

LINEAR BLOCK AND ARRAY CODES CORRECTING REPEATED CT BURST ERRORS

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ABSTRACT. Burst errors are very common in practice. There have been many designs in order to correct or at least detect such errors. Recently, a new kind of burst error which is termed as repeated burst error has been introduced in order to detect or correct errors that occurs in very busy communication channels. In this paper, we extend the definition of repeated burst errors for block and array codes endowed with a homogeneous metric. We also obtain some upper bounds on the number of parity check digits for these codes correcting all repeated burst errors.

1. INTRODUCTION

The early studies in coding theory based on the detection and correction of errors have been introduced for detection and correction of random errors [7]. In the applications of codes to various communication channels, errors do not occur independently but are clustered, that is, the error patterns are mostly in the form of bursts. This led to the study of burst error correcting codes, depicted by Fire [6] and Reiger [8]. Because of the nature of applications to communication channels, several definitions regarding the concept of burst error have been introduced by many researchers. Chien and Tang [3] introduced the concept of Chien and Tang (shortly CT) burst errors for block codes. These burst errors have found applications in error analysis experiment on telephone lines [1]. Later, Jain [10] extended the notion of CT burst errors for array codes by endowing a non-Hamming metric [9]. In order to solve the same problem in [10] with a novel approach Siap [11] introduced a CT burst error weight enumerator. The result obtained over finite fields in [10] was extended to array codes over finite rings [12].

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During the process of transmission over very noisy communication channels, errors repeat themselves. Dass [4] *et al.* introduced another type of error pattern called 2–repeated burst error for block codes and obtained an upper bound on the number of parity check digits for codes correcting such errors. It is pointed out that 2–repeated burst error correcting linear block codes provide a good source for detecting and correcting these burst errors in very busy communication channels. In these type of channels, burst errors may repeat more frequently. Therefore, it is useful to consider more than 2–repeated burst errors. Dass & Verma [5] termed such a burst error as m –repeated burst error for block codes. In this paper, firstly, we obtain some bounds on the parameters of linear block codes correcting all CT burst errors and all m –repeated burst errors with respect to homogeneous metric, respectively. Moreover, the study of these burst errors in terms of homogeneous weight is given. Later, we derive some bounds on the parameters of array codes with respect to this metric by using the definition of the extended CT burst error given in [10] for array coding systems. Furthermore, we combine these two interesting topics: CT burst error and repeated burst error for array codes and we introduce the concept of 2–repeated CT burst error for array codes. Finally, we obtain an upper bound on the parameters of linear array codes correcting all 2–repeated CT burst errors in terms of homogeneous weight.

The organization of this paper is as follows: In Section 2, we develop some basic terminology and cover some preliminary definitions. In Section 3, we study on m –repeated CT burst errors and CT burst errors with homogeneous weight constraint in linear block codes. In Sections 4 and 5, some new bounds on the parameters of array linear codes correcting all CT burst errors and all 2–repeated CT burst errors with respect to homogeneous metric are given, respectively.

2. DEFINITIONS AND NOTATIONS

Let \mathbb{Z}_{q^l} be the ring of integer modulo q^l , where q is a prime. Let $\mathbb{Z}_{q^l}^n$ be the space of all n –tuples with entries from a ring \mathbb{Z}_{q^l} . Then $\mathbb{Z}_{q^l}^n$ is a module over \mathbb{Z}_{q^l} . C is said to be an (n, M) –linear block code if and only if C is a submodule of $\mathbb{Z}_{q^l}^n$ of size M . If C is a k –free (with a basis of k elements) submodule with length n , then C is called an $[n, k]$ –linear block code. A linear array code C is a linear \mathbb{Z}_{q^l} –submodule of the space $Mat_{m \times s}(\mathbb{Z}_{q^l})$, the space of all $m \times s$ matrices with entries from a ring \mathbb{Z}_{q^l} .

The homogeneous weight w_{hom} on \mathbb{Z}_{q^l} is defined as

$$(1) \quad w_{hom}(x) = \begin{cases} 0 & \text{if } x = 0 \\ q^{l-1} & \text{if } x \in (q^{l-1}) \setminus \{0\} \\ (q-1)q^{l-2} & \text{otherwise} \end{cases}$$

where (q^{l-1}) denotes the ideal of \mathbb{Z}_{q^l} generated by q^{l-1} .

For $u = (u_1, u_2, \dots, u_n) \in \mathbb{Z}_{q^l}^n$, we have

$$(2) \quad w_{hom}(u) = \sum_{i=1}^n w_{hom}(u_i).$$

For any $u, v \in \mathbb{Z}_{q^l}^n$, the homogeneous distance d_{hom} is given by

$$(3) \quad d_{hom}(u, v) = w_{hom}(u - v).$$

Note that there are $q - 1$ elements of weight q^{l-1} and $q^l - q$ elements of weight $(q - 1)q^{l-2}$ in \mathbb{Z}_{q^l} .

There exist various types of burst errors in order to construct error detecting/correcting codes in literature. We first give the following definition of CT burst error.

Definition 2.1. [3] *A CT burst error of length b is a vector whose nonzero components are confined to some b consecutive components, with the first component being nonzero.*

A 2-repeated burst error of length b is defined as follows:

Definition 2.2. *A 2-repeated burst error of length b is a vector of length n whose only nonzero components are confined to two distinct sets of b consecutive components, the first and the last component of each set being nonzero [4].*

Dass & Verma [5] extended the above definition into the definition of m -repeated CT burst error as follows:

Definition 2.3. *An m -repeated CT burst error of length b is a vector of length n whose only non-zero components are confined to m distinct sets of b consecutive components, the first component of each set being non-zero.*

3. LINEAR BLOCK CODES WITH REPEATED CT BURST ERRORS

In this section, we consider the definitions of CT burst errors given in Definition 2.1 and m -repeated CT burst errors given in Definition 2.3 for linear block codes with respect to homogeneous metric, respectively. We determine the number of these burst errors of a given weight w_{hom} and also derive some bounds on the parameters of linear block codes correcting all these burst errors.

We first introduce the following lemma which will be used in the proof of Theorem 3.2.

Lemma 3.1. *The number of all CT burst errors of length b in $\mathbb{Z}_{q^l}^n$ is given by*

$$(4) \quad \mathbf{B}_n^b(\mathbb{Z}_{q^l}) = (n - b + 1) (q^l - 1) (q^l)^{b-1}.$$

Proof. Choosing b consecutive positions among n positions can be done in $n - b + 1$ ways. Then, by Definition 2.1, the first component of these b components can be any $q^l - 1$ nonzero elements of the ring \mathbb{Z}_{q^l} and the other $b - 1$ elements can be any q^l elements of the ring \mathbb{Z}_{q^l} . \square

We present some properties for an $[n, k]$ -linear code over \mathbb{Z}_{q^l} that is going to appear in the statement of Theorem 3.2.

Theorem 3.1. [2] *A nonzero $[n, k]$ -linear code C over \mathbb{Z}_{q^l} has a generator matrix which after a suitable permutation of the coordinates can be written in the form*

$$(5) \quad G = \begin{pmatrix} I & A_{0,1} & A_{0,2} & A_{0,3} & \dots & A_{0,l-1} & A_{0,l} \\ 0 & qI & qA_{1,2} & qA_{1,3} & \dots & qA_{1,l-1} & qA_{1,l} \\ 0 & 0 & q^2I & q^2A_{2,3} & \dots & q^2A_{2,l-1} & q^2A_{2,l} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & q^{l-1}I & q^{l-1}A_{l-1,l} \end{pmatrix}.$$

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Here the columns are grouped into blocks of sizes $k_0, k_1, \dots, k_{l-1}, k_l$, and k_i are non-negative integers adding to n . This means that C consists of all codewords

$$[v_0 \ v_1 \ v_2 \ \dots \ v_{l-1}] G,$$

where each v_i is a vector of length k_i with components from $\mathbb{Z}_{q^{l-i}}$, so that C contains q^k codewords, $A_{i,j}$ ($0 \leq i < j \leq l$) are matrices over $\mathbb{Z}_{q^{l-i}}$ and

$$k = \sum_{i=0}^{l-1} (l-i) k_i.$$

We say that C has type

$$1^{k_0} q^{k_1} (q^2)^{k_2} \dots (q^{l-1})^{k_{l-1}}.$$

The following theorem gives a bound in order to correct of all CT burst errors in linear codes with respect to homogeneous metric.

Theorem 3.2. An $[n, k]$ -linear code C over \mathbb{Z}_{q^l} that corrects all CT burst errors of length b must satisfy the bound

$$(6) \quad q^{ln-k} \geq 1 + \mathbf{B}_n^b(\mathbb{Z}_{q^l}),$$

where $\mathbf{B}_n^b(\mathbb{Z}_{q^l})$ is the number of all CT burst errors of length b in $\mathbb{Z}_{q^l}^n$.

Proof. The proof is based on the fact that the number of available cosets must be greater than or equal to the number of correctable CT burst errors having the vector of all zeros. By Lemma 3.1, the number of correctable CT burst errors having the vector of all zeros is

$$1 + \mathbf{B}_n^b(\mathbb{Z}_{q^l}).$$

By Theorem 3.1, an $[n, k]$ -linear code C over \mathbb{Z}_{q^l} has q^k codewords. So the number of available cosets is q^{ln-k} . \square

We now derive a bound for the correction of all CT burst errors of length b or less with respect to homogeneous metric.

Theorem 3.3. An $[n, k]$ -linear code C over \mathbb{Z}_{q^l} that corrects all CT burst errors of length b or less must satisfy the bound

$$(7) \quad q^{ln-k} \geq 1 + \sum_{a=1}^b \mathbf{B}_n^a(\mathbb{Z}_{q^l}),$$

$\mathbf{B}_n^a(\mathbb{Z}_{q^l})$ is the number of all CT burst errors of length a in $\mathbb{Z}_{q^l}^n$.

Proof. Using Theorem 3.2, its proof is straightforward. \square

In Theorem 3.4, we obtain a bound on the number of parity check digits for a linear code over \mathbb{Z}_{q^l} correcting all CT burst errors of length b or less having homogeneous weight w_{hom} or less. For this purpose, we first prove the following lemma that counts the number of all CT burst errors of length b having homogeneous weight w_{hom} .

Lemma 3.2. *The number of all CT burst errors of length b having homogeneous weight w_{hom} in $\mathbb{Z}_{q^l}^n$ is given by*

$$(8) \quad \mathbf{B}_n^b(\mathbb{Z}_{q^l}, w_{hom}) = (n - b + 1) \left[\sum_{u,v} \binom{b-1}{u} \binom{b-1-u}{v} (q-1)^{u+1} (q^l - q)^v \right. \\ \left. + \sum_{u',v'} \binom{b-1}{u'} \binom{b-1-u'}{v'} (q-1)^{u'} (q^l - q)^{v'+1} \right],$$

where u, v and u', v' are nonnegative integers such that

$$u + v \leq b - 1, \\ u' + v' \leq b - 1,$$

$$w_{hom} - q^{l-1} = u(q^{l-1}) + v(q-1)(q^{l-2}), \\ w_{hom} - (q-1)(q^{l-2}) = u'(q^{l-1}) + v'(q-1)(q^{l-2}).$$

Note that

$$\sum_{u,v} \binom{b-1}{u} \binom{b-1-u}{v} (q-1)^{u+1} (q^l - q)^v = 0 \quad \text{if } A, \\ \sum_{u',v'} \binom{b-1}{u'} \binom{b-1-u'}{v'} (q-1)^{u'} (q^l - q)^{v'+1} = 0 \quad \text{if } B,$$

where

$$A := w_{hom} \neq (u+1)(q^{l-1}) + v(q-1)(q^{l-2}) \\ B := w_{hom} \neq (u')(q^{l-1}) + (v'+1)(q-1)(q^{l-2}).$$

Proof. Consider a CT burst error of length b having homogeneous weight w_{hom} . Since the first component of the b consecutive positions in which nonzero components are clustered, it must be nonzero and since there exist two different nonzero homogeneous weights in \mathbb{Z}_{q^l} , we can investigate our proof in two cases:

Case 1: If the first nonzero component is of weight q^{l-1} then there can be u components of weight q^{l-1} in rest of b components and there can be v components of weight $(q-1)(q^{l-2})$ in rest of $b-u$ components such that $u+v \leq b-1$ and $w_{hom} = (u+1)(q^{l-1}) + v(q-1)(q^{l-2})$. For every values of u and v satisfying inequality and equality at the same time, we have some CT burst errors of length b having homogeneous weight w_{hom} in $\mathbb{Z}_{q^l}^n$. For convenient values of u and v we can choose u positions among $b-1$ positions and then we can choose v positions among $b-1-u$ positions. Afterwards, since there are $u+1$ components of weight q^{l-1} including the first component and since there are v components of weight $(q-1)q^{l-2}$, regarding there are $q-1$ elements of weight q^{l-1} and $q^l - q$ elements of $(q-1)q^{l-2}$ in \mathbb{Z}_{q^l} , we have

$$(n - b + 1) \binom{b-1}{u} \binom{b-1-u}{v} (q-1)^{u+1} (q^l - q)^v$$

CT burst errors of length b having homogeneous weight w_{hom} .

Case 2: When the first nonzero component is of weight $(q - 1)q^{l-2}$ similarly, for convenient nonzero integers u' and v' satisfying $u' + v' \leq b - 1$ and $w = u(q^{l-1}) + (v + 1)(q - 1)(q^{l-2})$, we have

$$(n - b + 1) \binom{b - 1}{u'} \binom{b - 1 - u}{v'} (q - 1)^u (q^l - q)(v + 1)$$

CT burst errors of length b having homogeneous weight w_{hom} . □

We give an example in order to illustrate Lemma 3.2.

Example 3.1. Let $q = l = 2, b = 3, n = 5$ and $w_{hom} = 4$. We first determine nonnegative integers u, v and u', v' satisfying the conditions in Lemma 3.2 in order to obtain the number of all CT burst errors of length 3 having homogeneous weight 4 in \mathbb{Z}_2^5 . For this we consider the following conditions for integers u, v and u', v' :

$$(9) \quad \begin{aligned} 4 &= (u + 1)2^{2-1} + v(2 - 1)(2^{2-2}), \\ 4 &= u'2^{2-1} + (v' + 1)(2 - 1)(2^{2-2}), \\ 2 &\geq u + v, \\ 2 &\geq u' + v'. \end{aligned}$$

Then, nonnegative integer solutions u, v and u', v' satisfying the conditions given in (9) are as follows, respectively:

$$u = 0, v = 2 \text{ and } u = 1, v = 0$$

and also

$$u' = 1, v' = 1.$$

Substituting these nonnegative integers into Eq.(8) in Lemma 3.2, we get the following result:

$$3 \left[\binom{2}{0} \binom{2}{2} 1^1 2^2 + \binom{2}{1} \binom{1}{0} 1^2 2^0 + \binom{2}{1} \binom{1}{1} 1^1 2^2 \right] = 42.$$

In fact, all CT burst errors of length 3 having homogeneous weight 4 in \mathbb{Z}_4^5 are as follows:

11200	01120	00112	13200	01320	00132	12100
01210	00121	12300	01230	00123	31200	03120
00312	33200	03320	00332	32100	03210	00321
32300	03230	00323	22000	02200	00220	20200
02020	00202	21100	02110	00211	21300	02130
00213	23100	02310	00231	23300	02330	00233.

Theorem 3.4. An $[n, k]$ -linear code C over \mathbb{Z}_{q^l} that corrects all CT burst errors of length b or less and having homogeneous weight w_{hom} or less must satisfy the bound

$$(10) \quad q^{ln-k} \geq 1 + \sum_{a=1}^b \sum_{j=1}^{a(q^{l-1})} \mathbf{B}_n^a(\mathbb{Z}_{q^l}, j)$$

where $\mathbf{B}_n^a(\mathbb{Z}_{q^l}, j)$ is the number of all CT burst errors of length a in $\mathbb{Z}_{q^l}^n$ having homogeneous weight j .

Proof. The proof is straightforward from Lemma 3.2. □

Now, we can obtain some results on the correction of m -repeated CT burst errors in linear codes with respect to homogeneous metric.

Lemma 3.3. *The number of all m -repeated CT burst errors of length b in $\mathbb{Z}_{q^l}^n$ is given by*

$$\mathbf{B}_n^{m,b}(\mathbb{Z}_{q^l}) = \frac{(n - mb + 1) \cdots (n - mb + m)}{m!} (q^l - 1)^m (q^l)^{mb-m}.$$

Proof. Clearly, m distinct sets of b consecutive positions can be chosen in

$$\frac{(n - mb + 1) \cdots (n - mb + m)}{m!}$$

ways among n positions. Later, each vector can be constructed such a way that the first components of m sets of b consecutive positions in which all the nonzero components are clustered must be nonzero, and the rest $b - 1$ components may be any element of \mathbb{Z}_{q^l} for each m distinct set. Hence the lemma. □

We enumerate all m -repeated CT burst errors with weight constraint in the following lemma.

Lemma 3.4. *The number of all m -repeated CT burst errors of length b having homogeneous weight w_{hom} in $\mathbb{Z}_{q^l}^n$ is given by*

$$\mathbf{B}_n^{m,b}(\mathbb{Z}_{q^l}, w_{hom}) = \frac{(n - mb + 1) \cdots (n - mb + m)}{m!} \times \left[\sum_{i=0}^m \sum_{u_i, v_i} \binom{m}{i} \binom{mb - m}{u_i} \binom{mb - m - u_i}{v_i} (q - 1)^{u_i+i} (q^l - q)^{v_i+m-i} \right],$$

where u_i and v_i are nonnegative integers for each $0 \leq i \leq m$ such that

$$u_i + v_i \leq m(b - 1),$$

and for each $0 \leq i \leq m$

$$w_{hom} = (u_i + i)(q^{l-1}) + (v_i + m - i)(q - 1)(q^{l-2}).$$

Say

$$\mathbf{B}_n^{m,b}(\mathbb{Z}_{q^l}, w_{hom}) = \frac{(n - mb + 1) \cdots (n - mb + m)}{m!} \times [\mathbf{V}_b^m(\mathbb{Z}_{q^l}, w_{hom})].$$

Proof. Considering m distinct sets of b consecutive components, the sketch of the proof can be investigated in $m + 1$ cases depending upon weights of the first components of each b consecutive positions.

Case 1: Assuming all the first components of each m distinct sets of b consecutive positions are of weight $(q - 1)q^{l-2}$ we have

$$\binom{mb - m}{u_0} \binom{mb - m - u_0}{v_0} (q - 1)^{u_0} (q^l)^{v_0+m}$$

m -repeated CT burst errors of length b for convenient pairwise u_0 and v_0 .

Case 2: Assuming one of the first components of m distinct sets of b consecutive positions is of weight q^{l-1} and all the others are of weight $(q-1)q^{l-2}$ we have

$$\binom{m}{1} \binom{mb-m}{u_1} \binom{mb-m-u}{v_1} (q-1)^{u_1+1} (q^l)^{v_1+m-1}$$

m -repeated CT burst errors of length b for convenient pairwise u_1 and v_1 .

Applying similar arguments, we have

$$\binom{m}{i} \binom{mb-m}{u_i} \binom{mb-m-u_i}{v_i} (q-1)^{u_i+i} (q^l)^{v_i+m-i}$$

m -repeated CT burst errors of length b for convenient u_i and v_i whenever i ($0 \leq i \leq m$) of the first components of m distinct sets of b consecutive positions is of weight q^{l-1} and the rest are of weight $(q-1)q^{l-2}$. Hence the result. \square

Example 3.2. Take $q = l = m = b = 2, w = 3,$ and $n = 5$ in Lemma 3.4. Then, nonnegative integers u_i and v_i for each $0 \leq i \leq m$ such that

$$3 = (u_i + i)(2^{2-1}) + (v_i + 2 - i)(2 - 1)(2^{2-2}), \text{ and } 2(2 - 1) \geq u_i + v_i$$

are given by

$$u_0 = 0, v_0 = 1 \text{ and } u_1 = v_1 = 0$$

and there is no solution for u_2 and v_2 and u_3 and v_3 . Therefore, our formula gives a direct computation

$$\frac{2 \cdot 3}{2!} \left[\binom{2}{0} \binom{2}{0} \binom{2}{1} 1^0 2^3 + \binom{2}{1} \binom{2}{0} \binom{2}{0} 1^1 2^1 \right] = 60.$$

In fact, all 2-repeated CT burst errors of length 2 having homogeneous weight 3 in \mathbb{Z}_4^5 are as follows:

01020	10020	10200
02010	20010	20100
03020	30020	30200
02030	30020	30200
01110	11010	11100
03330	33030	33300
01011	10011	10110
03033	30033	30330
01130	11030	11300
03310	33010	33100
01013	10013	10130
03031	30031	30310
01310	13010	13100
03130	31030	31300
01013	10013	10130
03031	30031	30310
03110	31010	31100
01310	13010	13100
03031	30031	30310
01013	10013	10130.

In Theorem 3.5, we obtain a bound for the correction of all m -repeated CT burst errors of length b or less and having homogeneous weight w or less.

Theorem 3.5. *An $[n, k]$ -linear code C over \mathbb{Z}_{q^l} that corrects all m -repeated CT burst errors of length b or less and having homogeneous weight w_{hom} or less must satisfy the bound*

$$(11) \quad q^{ln-k} \geq 1 + \sum_{a=1}^b \sum_{j=m(q-1)q^{l-2}}^{ma(q^{l-1})} \mathbf{B}_n^{m,a}(\mathbb{Z}_{q^l}, j).$$

Proof. It follows directly from Lemma 3.4. □

4. LINEAR ARRAY CODES WITH RESPECT TO HOMOGENEOUS WEIGHT

In this section, we extend the notion of CT burst errors for linear array codes with respect to homogeneous metric, originally in given in [10].

Definition 4.1. *A CT burst error of order $p \times r$ ($1 \leq p \leq m$, $1 \leq r \leq s$) in $Mat_{m \times s}(\mathbb{Z}_{q^l})$ is an $m \times s$ matrix in which all the nonzero entries are confined to some $p \times r$ submatrix which has nonzero first row and first column.*

We first enumerate all CT burst errors having homogeneous weight w_{hom} for linear array codes.

Lemma 4.1. *The number of all CT burst errors of order $p \times r$ having homogeneous weight w_{hom} in $Mat_{m \times s}(\mathbb{Z}_{q^l})$ is given by*

$$\begin{aligned} \mathbf{B}_{m \times s}^{p \times r}(\mathbb{Z}_{q^l}, w) &= (m - p + 1)(s - r + 1) \\ &\times \left[\mathbf{V}_{pr}^1(\mathbb{Z}_{q^l}, w_{hom}) + (p - 1)(r - 1) \right. \\ &\times \left[\sum_{a,b} \binom{pr - 3}{a} \binom{pr - 3 - a}{b} (q - 1)^{a+1} (q^l - q)^{b+1} \right. \\ &+ \sum_{c,d} \binom{pr - 3}{c} \binom{pr - 3 - c}{d} (q - 1)^{c+2} (q^l - q)^d \\ &\left. \left. + \sum_{e,f} \binom{pr - 3}{e} \binom{pr - 3 - e}{f} (q - 1)^e (q^l - q)^{f+2} \right] \right] \end{aligned}$$

where a, b, c, d, e, f are nonnegative integers satisfying

$$\begin{aligned} a + b &\leq pr - 3, \\ c + d &\leq pr - 3, \\ e + f &\leq pr - 3, \end{aligned}$$

and

$$\begin{aligned} w_{hom} &= (a + 1)(q^{l-1}) + (b + 1)(q - 1)(q^{l-2}), \\ w_{hom} &= (c + 2)(q^{l-1}) + d(q - 1)(q^{l-2}), \\ w_{hom} &= c(q^{l-1}) + (d + 2)(q - 1)(q^{l-2}). \end{aligned}$$

Proof. Take any CT burst error $A \in \text{Mat}_{m \times s}(\mathbb{Z}_{q^l})$ of order $p \times r$. Let us denote $p \times r$ nonzero submatrix whose first row and first column being nonzero by B . The row number of starting positions for B can vary between 1 and $m - p + 1$ and the column number of starting positions for B can vary between 1 and $s - r + 1$. Therefore, choosing the location of B regardless of its entries can be done in $(m - p + 1)(s - r + 1)$ ways.

The selection of entries of the submatrix B such that having homogeneous weight w_{hom} can be considered in two cases.

Case 1: If the entry b_{11} is different from zero, then the first row and the first column of B will be nonzero automatically. For this case, constructing the submatrix B can be achieved in \mathbf{V}_{pr}^1 ways since the work is the same that constructing a CT burst error of length pr in one array.

Case 2: If the entry b_{11} is zero, then we must have at least one nonzero component in the first row and at least one nonzero component in the first column of B . There are $(p - 1)(r - 1)$ options for choosing these nonzero components. Since the rest of $pr - 3$ entries will be selected depending upon the weights of these two nonzero components, the constructing of the submatrix B can be considered in three cases as given in the lemma. Hence, the proof is completed. \square

Theorem 4.1. *An $[n, k]$ -linear array code C over \mathbb{Z}_{q^l} that corrects all CT burst errors of order $p \times r$ or less and having homogeneous weight w_{hom} or less must satisfy the bound*

$$(12) \quad q^{lms-k} \geq 1 + \sum_{a=1}^p \sum_{b=1}^r \sum_{j=(q-1)q^{l-2}}^{ab(q^{l-1})} \mathbf{B}_{m \times s}^{a \times b}(\mathbb{Z}_{q^l}, j).$$

Proof. It follows directly from Lemma 4.1. \square

5. LINEAR ARRAY CODES WITH 2-REPEATED CT BURST ERRORS

In this section, we present a new transmission model for linear array codes correcting all types of burst errors introduced in Definition 5.1. Suppose that a message is an s -tuple of m -tuples of symbols from \mathbb{Z}_{q^l} . We assume that this message is sent over m parallel channels. When this coded message is transmitted through m parallel channels, it may get corrupted and errors may occur. These errors are not scattered randomly but occur in clusters in the code matrix. In this code matrix, burst errors may repeat usually themselves in the submatrix parts of the code matrix. These errors appear due to very busy communication channels for array coding systems. So, we define the notion of 2-repeated CT burst errors for linear array codes.

Definition 5.1. *A 2-repeated CT burst error of order $p \times r$ in $\text{Mat}_{m \times s}(\mathbb{Z}_{q^l})$ is an $m \times s$ matrix in which all the nonzero entries are confined to two distinct $p \times r$ submatrices, with the first row and the first column of each submatrix being nonzero.*

Lemma 5.1. *The number of all 2-repeated CT burst errors of order $p \times r$ in $Mat_{m \times s}(\mathbb{Z}_{q^l})$ is given by*

$$\mathbf{B}_{m \times s}^{2,p \times r}(\mathbb{Z}_{q^l}) = \left[(m - 2p + 1) \binom{s - 2r + 1}{p \sum_{i=1}^{s - 2r + 1} i} + (m - 2p + 1) \binom{s - 2r + 1}{(p - 1) \sum_{j=1}^{s - 2r + 1} j} + (s - r + 1)^2 \binom{m - 2p + 1}{\sum_{k=1}^{m - 2p + 1} k} + \sum_{u=1}^{s - 2r + 1} u \sum_{v=1}^p v + \sum_{z=1}^{s - 2r + 1} z \sum_{t=0}^{p - 1} t \right] \times (q^l)^{2r(p-1)} \left[((q^l)^r - 1) - ((q^l)^{r-1} - 1) q^{l(1-p)} \right]^2$$

and say

$$\mathbf{B}_{m \times s}^{2,p \times r}(\mathbb{Z}_{q^l}) = \mathbf{B}_{m \times s}^{2,p \times r} \times (q^l)^{2r(p-1)} \left[((q^l)^r - 1) - ((q^l)^{r-1} - 1) q^{l(1-p)} \right]^2.$$

Proof. Consider a matrix A of order $m \times s$ having distinct submatrices B and C of order $p \times r$, with the first row and the first column of each submatrices being nonzero.

For the sake of avoiding complexity, we will take into account the arrangement of submatrices. For this purpose we will set a submatrix as the first one if the row number of its starting position is less than the row number of the starting position of the other one. If starting positions of submatrices are located at the same row then the left one is set as the first one.

In order to obtain 2-repeated burst errors of order $p \times r$, the starting position of the first submatrix can be a_{ij} where $1 \leq i \leq (m - p + 1)$ and $1 \leq j \leq (s - r + 1)$. Note that the column number of starting position of the first submatrix vary between 1 and $s - 2r + 1$ when its row number is $m - p + 1$.

For $m - p + 1$ possible row number of the possible starting positions of the first submatrix, we will consider $m - p + 1$ steps to count the number of starting positions of the second submatrix. We will also consider several cases in these steps. In this manner, first consider the matrix having two submatrices irrespective of their entries, in which the starting position of the first submatrix is a_{11} , then the number of all possible starting positions for the second matrix will be

$$(s - 2r + 1) \cdot p + (s - r + 1)(m - 2p + 1).$$

Afterwards consider the matrix including two submatrices, in which the starting position of the first submatrix is a_{12} , this time the number of all possible starting positions for the second submatrix will be

$$(s - 2r) \cdot p + (s - r + 1)(m - 2p + 1).$$

In this wise, until the starting position of the first submatrix will be a_{1r} , the number of possible starting positions for the second submatrix will decrease p according to the previous case as the column number increase. Because until this case, the second submatrix C can never appear on the left side of the first submatrix B entirely and hence the number of possible starting positions will decrease p for p rows of B .

After this step, the number of possible starting positions for C will decrease 1 according to the previous case each time. Because after this step, the second submatrix C can appear on the left side of B entirely on condition that the row number of its starting position is greater than the row number of the starting

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position of B . Here, we can think this number as decreasing p and increasing $p - 1$ for each case. Hence the number of all possible starting positions for the second submatrix will be

$$(s - 3r + 1) \cdot p + 1 \cdot (p - 1) + (s - r + 1)(m - 2p + 1)$$

whenever the first submatrix starts from $a_{1,r+1}$, and

$$1 \cdot p + (s - 3r + 1)(p - 1) + (s - r + 1)(m - 2p + 1)$$

whenever B starts from $a_{1,s-2r+1}$

After the case the starting position of the first submatrix is $a_{1,s-2r+2}$, the number of the possible starting positions for the second submatrix will not change until the last case of our first step that the starting position of the first submatrix is located in the first row of the matrix $A_{m \times s}$ i.e. the starting position of B is being $a_{1,s-r+1}$. Thus, the number of all possible starting positions for the second submatrix will be

$$(s - 2r + 1)(p - 1) + (s - r + 1)(m - 2p + 1)$$

in this last case.

Up to now, we have considered all 2-repeated burst errors where the starting position of the first submatrix varies in the first row of the matrix A .

In the same way, let us consider all 2-repeated burst errors where this time the starting position of B varies in the second row of the matrix A . In these cases, clearly, the number of possible starting positions for C will decrease $(s - r + 1)$ for each time. When B starts from a_{21} , for instance, the number of all possible starting positions for the second submatrix will be

$$(s - 2r + 1) \cdot p + (s - r + 1)(m - 2p)$$

and

$$(s - 2r + 1)(p - 1) + (s - r + 1)(m - 2p)$$

when B starts from $a_{2,s-r+1}$.

As the row number of the starting position of B increases until the row number $(m - 2p + 1)$ i.e. in each step, the number of possible starting positions for C will be decrease $(s - r + 1)$ each time according to the previous row number. Hence the number of all possible starting positions for C will be

$$(s - 2r + 1) \cdot p$$

when B starts from $a_{m-2p+2,1}$ and

$$(s - 2r + 1)(p - 1)$$

when B starts from $a_{m-2p+2,s-r+1}$.

After this step, the $(s - r + 1)$ decreasing of the number of possible starting positions for the second submatrix for each time will be invalid because the second submatrix will not be able to located under the first submatrix any more.

This combinatoric process proves that the number of all two distinct submatrices of order $p \times r$ in a matrix of order $m \times s$ is

$$\mathbf{B}_{m \times s}^{2,p \times r} = \left[(m - 2p + 1) \binom{s-2r+1}{p \sum_{i=1}^{s-2r+1} i} + (m - 2p + 1) \binom{s-2r+1}{(p-1) \sum_{j=1}^{s-2r+1} j} + (s - r + 1)^2 \binom{m-2p+1}{\sum_{k=1}^{m-2p+1} k} + \sum_{u=1}^{s-2r+1} u \sum_{v=1}^p v + \sum_{z=1}^{s-2r+1} z \sum_{t=0}^{p-1} t \right]$$

Now, let us compute the number of ways that elements of submatrices B and C can be selected. This selection can be done in

$$(13) \quad ((q^l)^r - 1) (q^l)^{r(p-1)}$$

ways if the first row of the submatrix is not entirely zero. However, to consider a CT burst error, the first column of the submatrix must also be nonzero. In this manner, we should subtract the number of cases included in the equation (13) which the first column of the submatrix being zero. This number is given by

$$(14) \quad ((q^l)^{r-1} - 1) (q^l)^{(r-1)(p-1)}.$$

By subtracting (13) from (14) we obtain

$$(15) \quad (q^l)^{r(p-1)} [((q^l)^r - 1) - ((q^l)^{r-1} - 1) (q^l)^{1-p}]$$

and also note that A is a 2-repeated burst error. Hence the lemma. □

Example 5.1. *Let us show all 2-repeated CT burst errors of order 2×2 in $Mat_{4 \times 4}(\mathbb{Z}_{q^l})$ such that all of the entries of the submatrices B and C are 1.*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Note that this number is precisely the same as $\mathbf{B}_{4 \times 4}^{2,2 \times 2}$ since the selection of entries was done uniquely.

Theorem 5.1. *The number of all 2-repeated CT burst errors of order $p \times r$ having homogeneous weight w_{hom} in $Mat_{m \times s}(\mathbb{Z}_{q^l})$ is given by*

$$\mathbf{B}_{m \times s}^{2,p \times r}(\mathbb{Z}_{q^l}, w_{hom}) = \mathbf{B}_{m \times s}^{2,p \times r}(\mathbb{Z}_{q^l}) \times \left[\mathbf{V}_{pr}^2(\mathbb{Z}_{q^l}, w) + \right. \\ \left. 2(p-1)(r-1) \times \sum_{i=0}^3 \sum_{u_i, v_i} \binom{2pr-3}{u_i} \binom{2pr-3-u_i}{v_i} (q-1)^{u_i+i} (q^l-q)^{v_i+3-i} + \right. \\ \left. (p-1)^2(r-1)^2 \times \sum_{j=0}^4 \sum_{u'_j, v'_j} \binom{2pr-4}{u'_j} \binom{2pr-4-u'_j}{v'_j} (q-1)^{u'_j+j} (q^l-q)^{v'_j+4-j} \right],$$

where u_i, v_i and u'_j, v'_j are all nonnegative pairwise integers for $0 \leq i \leq 3$ and $0 \leq j \leq 4$ satisfying

$$u_i + v_i \leq 2pr - 3$$

$$u'_j + v'_j \leq 2pr - 4$$

also for each $0 \leq i \leq 3$ and $0 \leq j \leq 4$ satisfying $w_{hom} = (u_i + i)(q^{l-1}) + (v_i + 3 - i)(q-1)(q^{l-2})$ and $w_{hom} = (u'_j + j)(q^{l-1}) + (v'_j + 4 - j)(q-1)(q^{l-2})$.

Proof. Let A be a matrix of order $m \times s$ having two distinct submatrices B and C with first row and first column of each being nonzero. The places of these two submatrices can be chosen in $\mathbf{B}_{m \times s}^{2,p \times r}$ ways by the lemma 5.1. After choosing arrays of submatrices we have three cases with respect to the weights of the starting positions of B and C .

Case 1: When the starting positions of B and C are both nonzero then we have $\mathbf{V}_{pr}^2(\mathbb{Z}_{q^l}, w)$ CT burst errors.

Case 2: When only one of the starting positions of B or C is zero then we choose one position from the first row and one position from the first column of submatrix whose starting position is zero. Afterwards, we use the same technique for counting burst errors by taking into account the weights of three chosen nonzero positions.

Case 3: When the starting positions of B and C are both zero, then after choosing one position from the first row and one position from the first column for each submatrix, again we count the burst errors taking into account the weights of four chosen nonzero positions. \square

Example 5.2. Let us show all possible submatrices B and C each of order 2×2 and having homogeneous weight 3 altogether in $Mat_{m \times s}(\mathbb{Z}_{2^2})$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$$

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$$\begin{aligned} & \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \\ & \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \\ & \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Note that we showed that there are 16 possible places for locating these submatrices in a 4×4 matrix in example 5.1 and hence there are $16 \times 68 = 1088$ 2-repeated CT burst errors of order 2×2 in $\text{Mat}_{4 \times 4}(\mathbb{Z}_4)$.

Theorem 5.2. An $[n, k]$ -linear array code over \mathbb{Z}_{q^l} that corrects all 2-repeated CT burst errors of order $p \times r$ having homogeneous weight w_{hom} must satisfy the bound

$$(16) \quad q^{lms-k} \geq 1 + \sum_{a=1}^p \sum_{b=1}^r \sum_{j=2^{ab(q^l-1)}}^{2ab(q^l-1)} \mathbf{B}_{m \times s}^{p \times r, 2}(\mathbb{Z}_{q^l}, j).$$

Proof. It follows directly from Lemma 5.1 and Theorem 5.1. □

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