ABELIAN CROSSED MODULES AND STRICT PICARD CATEGORIES

NGUYEN TIEN QUANG

*Department of Mathematics,
Hanoi National University of Education,
Vietnam
E-mail: cn.nguyenquang@gmail.com*

CHE THI KIM PHUNG

*Department of Mathematics and Applications,
Saigon University, Vietnam
E-mail: kimphungk25@yahoo.com*

NGO SY TUNG

*Department of Mathematics,
Vinh University, Vietnam
E-mail: ngositung@yahoo.com*

Abstract. In this paper, we state the notion of morphisms in the category of abelian crossed modules and prove that this category is equivalent to the category of strict Picard categories and regular symmetric monoidal functors. The theory of obstructions for symmetric monoidal functors and symmetric cohomology groups are applied to show a treatment of the group extension problem of the type of an abelian crossed module.

1. INTRODUCTION

Crossed modules have been used widely, and in various contexts, since their definition by Whitehead [14] in his investigation of the algebraic structure of second relative homotopy groups. A brief summary of researches related to crossed modules was given in [4] in which Carrasco et al. obtained interesting results on the category

---

of *abelian* crossed modules. The notion of abelian crossed module was characterized by that of the *center* of a crossed module in the paper of Norrie [11].

Crossed modules are essentially the same as *strict categorical groups* (see [7, 3, 1, 10]). A *strict* categorical group is a categorical group in which the associativity, unit constraints are strict ($\mathbf{a} = id, \mathbf{l} = id = \mathbf{r}$) and, for each object x, there is an object y such that $x \otimes y = 1 = y \otimes x$. This concept is also called a $\mathcal{G}$-*groupoid* by Brown and Spencer [3], or a *2-group* by Noohi [10], or a *strict 2-group* by Baez and Lauda [1].

Brown and Spencer [3] (Theorem 1) published a proof that the category of $\mathcal{G}$-*groupoids* is equivalent to the category **CrossMd** of crossed modules (the morphisms in the first category are functors preserving the group structure, those in the second category are homomorphisms of crossed modules).

Another result on crossed modules, the group extension problem of the type of a crossed module, was presented by Brown and Mucuk in [2] (Theorem 5.2). This problem has attracted the attention of many mathematicians.

In our opinion, the above data can be considered for abelian crossed modules. At the beginning of Section 3, we show that each abelian crossed module is seen as a strict Picard category (as defined in Section 2). Therefore, we can apply Picard category theory to study abelian crossed modules and obtain results similar to the above results on crossed modules.

The content of this paper consists of two main results.

In Section 3, we prove that (Theorem 4) the category **Picstr** of strict Picard categories and regular symmetric monoidal functors is equivalent to the category **AbCross** of abelian crossed modules. Every morphism in the category **AbCross** consists of a homomorphism $(f_1, f_0) : \mathcal{M} \to \mathcal{M}'$ of abelian crossed modules and an element of the group of symmetric 2-cocycles $Z_s^2(\pi_0\mathcal{M}, \pi_1\mathcal{M}')$. This theorem is analogous to [3, Theorem 1].

In Section 4, we study the group extension problem of the type of an abelian crossed module. The theory of obstructions for symmetric monoidal functors is applied to show a treatment of this problem. Each abelian crossed module $B \xrightarrow{d} D$ defines a strict Picard category $\mathbb{P}$. The third invariant of $\mathbb{P}$ is an element $\bar{k} \in H_s^3(\operatorname{Coker} d, \operatorname{Ker} d)$. Then a group homomorphism $\psi : Q \to \operatorname{Coker} d$ induces $\overline{\psi^* k} \in Z_s^3(Q, \operatorname{Ker} d)$. Theorem 7 shows that the vanishing of $\psi^* k$ in $H_s^3(Q, \operatorname{Ker} d)$ is necessary and sufficient for there to exist a group extension of the type of an abelian crossed module $B \xrightarrow{d} D$. Each such extension induces a symmetric monoidal functor $F \colon \operatorname{Dis} Q \to \mathbb{P}$. This correspondence determines a bijection (Theorem 6)

$$\Omega : \operatorname{Hom}_{(\psi,0)}^{Pic}[\operatorname{Dis}Q, \mathbb{P}_{B \to D}] \to \operatorname{Ext}_{B \to D}^{ab}(Q, B, \psi).$$

Theorem 7 is analogous to Theorem 5.2 [2].

2. Preliminaries

A *symmetric monoidal category* $\mathbb{P} := (\mathbb{P}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r}, \mathbf{c})$ consists of a category $\mathbb{P}$, a functor $\otimes : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ and natural isomorphisms $\mathbf{a}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \mathbf{l}_X : I \otimes X \xrightarrow{\sim} X, \mathbf{r}_X : X \otimes I \xrightarrow{\sim} X$ and $\mathbf{c}_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ such that, for any objects X, Y, Z, T of $\mathbb{P}$, the following coherence conditions hold:

i) $\mathbf{a}_{X,Y,Z \otimes T} \mathbf{a}_{X \otimes Y, Z, T} = (id_X \otimes \mathbf{a}_{Y,Z,T}) \mathbf{a}_{X, Y \otimes Z, T} (\mathbf{a}_{X,Y,Z} \otimes id_T)$,

ii) $\mathbf{c}_{X,Y} \cdot \mathbf{c}_{Y,X} = id_{Y \otimes X}$,

iii) $(id_X \otimes \mathbf{l}_Y)\mathbf{a}_{X,I,Y} = \mathbf{r}_X \otimes id_Y$,

iv) $(id_Y \otimes \mathbf{c}_{X,Z})\mathbf{a}_{Y,X,Z}(\mathbf{c}_{X,Y} \otimes id_Z) = \mathbf{a}_{Y,Z,X}\mathbf{c}_{X,Y\otimes Z}\mathbf{a}_{X,Y,Z}$.

A *Picard category* is a symmetric monoidal category in which every morphism is invertible and, for each object X, there is an object Y with a morphism $X \otimes Y \to I$.

A Picard category is said to be *strict* when the constraints $\mathbf{a} = id$, $\mathbf{c} = id$, $\mathbf{l} = id = \mathbf{r}$ and, for each object X, there is an object Y such that $X \otimes Y = I$.

If $\mathbb{P}$, $\mathbb{P}'$ are symmetric monoidal categories, then a *symmetric monoidal functor* $F := (F, \widetilde{F}, F_*) : \mathbb{P} \to \mathbb{P}'$ consists of a functor $F : \mathbb{P} \to \mathbb{P}'$, natural isomorphisms $\widetilde{F}_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ and an isomorphism $F_* : I' \to FI$, such that, for any objects X, Y, Z of $\mathbb{P}$, the following coherence conditions hold:

$$\widetilde{F}_{X,Y\otimes Z}(id_{FX} \otimes \widetilde{F}_{Y,Z})\mathbf{a}_{FX,FY,FZ} = F(\mathbf{a}_{X,Y,Z})\widetilde{F}_{X\otimes Y,Z}(\widetilde{F}_{X,Y} \otimes id_{FZ}),$$

$$F(\mathbf{r}_X)\widetilde{F}_{X,I}(id_{FX} \otimes F_*) = \mathbf{r}_{FX} \ , \ F(\mathbf{l}_X)\widetilde{F}_{I,X}(F_* \otimes id_{FX}) = \mathbf{l}_{FX},$$

$$\widetilde{F}_{Y,X}\mathbf{c}_{FX,FY} = F(\mathbf{c}_{X,Y})\widetilde{F}_{X,Y}.$$

Note that if $F := (F, \widetilde{F}, F_*)$ is a symmetric monoidal functor between Picard categories, then the isomorphism $F_* : I' \to FI$ is implied from F and $\widetilde{F}$, so we can omit F_* when not necessary.

A *symmetric monoidal natural equivalence* between symmetric monoidal functors $(F, \widetilde{F}, F_*), (F', \widetilde{F}', F'_*) : \mathbb{P} \to \mathbb{P}'$ is a natural equivalence $\theta : F \xrightarrow{\sim} F'$ such that, for any objects X, Y of $\mathbb{P}$, the following coherence conditions hold:

$$\widetilde{F}'_{X,Y}(\theta_X \otimes \theta_Y) = \theta_{X\otimes Y}\widetilde{F}_{X,Y}, \ \theta_I F_* = F'_*.$$

Let $\mathbb{P} := (\mathbb{P}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r}, \mathbf{c})$ be a Picard category. According to Sinh [13], $\mathbb{P}$ is equivalent to its reduced Picard category $\mathbb{S} = S_{\mathbb{P}}$ thanks to *canonical* equivalences

$$G : \mathbb{P} \to \mathbb{S}, \quad H : \mathbb{S} \to \mathbb{P}.$$

For convenience, we briefly recall the construction of $\mathbb{S}$. Let $M = \pi_0\mathbb{P}$ be the abelian group of isomorphism classes of the objects in $\mathbb{P}$ where the operation is induced by the tensor product, $N = \pi_1\mathbb{P}$ be the abelian group of automorphisms of the unit object I of $\mathbb{P}$ where the operation is composition. Then, objects of $\mathbb{S}$ are elements $x \in M$, and its morphisms are automorphisms $(a, x) : x \to x$, $a \in N$. The composition of morphisms is given by

$$(a, x) \circ (b, x) = (a + b, x).$$

The tensor product is defined by

$$x \otimes y = x + y,$$

$$(a, x) \otimes (b, y) = (a + b, x + y).$$

The unit constraints in $\mathbb{S}$ are strict (in the sense that $\mathbf{l}_x = \mathbf{r}_x = id_x$), the associativity constraint ξ and the symmetry constraint η are, respectively, functions $M^3 \to N$, $M^2 \to N$ satisfying normalized condition:

$$\xi(0, y, z) = \xi(x, 0, z) = \xi(x, y, 0) = 0.$$

and satisfying the following relations:

i) $\xi(y, z, t) - \xi(x + y, z, t) + \xi(x, y + z, t) - \xi(x, y, z + t) + \xi(x, y, z) = 0$,

ii) $\eta(x, y) + \eta(y, x) = 0$,

iii) $\xi(x, y, z) - \xi(y, x, z) + \xi(y, z, x) + \eta(x, y + z) - \eta(x, y) - \eta(x, z) = 0$,

The pair (ξ, η) satisfying these relations is just an element in the group $Z_s^3(M, N)$ of *symmetric* 3-cocycles in the sense of [8]. We refer to $\mathbb{S}$ as *Picard category of type* (M, N).

Let $\mathbb{S} = (M, N, \xi, \eta), \mathbb{S}' = (M', N', \xi', \eta')$ be Picard categories. A functor $F : \mathbb{S} \to \mathbb{S}'$ is called a *functor of type* (φ, f) if there are group homomorphisms $\varphi : M \to M'$, $f : N \to N'$ satisfying

$$F(x) = \varphi(x), \quad F(a, x) = (f(a), \varphi(x)).$$

In this case, (φ, f) is called a *pair of homomorphisms*, and the function

(1) $$k = \varphi^*(\xi', \eta') - f_*(\xi, \eta)$$

is called an *obstruction* of the functor $F : \mathbb{S} \to \mathbb{S}'$ of type (φ, f).

The following proposition is implied from the results on monoidal functors of type (φ, f) in [12] .

Proposition 1. *Let* $\mathbb{P}, \mathbb{P}'$ *be Picard categories and* $\mathbb{S}, \mathbb{S}'$ *be their reduced Picard categories, respectively.*

i) Any symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{P} \to \mathbb{P}'$ *induces a symmetric monoidal functor* $\mathbb{S}_F : \mathbb{S} \to \mathbb{S}'$ *of type* (φ, f)*. Further,* $\mathbb{S}_F = G'FH$*, where* H, G' *are canonical equivalences.*

ii) Any symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{S} \to \mathbb{S}'$ *is a functor of type* (φ, f)*.*

iii) The functor $F : \mathbb{S} \to \mathbb{S}'$ *of type* (φ, f) *is realizable, i.e., there are isomorphisms* $\widetilde{F}_{x,y}$ *so that* $(F, \widetilde{F})$ *is a symmetric monoidal functor, if and only if its obstruction* $\overline{k}$ *vanishes in* $H_s^3(M, N')$*. Then, there is a bijection*

$$\mathrm{Hom}_{(\varphi, f)}^{Pic}[\mathbb{S}, \mathbb{S}'] \leftrightarrow H_s^2(M, N'),$$

where $\mathrm{Hom}_{(\varphi, f)}^{Pic}[\mathbb{S}, \mathbb{S}']$ *denotes the set of homotopy classes of symmetric monoidal functors of type* (φ, f) *from* $\mathbb{S}$ *to* $\mathbb{S}'$*.*

3. Classification of abelian crossed modules by strict Picard categories

In this section, we will show a treatment of the problem on classification of abelian crossed modules due to 2-dimensional symmetric cohomology groups and regular symmetric monoidal functors.

We recall that a *crossed module* is a quadruple $\mathcal{M} = (B, D, d, \theta)$, where $d : B \to D$, $\theta : D \to \mathrm{Aut}B$ are group homomorphisms satisfying the following relations:

C_1. $\theta d = \mu$,

C_2. $d(\theta_x(b)) = \mu_x(d(b)), \quad x \in D, b \in B$,

where μ_x is an inner automorphism given by x.

In this paper, the crossed module (B, D, d, θ) is sometimes denoted by $B \xrightarrow{d} D$, or simply $B \to D$.

Standard consequences of the axioms are that $\mathrm{Ker}\, d$ is a left $\mathrm{Coker}\, d$-module under the action

$$sa = \theta_x(a), \quad a \in \mathrm{Ker}\, d, \ x \in s \in \mathrm{Coker}\, d.$$

The groups $\mathrm{Coker}\, d, \mathrm{Ker}\, d$ are also denoted by $\pi_0\mathcal{M}, \pi_1\mathcal{M}$, respectively.

We are interested in the case when B, D are abelian groups. Then, it follows from the condition C_1 that $\theta d = id$ (and hence $\mathrm{Im}d$ acts trivially on B). The

condition C_2 leads to $\theta_x(b) - b \in \operatorname{Ker} d$. Therefore, θ determines a function $g :$ $\operatorname{Coker} d \times \operatorname{Ker} d \to \operatorname{Ker} d$ by

$$g(s, b) = sb - b.$$

It is straightforward to see that g is a biadditive normalized function. Conversely, the data $(B \xrightarrow{d} D, g)$, where B, D are abelian, determines completely a crossed module. Particularly, if $g = 0$ we obtain the notion of *abelian crossed module*. In other words, abelian crossed modules are defined as follows.

Definition. A crossed module $\mathcal{M} = (B, D, d, \theta)$ is said to be *abelian* when B, D are abelian and $\theta = 0$.

For example, if $\mathcal{M}$ is a crossed module in which B, D are abelian and d is a monomorphism, then $\theta = 0$. Therefore, $\mathcal{M}$ is an abelian crossed module.

The notion of abelian crossed modules can be characterized by that of the center of crossed modules as in Norrie's work [11]. We say that the *center* $\xi\mathcal{M}$ of a crossed module $\mathcal{M} = (B, D, d, \theta)$ is a subcrossed module of $\mathcal{M}$ and defined by $(B^D, st_D(B) \cap Z(D), d, \theta)$, where B^D is the *fixed point subgroup* of B, $st_D(B)$ is the *stabilizer* in D of B, that is,

$$B^D = \{b \in B : \theta_x b = b \text{ for all } x \in D\},$$
$$st_D(B) = \{x \in D : \theta_x b = b \text{ for all } b \in B\},$$

and $Z(D)$ is the center of D (note that B^D is in the center of B). Then, the crossed module is termed *abelian* if $\xi(B, D, d, \theta) = (B, D, d, \theta)$.

It is well-known that crossed modules are the same as strict categorical groups (see [7], Remark 3.1). Now, we show that abelian crossed modules can be seen as strict Picard categories. We state this in detail.

• For any abelian crossed module $B \to D$, we can construct a strict Picard category $\mathbb{P}_{B \to D} = \mathbb{P}$, called the Picard category *associated to* the abelian crossed module $B \to D$, as follows.

$$\operatorname{Ob}(\mathbb{P}) = D, \quad \operatorname{Hom}(x, y) = \{b \in B \mid x = d(b) + y\},$$

for objects $x, y \in D$. The composition of two morphisms is given by

$$(2) \qquad\qquad (x \xrightarrow{b} y \xrightarrow{c} z) = (x \xrightarrow{b+c} z).$$

The tensor operation on objects is given by the addition in the group D and, for two morphisms $(x \xrightarrow{b} y), (x' \xrightarrow{b'} y')$ in $\mathbb{P}$, one defines

$$(3) \qquad\qquad (x \xrightarrow{b} y) \otimes (x' \xrightarrow{b'} y') = (x + x' \xrightarrow{b+b'} y + y').$$

Associativity, commutativity and unit constraints are identities ($\mathbf{a} = id, \mathbf{c} = id, \mathbf{l} = id = \mathbf{r}$). By the definition of an abelian crossed module, it is easy to check that $\mathbb{P}$ is a strict Picard category.

• Conversely, for a strict Picard category $(\mathbb{P}, \otimes)$, we determine an *associated* abelian crossed module $\mathcal{M}_{\mathbb{P}} = (B, D, d)$ as follows. Set

$$D = \operatorname{Ob}(\mathbb{P}), \quad B = \{x \xrightarrow{b} 0 | x \in D\}.$$

The operations in D and B are, respectively, given by

$$x + y = x \otimes y, \quad b + c = b \otimes c.$$

Then D becomes an abelian group whose zero element is 0, and the inverse of x is $-x$ $(x \otimes (-x) = 0)$. B is a group whose zero element is id_0, and the inverse of $(x \xrightarrow{b} 0)$ is the morphism $(-x \xrightarrow{\bar{b}} 0)(b \otimes \bar{b} = id_0)$. Further, B is abelian due to the naturality of the commutativity constraint $\mathbf{c} = id$.

The homomorphism $d : B \to D$ is given by

$$d(x \xrightarrow{b} 0) = x.$$

Definition. A *homomorphism* $(f_1, f_0) : (B, D, d) \to (B', D', d')$ of abelian crossed modules consists of group homomorphisms $f_1 : B \to B'$, $f_0 : D \to D'$ such that

$$f_0 d = d' f_1.$$

Clearly, the category of abelian crossed modules is a full subcategory of the category of crossed modules.

In order to classify abelian crossed modules we establish the following lemmas.

Lemma 2. *Let* $(f_1, f_0) : \mathcal{M} = (B, D, d) \to \mathcal{M}' = (B', D', d')$ *be a homomorphism of abelian crossed modules. Let* $\mathbb{P}$ *and* $\mathbb{P}'$ *be strict Picard categories associated to* $\mathcal{M}$ *and* $\mathcal{M}'$, *respectively.*

i) There exists the functor $F : \mathbb{P} \to \mathbb{P}'$ *defined by* $F(x) = f_0(x)$, $F(b) = f_1(b)$, *for* $x \in D, b \in B$.

ii) Natural isomorphisms $\widetilde{F}_{x,y} : F(x) + F(y) \to F(x + y)$ *together with* F *is a symmetric monoidal functor if and only if* $\widetilde{F}_{x,y} = \varphi(\overline{x}, \overline{y})$, *where* φ *is a symmetric 2-cocycle of the group* $Z_s^2(\operatorname{Coker} d, \operatorname{Ker} d')$.

Proof. i) By the determination of a strict Picard category associated to an abelian crossed module and the fact that f_1 is a homomorphism, F is a functor.

ii) Since f_1, f_0 are group homomorphisms, for two morphisms $(x \xrightarrow{b} x')$, $(y \xrightarrow{c} y')$ in $\mathbb{P}$, we have

$$F(b \otimes c) = F(b) \otimes F(c).$$

On the other hand, since f_0 is a homomorphism and $F(x) = f_0(x)$, $\widetilde{F}_{x,y} : F(x) + F(y) \to F(x + y)$ is a morphism in $\mathbb{P}'$ if and only if $d'(\widetilde{F}_{x,y}) = 0'$, i.e.,

$$\widetilde{F}_{x,y} \in \operatorname{Ker} d'.$$

Then the naturality of $(F, \widetilde{F})$, that is the commutativity of the following diagram

$$
\begin{array}{ccc}
F(x) + F(y) & \xrightarrow{\widetilde{F}_{x,y}} & F(x + y) \\
{\scriptstyle F(b) \otimes F(c)} \downarrow & & \downarrow {\scriptstyle F(b \otimes c)} \\
F(x') + F(y') & \xrightarrow[\widetilde{F}_{x',y'}]{} & F(x' + y'),
\end{array}
$$

is equivalent to the relation $\widetilde{F}_{x,y} = \widetilde{F}_{x',y'}$, where $x = d(b) + x', y = d(c) + y'$. This determines a function $\varphi : \operatorname{Coker} d \times \operatorname{Coker} d \to \operatorname{Ker} d'$ by

$$\varphi(\overline{x}, \overline{y}) = \widetilde{F}_{x,y}.$$

By $F(0) = 0'$, the compatibility of $(F, \widetilde{F})$ with unit constraints is equivalent to the normalization of φ. From the relations (2) and (3), the compatibility of $(F, \widetilde{F})$ with associativity, commutativity constraints are, respectively, equivalent to relations

$$\widetilde{F}_{y,z} + \widetilde{F}_{x,y+z} = \widetilde{F}_{x,y} + \widetilde{F}_{x+y,z},$$

$$\widetilde{F}_{x,y} = \widetilde{F}_{y,x}.$$

This shows that $\varphi \in Z_s^2(\operatorname{Coker} d, \operatorname{Ker} d')$. $\qquad\square$

Definition. A symmetric monoidal functor $(F, \widetilde{F}) : \mathbb{P} \to \mathbb{P}'$ between Picard categories $\mathbb{P}, \mathbb{P}'$ is termed *regular* if $F(x) \otimes F(y) = F(x \otimes y)$ for $x, y \in \operatorname{Ob}\mathbb{P}$.

Thanks to Lemma 2, we determine the category **AbCross** whose objects are abelian crossed modules and morphisms are triples (f_1, f_0, φ), where (f_1, f_0) : $(B \xrightarrow{d} D) \to (B' \xrightarrow{d'} D')$ is a homomorphism of abelian crossed modules and $\varphi \in Z_s^2(\operatorname{Coker} d, \operatorname{Ker} d')$.

Lemma 3. *Let* $\mathbb{P}$, $\mathbb{P}'$ *be corresponding strict Picard categories associated to abelian crossed modules* (B, D, d), (B', D', d'), *and let* $(F, \widetilde{F}) : \mathbb{P} \to \mathbb{P}'$ *be a regular symmetric monoidal functor. Then, the triple* (f_1, f_0, φ), *where*

$$f_1(b) = F(b), \ f_0(x) = F(x), \ \varphi(s_1, s_2) = \widetilde{F}_{x_1,x_2},$$

for $b \in B$, $x \in D$, $x_i \in s_i \in \operatorname{Coker} d, i = 1, 2$, *is a morphism in* **AbCross**.

Proof. Since F is regular, f_0 is a group homomorphism. Since F preserves the composition of morphisms, f_1 is a group homomorphism.

Any $b \in B$ can be considered as a morphism $(db \xrightarrow{b} 0)$ in $\mathbb{P}$, and hence $(F(db) \xrightarrow{F(b)} 0')$ is a morphism in $\mathbb{P}'$. This means that $f_0(d(b)) = d'(f_1(b))$, for all $b \in B$. Thus, (f_1, f_0) is a homomorphism of abelian crossed modules.

According to Lemma 2, $\widetilde{F}_{x_1,x_2}$ determines a function $\varphi \in Z_s^2(\operatorname{Coker} d, \operatorname{Ker} d')$ by

$$\varphi(s_1, s_2) = \widetilde{F}_{x_1,x_2}, \ \ x_i \in s_i \in \operatorname{Coker} d, \ i = 1, 2.$$

Therefore, (f_1, f_0, φ) is a morphism in **AbCross**. $\qquad\square$

Let **Picstr** denote the category of strict Picard categories and regular symmetric monoidal functors, we obtain the following theorem.

Theorem 4 (Classification Theorem)**.** *There exists an equivalence*

$$\begin{aligned}
\Phi : \mathbf{AbCross} \quad &\to \quad \mathbf{Picstr}, \\
(B \to D) \quad &\mapsto \quad \mathbb{P}_{B \to D}, \\
(f_1, f_0, \varphi) \quad &\mapsto \quad (F, \widetilde{F}),
\end{aligned}$$

where $F(x) = f_0(x)$, $F(b) = f_1(b)$, $\widetilde{F}_{x_1,x_2} = \varphi(s_1, s_2)$ *for* $x \in D$, $b \in B$, $x_i \in s_i \in \operatorname{Coker} d$, $i = 1, 2$.

Proof. Suppose that $\mathbb{P}$ and $\mathbb{P}'$ are Picard categories associated to abelian crossed modules $B \to D$ and $B' \to D'$, respectively. By Lemma 2, the correspondence $(f_1, f_0, \varphi) \mapsto (F, \widetilde{F})$ determines an injection on the homsets

$$\Phi : \operatorname{Hom}_{\mathbf{AbCross}}(B \to D, B' \to D') \to \operatorname{Hom}_{\mathbf{Picstr}}(\mathbb{P}_{B \to D}, \mathbb{P}_{B' \to D'}).$$

According to Lemma 3, Φ is surjective.

If $\mathbb{P}$ is a strict Picard category, and $\mathcal{M}_{\mathbb{P}}$ is an abelian crossed module associated to it, then $\Phi(\mathcal{M}_{\mathbb{P}}) = \mathbb{P}$ (not only isomorphic). Therefore, Φ is an equivalence. $\qquad\square$

Remark. Denote by **AbCross*** the subcategory of **AbCross** whose morphisms are homomorphisms of abelian crossed modules ($\varphi = 0$), and denote by **Picstr*** the subcategory of **Picstr** whose morphisms are *strict* symmetric monoidal functors ($\widetilde{F} = id$). Then these two categories are equivalent via Φ. This result is analogous to Theorem 1 [3].

4. Classification of group extensions of the type of an abelian crossed module

The concept of group extension of the type of a crossed module was introduced by Dedecker [5] (see also [2]). This concept has a version for abelian crossed modules as follows.

Definition. Let $\mathcal{M} = (B \xrightarrow{d} D)$ be an abelian crossed module, and let Q be an abelian group. An *abelian extension of B by Q of type $\mathcal{M}$* is the diagram of group homomorphisms

(4)
$$\mathcal{E}: \quad 0 \longrightarrow B \xrightarrow{j} E \xrightarrow{p} Q \longrightarrow 0,$$
$$B \xrightarrow{d} D$$

where the top row is exact and (id_B, ε) is a homomorphism of abelian crossed modules.

So, any extension of the type of an abelian crossed module is an extension of the type of a crossed module.

Two extensions $\mathcal{E}, \mathcal{E}'$ of B by Q of type $\mathcal{M}$ are said to be *equivalent* if the following diagram commutes

(5)
$$\mathcal{E}: \quad 0 \longrightarrow B \xrightarrow{j} E \xrightarrow{p} Q \longrightarrow 0, \qquad E \xrightarrow{\varepsilon} D$$
$$\mathcal{E}': \quad 0 \longrightarrow B \xrightarrow{j'} E' \xrightarrow{p'} Q \longrightarrow 0, \qquad E' \xrightarrow{\varepsilon'} D$$

and $\varepsilon'\alpha = \varepsilon$. Obviously, α is an isomorphism.

In the diagram

$$\mathcal{E}: \quad 0 \longrightarrow B \xrightarrow{j} E \xrightarrow{p} Q \longrightarrow 0,$$
$$B \xrightarrow{d} D \xrightarrow{q} \mathrm{Coker}\,d$$

since the top row is exact and $q \circ \varepsilon \circ j = q \circ d = 0$, there is a homomorphism $\psi : Q \to \mathrm{Coker}\,d$ such that the right hand side square commutes. Moreover, ψ is dependent only on the equivalence class of the extension $\mathcal{E}$.

Our objective is to study the set

$$\mathrm{Ext}^{ab}_{B \to D}(Q, B, \psi)$$

of equivalence classes of extensions of B by Q of type $B \xrightarrow{d} D$ inducing $\psi : Q \to \mathrm{Coker}\,d$. It is well-known that the set $\mathrm{Ext}_{B \to D}(Q, B, \psi)$ for extensions of the type of a (not necessarily abelian) crossed module was classified by Brown and Mucuk. In the present paper, we use the obstruction theory of symmetric monoidal functors to prove Theorem 7 which is an abelian analogue of Theorem 5.2 in [2]. The second assertion of this theorem can be seen as a consequence of Schreier Theory (Theorem 6) due to symmetric monoidal functors between strict Picard categories $\mathbb{P}_{B \to D}$ and $\mathrm{Dis}\,Q$, where $\mathrm{Dis}\,Q$ is a Picard category of type $(Q, 0, 0)$.

Lemma 5. *Let $B \xrightarrow{d} D$ be an abelian crossed module, Q be an abelian group and $\psi : Q \to \operatorname{Coker} d$ be a group homomorphism. Then, for each symmetric monoidal functor $(F, \widetilde{F}) : \operatorname{Dis} Q \to \mathbb{P}$ which satisfies $F(0) = 0$ and induces the pair $(\psi, 0) : (Q, 0) \to (\operatorname{Coker} d, \operatorname{Ker} d)$, there exists an extension $\mathcal{E}_F$ of B by Q of type $B \to D$ inducing ψ.*

Such an extension $\mathcal{E}_F$ is called *associated* to a symmetric monoidal functor $(F, \widetilde{F})$.

Proof. Suppose that $(F, \widetilde{F}) : \operatorname{Dis} Q \to \mathbb{P}$ is a symmetric monoidal functor. Then, we set a function $f : Q \times Q \to B$ as follows

$$f(u, v) = \widetilde{F}_{u,v}.$$

Because $\widetilde{F}_{u,v}$ is a morphism in $\mathbb{P}$, one has

$$F(u) + F(v) = df(u, v) + F(u + v).$$

Since $F(0) = 0$ and $(F, \widetilde{F})$ is compatible with the strict constraints of $\operatorname{Dis} Q$ and $\mathbb{P}$, f is a normalized function satisfying

(6) $$f(v, t) + f(u, v + t) = f(u, v) + f(u + v, t),$$

(7) $$f(u, v) = f(v, u).$$

Now we construct the semidirect product $E_0 = [B, f, Q]$, that is, $E_0 = B \times Q$ with the operation

$$(b, u) + (c, v) = (b + c + f(u, v), u + v).$$

The set E_0 is an abelian group due to the normalization of f and the relations (6), (7), the zero element is $(0, 0)$ and $-(b, u) = (-b - f(u, -u), -u)$. Then, we have an exact sequence of abelian groups

$$\mathcal{E}_F : \quad 0 \to B \xrightarrow{j_0} E_0 \xrightarrow{p_0} Q \to 0,$$

where

$$j_0(b) = (b, 0), \ p_0(b, u) = u, \ b \in B, u \in Q.$$

The map $\varepsilon : E_0 \to D$ given by

$$\varepsilon(b, u) = db + F(u), \ (b, u) \in E_0,$$

is a homomorphism, and hence the pair (id_B, ε) is a homomorphism of abelian crossed modules. Therefore, one obtains an extension of the type of an abelian crossed module $\mathcal{E}_F$ satisfying the diagram (4). For all $u \in Q$, one has

$$q\varepsilon(b, u) = q(db + F(u)) = qF(u) = \psi(u),$$

i.e., this extension induces $\psi : Q \to \operatorname{Coker} d$. $\qquad\qquad\square$

Under the assumptions of Lemma 5, we have

Theorem 6 (Schreier Theory for group extensions of the type of an abelian crossed module)**.** *There exists a bijection*

$$\Omega : \operatorname{Hom}^{Pic}_{(\psi, 0)}[\operatorname{Dis} Q, \mathbb{P}_{B \to D}] \to \operatorname{Ext}^{ab}_{B \to D}(Q, B, \psi)$$

if one of the above sets is nonempty.

Proof. Step 1: Symmetric monoidal functors $(F, \widetilde{F})$, $(F', \widetilde{F}')$ are homotopic if and only if the corresponding associated extensions $\mathcal{E}_F, \mathcal{E}_{F'}$ are equivalent.

First, since every symmetric monoidal functor $(F, \widetilde{F})$ is homotopic to one $(G, \widetilde{G})$ in which $G(0) = 0$, the following symmetric monoidal functors are regarded as the functors which have this property .

Suppose that $F, F' : \mathrm{Dis}Q \to \mathbb{P}_{B \to D}$ are homotopic by a homotopy $\alpha : F \to F'$. By Lemma 5, there exist the extensions $\mathcal{E}_F$ and $\mathcal{E}_{F'}$ associated to F and F', respectively. Then, it follows from the definition of a homotopy that $\alpha_0 = 0$ and the following diagram commutes

$$
\begin{array}{ccc}
Fu + Fv & \xrightarrow{\widetilde{F}_{u,v}} & F(u+v) \\
{\scriptstyle \alpha_u \otimes \alpha_v} \downarrow & & \downarrow {\scriptstyle \alpha_{u+v}} \\
F'u + F'v & \xrightarrow[\widetilde{F}'_{u,v}]{} & F'(u+v),
\end{array}
$$

that is,

$$
\widetilde{F}_{u,v} + \alpha_{u+v} = \alpha_u \otimes \alpha_v + \widetilde{F}'_{u,v}.
$$

By the relation (3), one has

$$
(8) \qquad\qquad f(u,v) + \alpha_{u+v} = \alpha_u + \alpha_v + f'(u,v),
$$

where $f(u,v) = \widetilde{F}_{u,v}, f'(u,v) = \widetilde{F}'_{u,v}$. Now we set

$$
\alpha^* : E_F \to E_{F'}
$$
$$
(b, u) \mapsto (b + \alpha_u, u).
$$

Then α^* is a homomorphism thanks to the relation (8). Further, the diagram (5) commutes. It remains to show that $\varepsilon' \alpha^* = \varepsilon$. Since $\alpha : F \to F'$ is a homotopy, $F(u) = d(\alpha_u) + F'(u)$. Then,

$$
\varepsilon' \alpha^*(b, u) = \varepsilon'(b + \alpha_u, u) = d(b + \alpha_u) + F'(u)
$$
$$
= d(b) + d(\alpha_u) + F'(u) = d(b) + F(u) = \varepsilon(b, u).
$$

Therefore, $\mathcal{E}_F$ and $\mathcal{E}_{F'}$ are equivalent.

Conversely, if an isomorphism $\alpha^* : E_F \to E_{F'}$ satisfying the triple (id_B, α^*, id_Q) is an equivalence of extensions, then it is easy to see that

$$
\alpha^*(b, u) = (b + \alpha_u, u),
$$

where $\alpha : Q \to B$ is a function satisfying $\alpha_0 = 0$. By retracing our steps, α is a homotopy between F and F'.

Step 2: Ω is surjective.

Assume that $\mathcal{E}$ is an extension E of B by Q of type $B \to D$ inducing $\psi : Q \to \mathrm{Coker}\, d$. We prove that $\mathcal{E}$ is equivalent to the semidirect product extension $\mathcal{E}_F$ which is associated to a symmetric monoidal functor $(F, \widetilde{F}) : \mathrm{Dis}Q \to \mathbb{P}_{B \to D}$.

For any $u \in Q$, choose a representative $e_u \in E$ such that $p(e_u) = u$, $e_0 = 0$. Each element of E can be represented uniquely as $b + e_u$ for $b \in B, u \in Q$. The representatives $\{e_u\}$ induces a normalized function $f : Q \times Q \to B$ by

$$
(9) \qquad\qquad e_u + e_v = f(u,v) + e_{u+v}.
$$

Then, the group structure of E can be described by

$$
(b + e_u) + (c + e_v) = b + c + f(u,v) + e_{u+v}.
$$

Now, we construct a symmetric monoidal functor $(F, \widetilde{F}) : \text{Dis}\, Q \to \mathbb{P}$ as follows. Since $\psi(u) = \psi p(e_u) = q\varepsilon(e_u)$, $\varepsilon(e_u)$ is a representative of $\psi(u)$ in D. Thus, we set

$$F(u) = \varepsilon(e_u), \ \widetilde{F}_{u,v} = f(u,v).$$

The relation (9) shows that $\widetilde{F}_{u,v}$ are actually morphisms in $\mathbb{P}$. Obviously, $F(0) = 0$. This together with the normalization condition of the function f implies the compatibility of $(F, \widetilde{F})$ with the unit constraints. The associativity and commutativity laws of the operation in E lead to the relations (6), (7), respectively. These relations prove that $(F, \widetilde{F})$ is compatible with the associativity and commutativity constraints of $\text{Dis}Q$ and $\mathbb{P}$, respectively. The naturality of $\widetilde{F}_{u,v}$ and the condition of F preserving the composition of morphisms are obvious.

Finally, it is easy to check that the semidirect product extension $\mathcal{E}_F$ associated to $(F, \widetilde{F})$ is equivalent to the extension $\mathcal{E}$ by the isomorphism $\beta : (b, u) \mapsto b + e_u$. $\quad\square$

Let $\mathbb{P} = \mathbb{P}_{B \to D}$ be a strict Picard category associated to an abelian crossed module $B \to D$. Since $\pi_0(\mathbb{P}) = \text{Coker}\, d$ and $\pi_1(\mathbb{P}) = \text{Ker}\, d$, the reduced Picard category $S_{\mathbb{P}}$ is of form

$$S_{\mathbb{P}} = (\text{Coker}d, \text{Ker}d, k), \ \overline{k} \in H^3_s(\text{Coker}d, \text{Ker}d).$$

Then, by the relation (1), the pair of homomorphisms $(\psi, 0) : (Q, 0) \to (\text{Coker}d, \text{Ker}\, d)$ induces an *obstruction*

$$\psi^* k \in Z^3_s(Q, \text{Ker}\, d).$$

Under this notion of obstruction, we state and prove the following theorem.

Theorem 7. *Let (B, D, d) be an abelian crossed module, and let $\psi : Q \to \text{Coker}\, d$ be a homomorphism of abelian groups. Then, the vanishing of $\overline{\psi^* k}$ in $H^3_s(Q, \text{Ker}\, d)$ is necessary and sufficient for there to exist an extension of B by Q of type $B \to D$ inducing ψ. Further, if $\overline{\psi^* k}$ vanishes, then the set of equivalence classes of such extensions is bijective with $H^2_s(Q, \text{Ker}\, d)$.*

Proof. By the assumption $\overline{\psi^* k} = 0$, it follows by Proposition 1 that there is a symmetric monoidal functor $(\Psi, \widetilde{\Psi}) : \text{Dis}\, Q \to S_{\mathbb{P}}$. Then the composition of $(\Psi, \widetilde{\Psi})$ and the canonical symmetric monoidal functor $(H, \widetilde{H}) : S_{\mathbb{P}} \to \mathbb{P}$ is a symmetric monoidal functor $(F, \widetilde{F}) : \text{Dis}\, Q \to \mathbb{P}$, and hence by Lemma 5, we obtain an associated extension $\mathcal{E}_F$.

Conversely, suppose that there is an extension as in the diagram (4). Let $\mathbb{P}'$ be a strict Picard category associated to the abelian crossed module $B \to E$. Then, according to Lemma 2, there is a symmetric monoidal functor $F : \mathbb{P}' \to \mathbb{P}$. Since the reduced Picard category of $\mathbb{P}'$ is $\text{Dis}\, Q$, by Proposition 1 $i)$, F induces a symmetric monoidal functor of type $(\psi, 0)$ from $\text{Dis}\, Q$ to $S_{\mathbb{P}} = (\text{Coker}\, d, \text{Ker}\, d, k)$. Now, thanks to Proposition 1 $iii)$, the obstruction of the pair $(\psi, 0)$ vanishes in $H^3_s(Q, \text{Ker}\, d)$, i.e., $\overline{\psi^* k} = 0$.

The final assertion of the theorem is obtained from Theorem 6. First, there is a natural bijection

$$\text{Hom}^{Pic}_{(\psi,0)}[\text{Dis}\, Q, \mathbb{P}] \leftrightarrow \text{Hom}^{Pic}_{(\psi,0)}[\text{Dis}\, Q, S_{\mathbb{P}}].$$

Since $\pi_0(\text{Dis}\, Q) = Q, \pi_1(S_{\mathbb{P}}) = \text{Ker}\, d$, the bijection

$$\text{Ext}^{ab}_{B \to D}(Q, B, \psi) \leftrightarrow H^2_s(Q, \text{Ker}\, d)$$

follows from Theorem 6 and Proposition 1. $\quad\square$

In the case when the homomorphism d of the abelian crossed module $\mathcal{M}$ is a monomorphism, then the diagram (4) shows that the extension $(\mathcal{E} : B \to E \to Q)$ is obtained from the extension $(\mathcal{D} : B \to D \to \operatorname{Coker} d)$ and ψ, i.e., $\mathcal{E} = \mathcal{D}\psi$ (see [9, 6]). Since $\operatorname{Ker} d = 0$, by Theorem 7, we obtain a well-known result as follows.

Corollary 8. *Let $(\mathcal{D} : B \to D \to C)$ be an extension of abelian groups and $\psi : Q \to C$ be a homomorphism of abelian groups. Then, there is an extension $\mathcal{D}\psi$ determined uniquely up to equivalence.*

REFERENCES

[1] J. C. Baez and A. D. Lauda, *Higher Dimensional Algebra V: 2-groups*, Theory Appl. Categ. 12 (2004) 423–491.

[2] R. Brown and O. Mucuk, *Covering groups of non-connected topological groups revisited*, Math. Proc. Camb. Phil. Soc. 115 (1994) 97–110.

[3] R. Brown and C. B. Spencer, $\mathcal{G}$-*groupoids, crossed modules and the fundamental groupoid of a topological group*, Proc. Konn. Ned. Akad. v. Wet. 79 (1976) 296–302.

[4] P. Carrasco, A. M. Cegarra and A. R. -Grandjeán, *(Co)homology of crossed modules*, Category theory 1999 (Coimbra). J. Pure Appl. Algebra 168 no. 2-3 (2002) 147–176.

[5] P. Dedecker, *Les foncteurs Ext_Π, H^2_Π et H^2_Π non abéliens*, C. R. Acad. Sci. Paris 258 (1964) 4891–4894.

[6] L. Fuchs, *Infinite Abelian Groups*, Vol. 1, Academic Press New York and London 1970.

[7] A. Joyal and R. Street, *Braided tensor categories,* Advances in Mathematics 102 (1993) 20–78.

[8] S. Mac Lane, *Cohomology theory of Abelian groups,* from: "Proceedings of the International Congress of Mathematicians, Cambridge, Mass., Vol. 2", Amer. Math. Soc. (1952) 8–14.

[9] S. Mac Lane, *Homology,* Springer-Verlag 1963.

[10] B. Noohi, *Notes on 2-groupoids, 2-groups and crossed modules,* Homology, Homotopy Appl. 9 (2007) 75–106.

[11] K. Norrie, *Actions and automrphisms of crossed modules*, Bull. Coc. Math. France Tome 118 No 2 (1990) 129–146.

[12] N. T. Quang, N. T. Thuy and P. T. Cuc, *Monoidal functors between (braided) Gr-categories and their applications*, East-West J. of Mathematics, Vol. 13 No 2 (2011) 170–193.

[13] H. X. Sinh, *Gr-catégories*, Thèse de doctorat, Université Paris VII 1975.

[14] J. H. C. Whitehead, *Combinatorial homotopy II*, Bull. Amer. Math. Soc 55 (1949) 453–496.