be obtained from the fibration #36 and the parameter

$$p = \frac{-Y}{w\left(X - w\left(1+4w\right)\right)}.$$

The usual transformations leading to a Weierstrass equation are summarized in the birational map $\phi : ((\mathfrak{x}, \mathfrak{y}, p) \mapsto (x, y, u)$ with

$$x=\frac{-G_3^2G_4^2}{p^2\mathfrak{x}\left(\mathfrak{x}^2-p\left(p^2+4p-1\right)\mathfrak{x}-4p^3\right)^4}\quad y=\frac{-G_2G_4^2G_3^3}{p^2\mathfrak{x}\left(\mathfrak{x}^2-p\left(p^2+4p-1\right)\mathfrak{x}-4p^3\right)^6}$$
$$u=\frac{-G_1}{p\left(\mathfrak{x}^2-p\left(p^2+4p-1\right)\mathfrak{x}-4p^3\right)}$$

where

$$G_1=\left(\mathfrak{x}-2p^2\right)\mathfrak{y}+p^2\left(p^2+1\right)\mathfrak{x}-4p^4, G_2=(p+1)\mathfrak{y}+\mathfrak{x}^2-p^2(p+3)\mathfrak{x},$$
$$G_3=\left(2p^2-\mathfrak{x}\right)\mathfrak{y}-p\mathfrak{x}^2+2p^2(2p-1)\mathfrak{x}+8p^4, G_4=\left(2p^2-\mathfrak{x}\right)\mathfrak{y}+p\mathfrak{x}^2-2p^3(p+2)\mathfrak{x}.$$

We find then the following Weierstrass equation

$$\mathfrak{y}^2-\left(p^2+1\right)\mathfrak{y}\mathfrak{x}+4p^2\mathfrak{y}=\mathfrak{x}\left(\mathfrak{x}-p^2\right)\left(\mathfrak{x}-4p^2\right). \tag{12}$$

We denote $V_1=\left(2p(p-1),2p(p-1)\right)$ and $V_2=\left(0,-4p^2\right)$.

The first and last line of Table 7 are computed using ϕ and also $\omega=\Theta_{\infty,3}$ the zero of the new fibration.

From the graph (Figure 3) we obtain the index of the component of the singular fibers (I_8 and I_{10}) which a given section S meets (line 2 and 3 of Table 7). Then we compute the heights as explained in 9.3. From formula (6), it follows $<\Theta_{1,1},2P_3+A_{22}>=\frac{1}{20}$. Thus the height matrix of $\Theta_{1,1}$ and $2P_3+A_{22}$ has determinant 3/20.; we recover the result:

The two sections V_1 and V_2 generate the Mordell-Weil lattice.

11. FIBRATION #40 BIS

The two divisors

$$D=\Theta_{\infty,1}+\Theta_{\infty,2}+\Theta_{\infty,3}+\Theta_{\infty,4}+\Theta_{\infty,5}+(P_3+A_{22})+\Theta_{0,1}+(A_{23}+2P_3)$$
$$D'=\Theta_{-1,2}+\Theta_{-1,3}+A_2+\Theta_{1,3}+\Theta_{1,4}+\Theta_{1,5}+\Theta_{1,0}+0+\Theta_{-1,0}+\Theta_{-1,1}$$

define two singular fibers of an elliptic fibration with elliptic parameter

$$t=\frac{1}{2}\frac{2y+(u-3)(u-1)x-(u-1)^3(u+1)^2}{(u^2-1)\left(x+\frac{1}{4}(u^2-1)^2\right)},$$

and Weierstrass equation

$$\mathfrak{y}^2+2\left(t^2-1\right)\mathfrak{y}\mathfrak{x}-2t^2\mathfrak{y}=\mathfrak{x}\left(\mathfrak{x}+t^2\right)\left(\mathfrak{x}+4t^2\right)$$

with the following birational transformations

TABLE 7. Heights for sections of fibration #40

sect.	$\Theta_{1,1}$	$\Theta_{1,2}$	$\Theta_{1,3}$	$\Theta_{-1,1}$	$\Theta_{3,1}$	$\Theta_{0,1}$	A_{22}	A_{23}	$2P_3+A_{22}$	$2P_3+A_2$
I_8	2	7	4	6	4	2	5	3	1	3
I_{10}	4	8	2	6	8	8	4	6	2	0
ht	$\frac{1}{10}$	$\frac{61}{40}$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{9}{10}$	$\frac{69}{40}$	$\frac{69}{40}$	$\frac{61}{40}$	$\frac{17}{8}$
	V_2	V_1-V_2	$-2V_2$	$-V_2$	$2V_2$	$-3V_2$	V_1-2V_2	V_1+V_2	V_1	V_1-3V_2

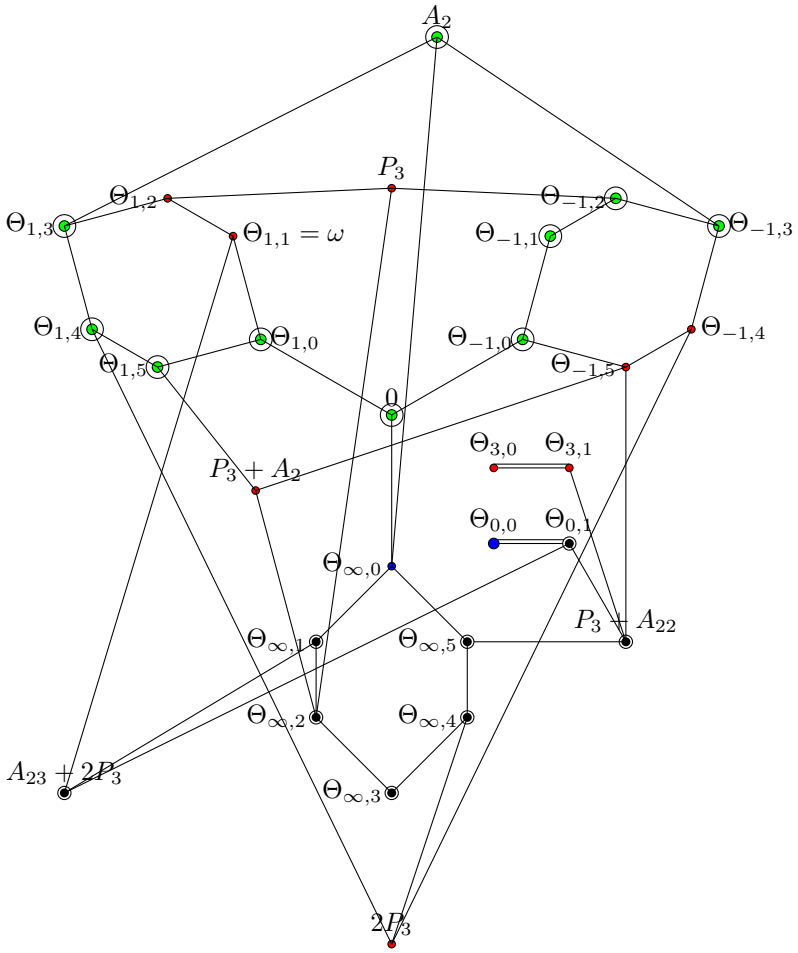


FIGURE 4. Fibration 40 bis

$$(\mathfrak{x},\mathfrak{y},t)\mapsto(x,y,u)$$

$$x=-8\frac{\mathfrak{y}(\mathfrak{x}+1)^2(\mathfrak{x}+4t^2)H_1^2}{((2t+1)\mathfrak{y}+(\mathfrak{x}+4t^2)(\mathfrak{x}-t))^4}$$

$$y=16\frac{(\mathfrak{x}+1)^2(\mathfrak{x}+4t^2)(2\mathfrak{y}+4t^2\mathfrak{x}+\mathfrak{x}^2)H_1^4}{((2t+1)\mathfrak{y}+(\mathfrak{x}+4t^2)(\mathfrak{x}-t))^6}$$

$$u=-\frac{(2t+1)\mathfrak{y}-(\mathfrak{x}+4t^2)(\mathfrak{x}+2+t)}{(2t+1)\mathfrak{y}+(\mathfrak{x}+4t^2)(\mathfrak{x}-t)}$$

where

$$H_1=-(2t+1)\mathfrak{y}+(t+1)(\mathfrak{x}+4t^2).$$

Notice also the relations

$$u-1=\frac{2H_1}{(2t+1)\mathfrak{y}+(\mathfrak{x}+4t^2)(\mathfrak{x}-t)}$$

TABLE 8. Heights for sections of fibration #40 bis

sect	$\Theta_{1,1}$	$\Theta_{1,2}$	$\Theta_{-1,4}$	$\Theta_{-1,5}$	$\Theta_{3,0}$	$\Theta_{3,1}$	P_3	$2P_3$	P_3+A_2	$2P_3+A_2$
I_8	0	2	0	2	0	2	6	4	6	4
I_{10}	0	7	5	2	1	6	4	8	9	3
ht	0	$\frac{12}{5}$	$\frac{3}{2}$	$\frac{9}{10}$	$\frac{31}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{8}{5}$	$\frac{19}{10}$
	0	$3Z_1-Z_5$	Z_5	$3Z_1$	$4Z_1-Z_5$	$-Z_1$	Z_1	$2Z_1$	Z_1-Z_5	$2Z_1-Z_5$

and

$$u+1=\frac{2(\mathfrak{x}+1)(\mathfrak{x}+4t^2)}{(2t+1)\mathfrak{y}+(\mathfrak{x}+4t^2)(\mathfrak{x}-t)}.$$

Let $Z_1=(0,0)$ and $Z_5=(-1,(2t-1)\,(t+1))$.

It follows from the previous formulae that the 0 section of the new fibration corresponds to $u=1$ and looking at $x/(u-1)$ and at the graph we find that the 0 section corresponds to $\Theta_{1,1}$. The correspondence between the sections of the two fibrations can be also derived and is shown on Table 8. On the same table are quoted the contributions and the heights of sections computed with the graph.

Moreover we find $<\Theta_{-1,4},P_3>=2+\Theta_{1,1}\cdot P_3+\Theta_{1,1}\cdot\Theta_{-1,4}-\Theta_{-1,4}\cdot P_3-\frac{5\times4}{10}=0$.

Thus the height matrix of the two sections $\Theta_{-1,4}$ and P_3 is diagonal with determinant $\frac{3}{20}$, so Z_1 and Z_5 generate the Mordell-Weil lattice.

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